Sydney Grammar School

2013 Trial Examination

FORM VI
MATHEMATICS EXTENSION 2

Thursday 1st August 2013

General Instructions

- Reading time — 5 minutes
- Writing time — 3 hours
- Write using black or blue pen.
- Board-approved calculators and templates may be used.
- A list of standard integrals is provided at the end of the examination paper.

Total — 100 Marks

- All questions may be attempted.

Section I – 10 Marks

- Questions 1–10 are of equal value.
- Record your solutions to the multiple choice on the sheet provided.

Section II – 90 Marks

- Questions 11–16 are of equal value.
- All necessary working should be shown.
- Start each question in a new booklet.

Collection

- Write your candidate number on each booklet and on your multiple choice answer sheet.
- Hand in the booklets in a single well-ordered pile.
- Hand in a booklet for each question in Section II, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Place your multiple choice answer sheet inside the answer booklet for Question Eleven.
- Write your candidate number on this question paper and submit it with your answers.

Checklist

- SGS booklets — 6 per boy
- Multiple choice answer sheet
- Candidature — 69 boys

Examiner
DS/REP

SECTION I - Multiple Choice

Answers for this section should be recorded on the separate answer sheet handed out with this examination paper.

QUESTION ONE

Which function is a primitive of $\dfrac{2x}{2x-1}$?

(A) $x + \ln(2x-1)$

(B) $\ln(2x-1)$

(C) $x + \frac{1}{2}\ln(2x-1)$

(D) $\frac{1}{2}\ln(2x-1)$

QUESTION TWO

Which expression is a correct factorisation of $z^3 - i$?

(A) $(z-i)(z^2+iz+1)$

(B) $(z+i)(z^2-iz-1)$

(C) $(z+i)(z-i)^2$

(D) $(z+i)^3$

QUESTION THREE

If $f(x)$ is an odd function, which statement

(A) $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

(B) $\displaystyle\int_{0}^{2a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

(C) $\displaystyle\int_{-2a}^{a} f(x)\,dx = \int_{-a}^{2a} f(x)\,dx$

(D) $\displaystyle\int_{-a}^{2a} f(x)\,dx = \int_{a}^{2a} f(x)\,dx$

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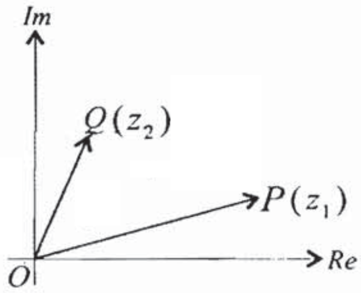
QUESTION FOUR



The points P and Q in the first quadrant represent the complex numbers z_1 and z_2 respectively, as shown in the diagram above. Which statement about the complex number $z_2 - z_1$ is true?

(A) It is represented by the vector QP.

(B) Its principal argument lies between $\frac{\pi}{2}$ and π.

(C) Its real part is positive.

(D) Its modulus is greater than $|z_1 + z_2|$.

QUESTION FIVE

An ellipse has foci at $(-6, 0)$ and $(6, 0)$, and its directrices have equations $x = -8$ and $x = 8$. What is the eccentricity of the ellipse?

(A) $\frac{1}{2}\sqrt{3}$ (B) $\frac{1}{3}\sqrt{3}$ (C) $\frac{2}{3}\sqrt{3}$ (D) $3\sqrt{3}$

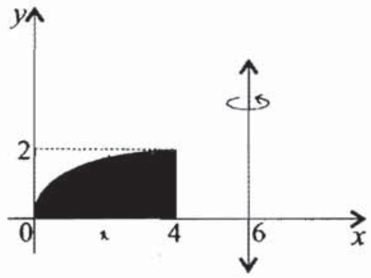
QUESTION SIX

The polynomial $P(x) = x^3 + 3x^2 - 24x + 28$ has a double zero. What is the value of the double zero?

(A) -7 (B) -4 (C) 4 (D) 2

Exam continues overleaf ...

QUESTION SEVEN



The diagram above shows the region bounded by the curve $y = \sqrt{x}$ and the x-axis, from $x = 0$ to $x = 4$. The region is rotated about the line $x = 6$ to form a solid of revolution. Which integral gives the volume of the solid?

(A) $\displaystyle\int_0^2 \pi(4 - y^2)(8 - y^2)\,dy$

(B) $\displaystyle\int_0^2 4\pi(5 - y^2)\,dy$

(C) $\displaystyle\int_0^2 \pi(4 - y^2)(6 - y^2)\,dy$

(D) $\displaystyle\int_0^2 \pi(2 - y^2)(6 - y^2)\,dy$

QUESTION EIGHT

The curve defined by the equation $x^2 - xy + 2y^2 = 4$ passes through the point $P(1, -1)$. What is the gradient of the tangent to the curve at P?

