Velocity & Acceleration in Terms of x

If $v = f(x)$;

$$\boxed{\frac{d^2x}{dt^2} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)}$$

Proof:

$$\frac{d^2x}{dt^2} = \frac{dv}{dt}$$

$$= \frac{dv}{dx} \cdot \frac{dx}{dt}$$

$$= \frac{dv}{dx} \cdot v$$

$$= \frac{dv}{dx} \cdot \frac{d}{dv}\left(\frac{1}{2}v^2\right)$$

$$= \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

e.g. (i) A particle moves in a straight line so that $\ddot{x} = 3 - 2x$

Find its velocity in terms of x given that $v = 2$ when $x = 1$.

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 3 - 2x$$

$$\frac{1}{2}v^2 = 3x - x^2 + c$$

when $x = 1, v = 2$

$$\text{i.e. } \frac{1}{2}(2)^2 = 3(1) - 1^2 + c$$

$$c = 0$$

$$\therefore v^2 = 6x - 2x^2$$

$$\underline{v = \pm\sqrt{6x - 2x^2}}$$

NOTE:

$$v^2 \geq 0$$

$$6x - 2x^2 \geq 0$$

$$2x(3 - x) \geq 0$$

$$0 \leq x \leq 3$$

$\therefore$ Particle moves between $x = 0$ and $x = 3$ and nowhere else.

(ii) A particle's acceleration is given by $\ddot{x} = 3x^2$. Initially, the particle is 1 unit to the right of O, and is traveling with a velocity of $\sqrt{2}$m/s in the negative direction. Find x in terms of t.

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 3x^2$$

$$\frac{1}{2}v^2 = x^3 + c$$

when $t = 0, x = 1, v = -\sqrt{2}$

$$\text{i.e.}\ \frac{1}{2}\left(-\sqrt{2}\right)^2 = 1^3 + c$$

$$c = 0$$

$$\therefore v^2 = 2x^3$$

$$v = \pm\sqrt{2x^3}$$

$$\frac{dx}{dt} = -\sqrt{2x^3}$$

(Choose –ve to satisfy the initial conditions)

$$= -\sqrt{2}x^{\frac{3}{2}}$$

$$\frac{dt}{dx} = -\frac{1}{\sqrt{2}}x^{-\frac{3}{2}}$$

$$\frac{dt}{dx} = -\frac{1}{\sqrt{2}}x^{-\frac{3}{2}}$$

$$t = -\frac{1}{\sqrt{2}} \cdot -2x^{-\frac{1}{2}} + c$$

$$= \sqrt{2}x^{-\frac{1}{2}} + c$$

$$= \sqrt{\frac{2}{x}} + c$$

when $t = 0$, $x = 1$

$$\text{i.e. } 0 = \sqrt{2} + c$$

$$c = -\sqrt{2}$$

$$t = \sqrt{\frac{2}{x}} - \sqrt{2}$$

OR

$$t = -\frac{1}{\sqrt{2}}\int_1^x x^{-\frac{3}{2}}dx$$

$$= -\frac{1}{\sqrt{2}}\left[-2x^{-\frac{1}{2}}\right]_1^x$$

$$= \sqrt{2}\left(\frac{1}{\sqrt{x}} - 1\right)$$

$$t + \sqrt{2} = \sqrt{\frac{2}{x}}$$

$$\frac{2}{x} = \left(t + \sqrt{2}\right)^2$$

$$\underline{x = \frac{2}{\left(t + \sqrt{2}\right)^2}}$$

2004 Extension 1 HSC Q5a)

A particle is moving along the x axis starting from a position 2 metres to the right of the origin (that is, $x = 2$ when $t = 0$) with an initial velocity of 5 m/s and an acceleration given by $\ddot{x} = 2x^3 + 2x$

(i) Show that $\dot{x} = x^2 + 1$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 2x^3 + 2x$$

$$\frac{1}{2}v^2 = \frac{1}{2}x^4 + x^2 + c$$

When $x = 2$, $v = 5$

$$\frac{1}{2}(25) = \frac{1}{2}(16) + (4) + c$$

$$c = \frac{1}{2}$$

$$v^2 = x^4 + 2x^2 + 1$$

$$v^2 = \left(x^2 + 1\right)^2$$

$$\underline{v = x^2 + 1}$$

Note: v > 0, in order to satisfy initial conditions

(ii) Hence find an expression for x in terms of t

$$\frac{dx}{dt} = x^2 + 1$$

$$\int_0^t dt = \int_2^x \frac{dx}{x^2+1}$$

$$t = \left[\tan^{-1} x\right]_2^x$$

$$t = \tan^{-1} x - \tan^{-1} 2$$

$$\tan^{-1} x = t + \tan^{-1} 2$$

$$x = \tan\left(t + \tan^{-1} 2\right)$$

$$x = \frac{\tan t + 2}{1 - 2\tan t}$$

Exercise 3E; 1 to 3 acfh, 7 , 9, 11, 13, 15, 17, 18, 20, 21, 24*