(A) $\frac{1}{2}$ (B) $-\frac{1}{5}$ (C) $\frac{3}{5}$ (D) $\frac{3}{4}$

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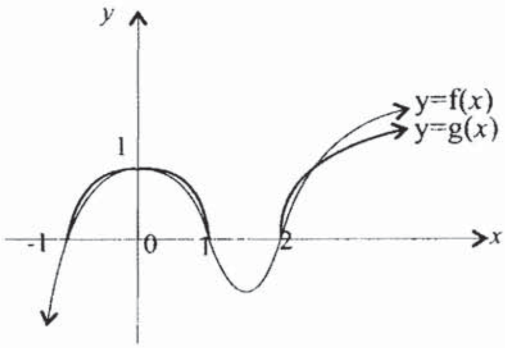
QUESTION NINE



In the diagram above, the relationship between the functions $f(x)$ and $g(x)$ could be represented by:

(A) $g(x) = \left(f(x)\right)^2$

(B) $g(x) = \log_e f(x)$

(C) $g(x) = \sqrt{f(x)}$

(D) $g(x) = |f(x)|$

QUESTION TEN

Without attempting to evaluate the integrals, determine which of the following inequalities is FALSE:

(A) $\displaystyle\int_1^2 \frac{1}{1+x}\,dx < \int_1^2 \frac{1}{x}\,dx$

(B) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x}{x}\,dx < \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{x}\,dx$

(C) $\displaystyle\int_0^{\frac{\pi}{4}} \tan^2 x\,dx < \int_0^{\frac{\pi}{4}} \tan^3 x\,dx$

(D) $\displaystyle\int_1^2 e^{-x^2}\,dx < \int_0^1 e^{-x^2}\,dx$

End of Section I

Exam continues overleaf ...

SECTION II - Written Response

Answers for this section should be recorded in the booklets provided.

Show all necessary working.

Start a new booklet for each question.

QUESTION ELEVEN (15 marks) Use a separate writing booklet. **Marks**

(a) Solve the quadratic equation $4z^2 + 4z + 5 = 0$. **2**

(b) Find the real values of x and y for which $\dfrac{x}{i} - \dfrac{y}{1+i} = -1 - 3i$. **2**

(c) Find $\displaystyle\int \frac{3x}{2x^2 - 5x + 2}\,dx$. **3**

(d) Find $\displaystyle\int \frac{1}{\sqrt{x^2 + 6x + 34}}\,dx$. **2**

(e) Evaluate $\displaystyle\int_0^{\frac{\pi}{3}} \tan^4 x\,dx$. **3**

(f) (i) Use the substitution $u = a - x$ to prove that $\displaystyle\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$. **1**

(ii) Hence find the value of $\displaystyle\int_0^1 x(1-x)^7\,dx$. **2**

Exam continues next page ...

QUESTION TWELVE (15 marks) Use a separate writing booklet. **Marks**

(a) Shade the region in the Argand diagram where $|z+3i| > 2|z|$. **3**

(b) (i) Express $-2+2i$ in modulus-argument form. **1**

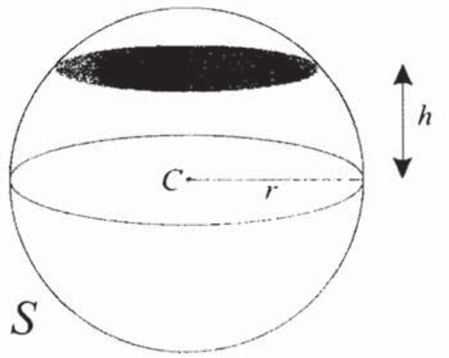
(ii) Simplify $(-2+2i)^{8k}$, where k is an integer. **2**

(c) A complex number z satisfies $\arg(z-1) = \frac{\pi}{6}$.

(i) Sketch the locus of z. **1**

(ii) Show that $|z-5| \geq 2$. **1**

(d)



In the diagram above, S is a sphere of radius r. The point C is the centre of the sphere. A typical horizontal cross-section h units above C is shown.

(i) Find the area of this cross-section as a function of h. **1**

(ii) Hence prove that the volume of S is $\frac{4}{3}\pi r^3$. **2**

(e) The polynomial $P(x) = x^4 - 4x^3 + 10x^2 - 12x - 40$ has zeroes α, β, γ and δ.

(i) Find a polynomial with zeroes $\alpha - 1$, $\beta - 1$, $\gamma - 1$ and $\delta - 1$. **2**

(ii) Hence find the zeroes of $P(x)$. **2**

Exam continues overleaf ...

QUESTION THIRTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) (i) Write down the equation of a line with gradient m that passes through the point $T(-4c, 2c)$. **1**

(ii) Solve the equation in part (i) simultaneously with the equation $xy = c^2$ to obtain a quadratic equation in x. **2**

(iii) Hence find the gradients of the two tangents to the rectangular hyperbola $xy = c^2$ that pass through the point $T(-4c, 2c)$. **2**

(b) Consider the polynomial $P(z) = z^4 + (1 - 2i)z^2 - 2i$.

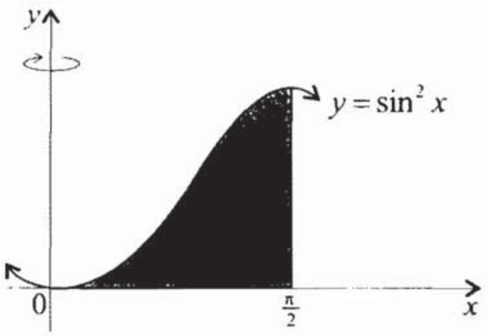
(i) Show that $P(i) = 0$. **1**

(ii) Explain why $P(-i)$ must also be zero. **1**

(iii) Suppose that the other two zeroes of $P(z)$ are w and $-w$. **3**
Use the product of the zeroes to find w.

(c) (i) Find $\displaystyle\int x \cos 2x \, dx$. **2**

(ii) **3**



The diagram above shows the region bounded by the curve $y = \sin^2 x$, the x-axis and the line $x = \frac{\pi}{2}$. The region is rotated about the y-axis through $360°$. Use the method of cylindrical shells to find the exact volume of the solid formed.

Exam continues next page ...

QUESTION FOURTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) An object of mass 2 kg is projected vertically upwards at 20 m/s and experiences air resistance of magnitude $\frac{1}{10}v^2$ Newtons, where v is the speed of the object after t seconds. Take $g = 10\,\text{m/s}^2$.

(i) Show that the maximum height reached by the object is $10\ln 3$ metres. **3**

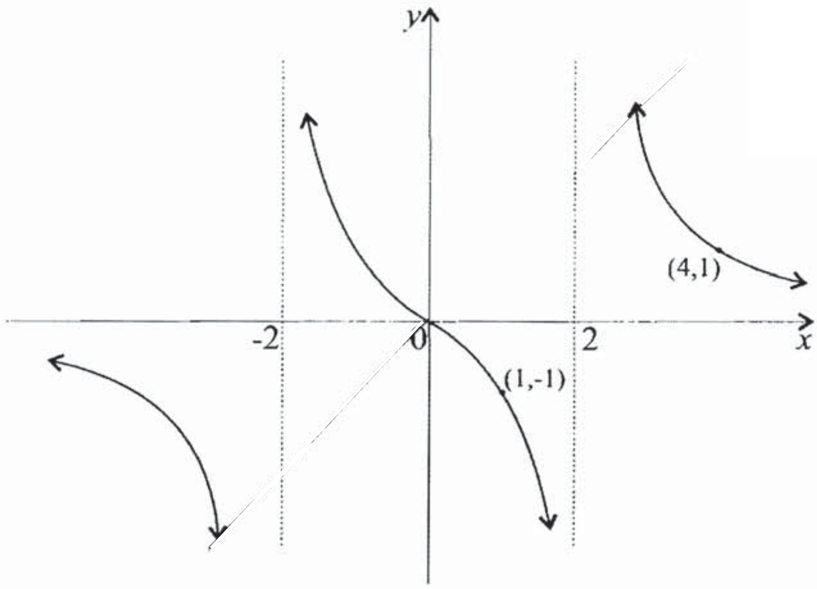
(ii) Find the speed of the object, correct to three significant figures, at the instant it reaches half its maximum height. **2**

(b) Let $I_n = \int_1^e (1 + \ln x)^n \, dx$, where $n \geq 0$.

(i) Use integration by parts to show that $I_n = (2^n)\,e - 1 - n\,I_{n-1}$. **2**

(ii) Hence find the exact value of $\int_1^e (2 + \ln x)\,(1 + \ln x)^2 \, dx$. **2**

(c)



The diagram above shows the graph of the odd function $y = f(x)$, where

$$f(x) = \frac{3x}{x^2 - 4}.$$

Sketch the graphs of each of the following functions on large separate diagrams, showing the x-intercepts and asymptotes. You are NOT expected to find any stationary points.

(i) $y = \dfrac{1}{f(x)}$ **2**

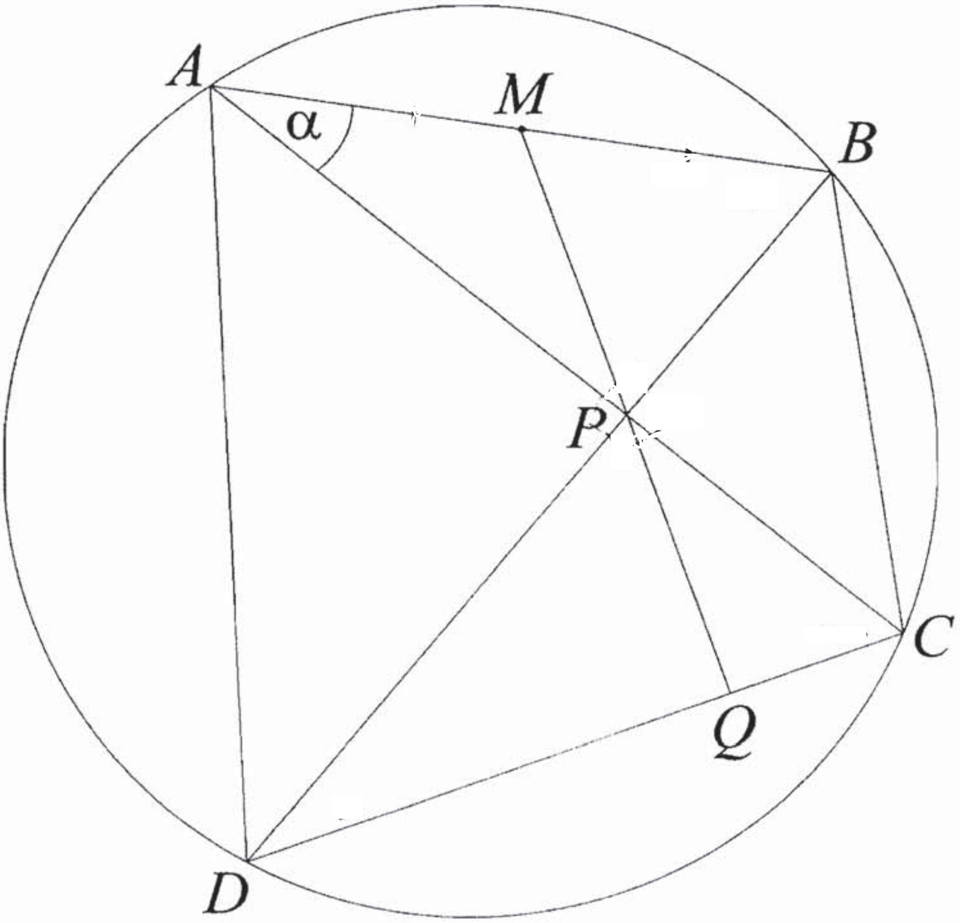
(ii) $y = \log_e f(x)$ **2**

(iii) $y = x + f(x)$ **2**

Exam continues overleaf ...

QUESTION FIFTEEN (15 marks) Use a separate writing booklet. **Marks**

(a)



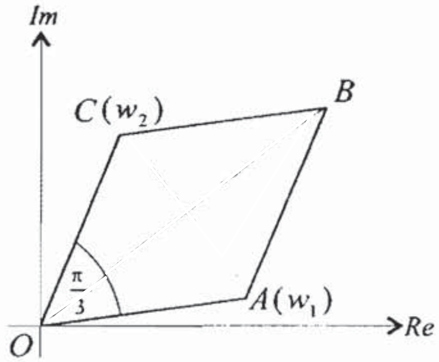
In the diagram above $ABCD$ is a cyclic quadrilateral whose diagonals are perpendicular and intersect at P. Let M be the midpoint of AB, and suppose that MP produced meets DC at Q. Let $\angle PAM = \alpha$.

(i) Explain why $AM = PM$. **1**

(ii) Prove that $MQ \perp DC$. **3**

QUESTION FIFTEEN (Continued)

(b)



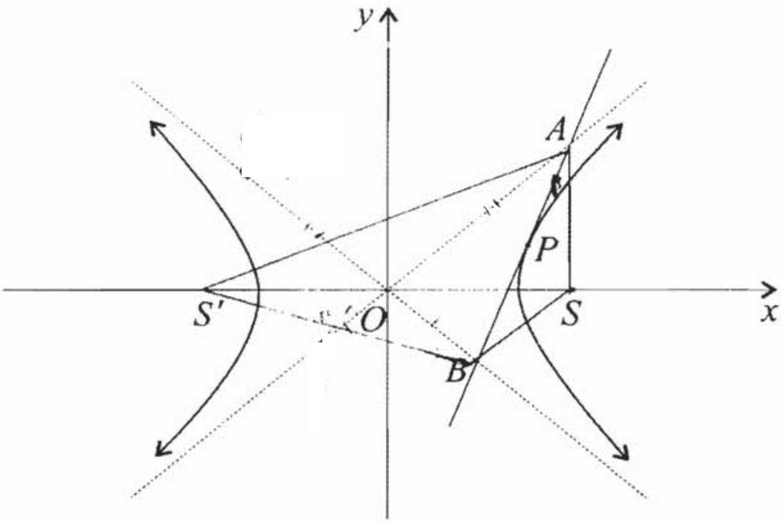
In the Argand diagram above, $OABC$ is a rhombus with $\angle COA = \frac{\pi}{3}$. The points A and C represent the complex numbers w_1 and w_2 respectively.

(i) Explain why $w_2 = w_1 \operatorname{cis} \frac{\pi}{3}$. **1**

(ii) Write down, in terms of w_1 only, the complex numbers represented by the vectors OB and AC. **2**

(iii) By considering $i(w_1 + w_2)$, show that the diagonals OB and AC of the rhombus are perpendicular. **2**

(c)



The diagram above shows the tangent at a point $P(a\sec\theta, b\tan\theta)$ on the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ meeting the asymptotes of the hyperbola at A and B. The points S and S' are the foci of the hyperbola.

(i) Show that the tangent at P has equation $bx\sec\theta - ay\tan\theta = ab$. **1**

(ii) Show that $OA \times OB = a^2 + b^2$, where O is the origin. **2**

(iii) Extend AO to A' so that $OA' = OB$ and extend BO to B' so that $OB' = OA$. Explain why the points A, B, A' and B' are concyclic. **1**

(iv) Hence show that the points A, S, B and S' are concyclic. **2**

Exam continues overleaf ...

QUESTION SIXTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) Let $z = \cos\theta + i\sin\theta$ and suppose that n is a positive integer.

(i) Show that $z^n + z^{-n} = 2\cos n\theta$. **1**

(ii) Hence use the identity $2\cos A\sin B = \sin(A+B) - \sin(A-B)$ to show that **2**

$$\left(z^{2n} + z^{2n-2} + z^{2n-4} + \cdots + z^{-2n}\right)\sin\theta = \sin(2n+1)\theta.$$

(iii) Use part (ii) and the identity $\cos 3A = 4\cos^3 A - 3\cos A$ to deduce that **1**

$$8\cos^3 2\theta + 4\cos^2 2\theta - 4\cos 2\theta - 1 = \frac{\sin 7\theta}{\sin\theta}.$$

(iv) Hence show that $\cos\frac{2\pi}{7}$ is a root of the equation $8x^3 + 4x^2 - 4x - 1 = 0$. **1**

(b) (i) Use a suitable double angle formula to show that $\tan\frac{\pi}{8} = \sqrt{2} - 1$. **1**

(ii) Find $\cos 4\theta$ in terms of $\cos\theta$. **1**

(iii) Let $I = \displaystyle\int_{-1}^{1} \frac{1}{\sqrt{1+x} + \sqrt{1-x} + 2}\, dx$.

(α) Show, by using the substitution $x = \sin 4\theta$, that **2**

$$I = \int_0^{\frac{\pi}{8}} \frac{2\cos 4\theta}{\cos^2\theta}\, d\theta.$$

(β) Hence find the exact value of I. **2**

(c) (i) Explain why **1**

$$\left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \cdots + \left(\frac{1}{2k-1} - \frac{1}{2k}\right) > \frac{k}{(2k+1)(2k+2)}.$$

(ii) Prove by mathematical induction, or otherwise, that for all $n \geq 2$, **3**

$$n\left(1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1}\right) > (n+1)\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{2n}\right).$$

End of Section II

END OF EXAMINATION

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① $\int \frac{2x}{2x-1}\,dx = \int \left[1 + \frac{1}{2x-1} \right] dx$

$= \underline{\underline{x + \frac{1}{2}\ln|2x-1| + c}}$ (C)

② $z^3 - i = z^3 + i^3$

$= (z+i)(z^2 - iz + i^2)$

$= \underline{\underline{(z+i)(z^2 - iz - 1)}}$ (B)

③ $\int_{-a}^{a} f(x)\,dx = f(a) - f(-a)$

$= f(a) + f(a)$

$= 2f(a)$

$2\int_{0}^{a} f(x)\,dx = 2f(a) - 2f(0)$

$= 2f(a)$

$\int_{0}^{2a} f(x)\,dx = f(2a) - f(0)$

$= f(2a)$

$\int_{-2a}^{a} f(x)\,dx = f(a) - f(-2a)$

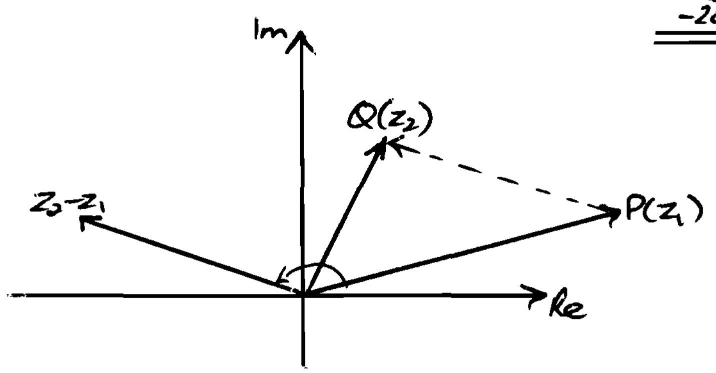
$= f(a) + f(2a)$

$\int_{-a}^{2a} f(x)\,dx = f(2a) - f(-a)$

$= f(2a) + f(a)$

$\therefore \underline{\underline{\int_{-2a}^{a} f(x)\,dx = \int_{-a}^{2a} f(x)\,dx}}$ (C)

④



$\arg(z_2 - z_1)$ is in Q_2

$\therefore \frac{\pi}{2} < \arg(z_2 - z_1) < \pi$

(B)

⑤ $2ae = 12$

$ae = 6$

$\frac{a}{e} = 8$

$a = 8e$

$\therefore 8e^2 = 6$

$e^2 = \frac{3}{4}$

$\underline{\underline{e = \frac{\sqrt{3}}{2}}}$ (A)

⑥ $P(x) = x^3 + 3x^2 - 24x + 28$

$P'(x) = 3x^2 + 6x - 24$

$= 3(x^2 + 2x - 8)$

$= 3(x-2)(x+4)$

$\therefore$ double zero is 2 or -4

$P(-4) = (-4)^3 + 3(-4)^2 - 24(-4) + 28$

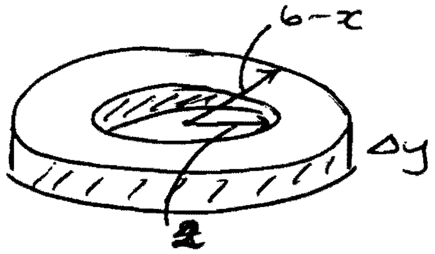
$= 108$

$P(2) = 2^3 + 3(2)^2 - 24(2) + 28$

$= 0$

$\therefore$ $\underline{\underline{\text{double root is } 2}}$ (D)

(7)



$$A(y) = \pi\{(6-y^2)^2 - 2^2\}$$
$$= \pi(36 - 12y^2 + y^4 - 4)$$
$$\Delta V = \pi(32 - 12y^2 + y^4)\,\Delta y$$
$$V = \lim_{\Delta y \to 0} \sum_{y=0}^{2} \pi(4-y)^2(8-y^2)\,\Delta y$$
$$= \int_0^2 \pi(4-y^2)(8-y^2)\,dy \qquad (A)$$

(8) $x^2 - xy + 2y^2 = 4$

$$2x - x\frac{dy}{dx} - y + 4y\frac{dy}{dx} = 0$$
$$(4y - x)\frac{dy}{dx} = y - 2x$$
$$\frac{dy}{dx} = \frac{y-2x}{4y-x}$$

At P, $\frac{dy}{dx} = \frac{-1-2}{-4-1}$
$= \frac{-3}{-5}$
$= \frac{3}{5}$

(C)

(9) $g(x) = \sqrt{f(x)}$ (C)

(10) $1 < x < 2$; $\frac{1}{1+x} < \frac{1}{x}$ ∴ A ✓

$\frac{\pi}{6} < x < \frac{\pi}{3}$; $\frac{\sin x}{x} < \frac{1}{x}$ ∴ B ✓

$0 < x < \frac{\pi}{4}$; $\tan^2 x > \tan^3 x$ ∴ C ✗ (C)

Question 11

a) $4z^2 + 4z + 5 \quad = \dfrac{-4 \pm \sqrt{-64}}{8}$

$= \dfrac{-4 \pm 8i}{8}$

$= -\frac{1}{2} \pm i$

b) $\dfrac{x}{i} - \dfrac{y}{1+i} = -1-3i$

$-ix \quad - \dfrac{(1-i)y}{2} = -1-3i$

$\therefore \; -\frac{y}{2} = -1 \qquad\qquad -x + \frac{y}{2} = -3$

$y = 2 \qquad\qquad -x + 1 = -3$

$-x = -4$

$x = 4 \qquad\qquad \therefore \; x = 4, \; y = 2$

c) $\displaystyle\int \frac{3x}{2x^2 - 5x + 2}\,dx$

$\displaystyle = \int \frac{3x}{(2x-1)(x-2)}\,dx$

$\displaystyle = \int \left[\frac{1}{2x-1} + \frac{2}{x-2} \right] dx$

$= \frac{1}{2}\ln(2x-1) + 2\ln(x-2) + c$

$A(x-2) + B(2x-1) = 3x$

$x = 2$: $3B = 6$, $B = 2$

$x = \frac{1}{2}$: $-\frac{3A}{2} = \frac{3}{2}$, $A = 1$

d) $\displaystyle\int \frac{dx}{\sqrt{x^2 + 6x + 34}} \quad = \int \frac{dx}{\sqrt{(x+3)^2 + 25}}$

$= \ln\left(x + 3 + \sqrt{x^2 + 6x + 34}\right) + c$

e) $\displaystyle\int_0^{\frac{\pi}{4}} \tan^4 x\,dx$

$\displaystyle = \int_0^{\frac{\pi}{4}} (\sec^2 x \tan^2 x)\,dx \; - \int_0^{\frac{\pi}{4}} \tan^2 x\,dx$

$\displaystyle = \int_0^{\frac{\pi}{4}} \tan^2 x \sec^2 x\,dx \; + \int_0^{\frac{\pi}{4}} (1 - \sec^2 x)\,dx$

$= \left[\frac{1}{3}\tan^3 x + x - \tan x \right]_0^{\frac{\pi}{4}}$

$= \frac{1}{3} + \frac{\pi}{4} - 1 - 0$

$= \frac{2}{3} + \frac{\pi}{4}$

f) (i) $\displaystyle\int_0^a f(x)\,dx \; = -\int_a^0 f(a-u)\,du$

$\displaystyle = \int_0^a f(a-u)\,du$

$\displaystyle = \int_0^a f(a-x)\,dx$

(ii) $\displaystyle\int_0^1 x(1-x)^7\,dx$

$\displaystyle = \int_0^1 (1-x)x^7\,dx$

$\displaystyle = \int_0^1 (x^7 - x^8)\,dx$

$= \left[\frac{1}{8}x^8 - \frac{1}{9}x^9 \right]_0^1$

$= \frac{1}{8} - \frac{1}{9} - 0$

$= \frac{1}{72}$

Question 12

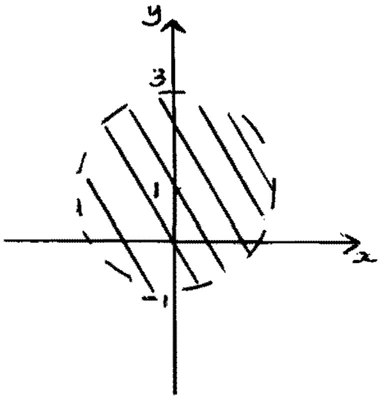
a) $|z+3i| > 2|z|$

$x^2 + (y+3)^2 > 4x^2 + 4y^2$

$3x^2 + 3y^2 - 6y < 9$

$x^2 + y^2 - 2y < 3$

$x^2 + (y-1)^2 < 4$



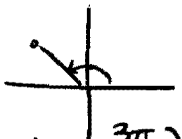
b) (i) $|-2+2i|$

$= \sqrt{2^2+2^2}$

$= 2\sqrt{2}$

$\arg(-2+2i)$

$= \frac{3\pi}{4}$

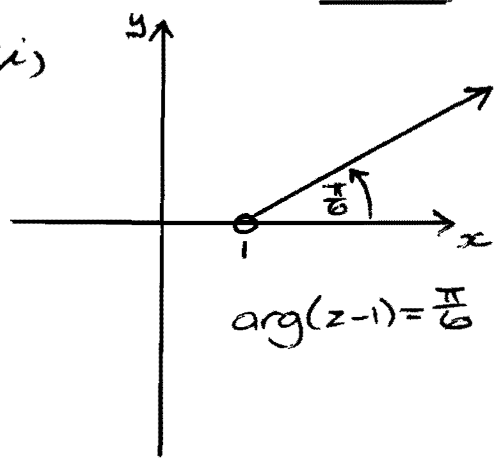


$-2+2i = \underline{\underline{2\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)}}$

(ii) $(-2+2i)^{8k} = (2\sqrt{2})^{8k}(\cos 6\pi k + i\sin 6\pi k)$

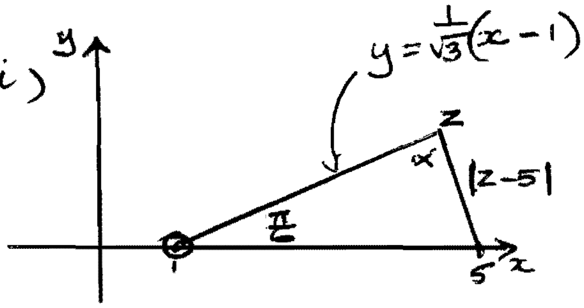
$= \underline{\underline{4096^k}}$

c) (i)



$\arg(z-1) = \frac{\pi}{6}$

(ii)



min $|z-5|$ occurs when $\alpha = \frac{\pi}{2}$

ie perpendicular distance from $(5,0)$ to $y = \frac{1}{\sqrt{3}}(x-1)$

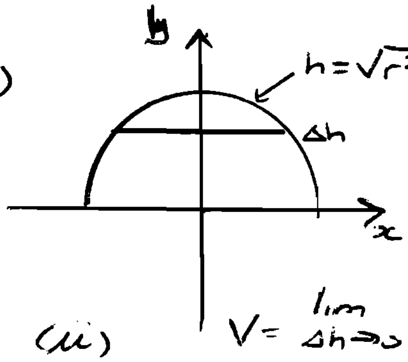
$$|z-5| \geq \frac{\left|\frac{1}{\sqrt{3}}(5) - (0) - \frac{1}{\sqrt{3}}\right|}{\sqrt{\left(\frac{1}{\sqrt{3}}\right)^2 + (1)^2}}$$

$$= \frac{\frac{5}{\sqrt{3}} - \frac{1}{\sqrt{3}}}{\sqrt{\frac{4}{3}}}$$

$$= \frac{4}{2}$$

$$= \underline{\underline{2}}$$

d)



$h=\sqrt{r^2-x^2} \Rightarrow x=\sqrt{r^2-h^2}$

$A(h) = \pi x^2$
$= \pi(r^2-h^2)$

(ii) $V = \lim_{\Delta h \to 0} \sum_{h=-r}^{r} \pi(r^2-h^2)\Delta h$

$= 2\pi \int_0^r (r^2-h^2)\,dh$

$= 2\pi \left[r^2 h - \frac{1}{3}h^3\right]_0^r$

$= 2\pi\left(r^3 - \frac{1}{3}r^3 - 0\right)$

$= \frac{4}{3}\pi r^3$

e) $P(x) = x^4 - 4x^3 + 10x^2 - 12x - 40$

(i) let $y = x-1$; $(y+1)^4 - 4(y+1)^3 + 10(y+1)^2 - 12(y+1) - 40$

$x = y+1$ $\quad = y^4 + 4y^3 + 6y^2 + 4y + 1 - 4y^3 - 12y^2 - 12y - 4$
$+ 10y^2 + 20y + 10 - 12y - 12 - 40$

$P(y) = y^4 + 4y^2 - 45$

(ii) $P(y) = 0$
$y^4 + 4y^2 - 45 = 0$
$(y^2+9)(y^2-5) = 0$
$y^2 = -9$ or $y^2 = 5$
$y = \pm 3i$ $\quad y = \pm\sqrt{5}$

$\therefore P(x) = 0$
$x = 1 \pm \sqrt{5},\ 1 \pm 3i$

Question 13

a) (i) $y - 2c = m(x+4c)$

$\underline{\underline{y = mx + 4mc + 2c}}$

(ii) $xy = c^2$

$\underline{\underline{mx^2 + 2c(2m+1)x - c^2 = 0}}$

(iii) tangents when $\Delta = 0$

$$4c^2(2m+1)^2 + 4mc^2 = 0$$
$$(2m+1)^2 + 4m = 0$$
$$4m^2 + 4m + 1 + 4m = 0$$
$$4m^2 + 8m + 1 = 0$$
$$m = \frac{-8 \pm \sqrt{48}}{8}$$
$$= \underline{\underline{\frac{-2 \pm \sqrt{3}}{2}}}$$

b) $P(z) = z^4 + (1-2i)z^2 - 2i$

$P(i) = i^4 + (1-2i)i^2 - 2i$

$= 1 - 1 + 2i - 2i$

$= \underline{\underline{0}}$

(ii) The order of the pronumerals are even

and $x^{2k} = (-x)^{2k}$ for all integer values of k

$\therefore$ $\underline{\underline{P(-i) = P(i) = 0}}$

(iii) $\alpha\beta\gamma\delta = -2i$

$i \times -i \times w \times -w = -2i$

$-w^2 = -2i$

$w^2 = 2i$

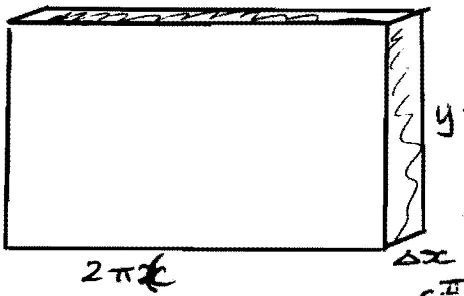
$\underline{\underline{w = \pm(1+i)}}$

c) (i) $\int x\cos 2x\,dx$ $\qquad u = x \qquad v = \frac{1}{2}\sin 2x$

$= \frac{1}{2}\sin 2x - \frac{1}{2}\int \sin 2x\,dx \qquad du = dx \qquad dv = \cos 2x\,dx$

$$= \frac{1}{2}\sin 2x + \frac{1}{4}\cos 2x + c$$

(ii)



$$A(x) = 2\pi x\sin^2 x$$

$$\Delta V = 2\pi x\sin^2 x\,\Delta x$$

$$V = \lim_{\Delta x \to 0} \sum_{x=0}^{\frac{\pi}{2}} 2\pi x\sin^2 x\,\Delta x$$

$$V = 2\pi\int_0^{\frac{\pi}{2}} x\sin^2 x\,dx$$

$$= 2\pi\int_0^{\frac{\pi}{2}} \frac{x}{2}(1-\cos 2x)\,dx$$

$$= \pi\int_0^{\frac{\pi}{2}} (x - x\cos 2x)\,dx$$

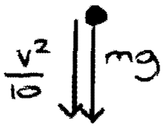
$$= \pi\left[\frac{x^2}{2} - \frac{\sin 2x}{2} - \frac{\cos 2x}{4}\right]_0^{\frac{\pi}{2}}$$

$$= \pi\left(\frac{\pi^2}{8} - 0 + \frac{1}{4} - 0 + 0 + \frac{1}{4}\right)$$

$$= \pi\left(\frac{\pi^2}{8} + \frac{1}{2}\right) \text{ units}^3$$

Question 14

a)



$$m\ddot{x} = -mg - \frac{v^2}{10}$$

$$\ddot{x} = -g - \frac{v^2}{10m}$$

$$\ddot{x} = -10 - \frac{v^2}{20}$$

$$v\frac{dv}{dx} = -\frac{200+v^2}{20}$$

$$\int_0^H dx = -\int_{20}^{0} \frac{20v}{200+v^2}\,dv$$

$$H = 10\Big[\ln|200+v^2|\Big]_0^{20}$$

$$= 10\ln\left(\frac{600}{200}\right)$$

$$\underline{\underline{= 10\ln 3 \text{ m}}}$$

(ii) $$\int_0^{\frac{H}{2}} dx = -\int_{20}^{v} \frac{20v}{200+v^2}\,dv$$

$$5\ln 3 = 10\ln\left(\frac{600}{200+v^2}\right)$$

$$\ln\left(\frac{600}{200+v^2}\right) = \frac{1}{2}\ln 3$$

$$\frac{600}{200+v^2} = \sqrt{3}$$

$$600 = 200\sqrt{3} + \sqrt{3}v^2$$

$$v^2 = \frac{600 - 200\sqrt{3}}{\sqrt{3}}$$

$$= 200(\sqrt{3}-1)$$

$$v = \sqrt{200(\sqrt{3}-1)} \qquad (v>0)$$

$$= 12.10000667\ldots$$

$$= \underline{\underline{12.1 \text{ m/s}}} \quad (\text{to 3 sig fig})$$

b) (i) $I_n = \int_1^e (1+\ln x)^n \, dx$ $\quad u = (1+\ln x)^n \quad v = x$

$du = \frac{n(1+\ln x)^{n-1}}{x} dx \quad dv = dx$

$= \left[x(1+\ln x)^n\right]_1^e - n\int_1^e (1+\ln x)^{n-1} dx$

$= \underline{\underline{e(2^n) - 1 - nI_{n-1}}}$

(ii) $\int_1^e (2+\ln x)(1+\ln x)^2 dx$

$= \int_1^e (1+\ln x)(1+\ln x)^2 dx + \int_1^e (1+\ln x)^2 dx$

$= I_3 + I_2$

$= e(2^3) - 1 - 3I_2 + I_2$

$= 8e - 1 - 2\left(e(2^2) - 1 - 2I_1\right)$

$= 8e - 1 - 8e + 2 + 4\left(e(2^1) - 1 - I_0\right)$

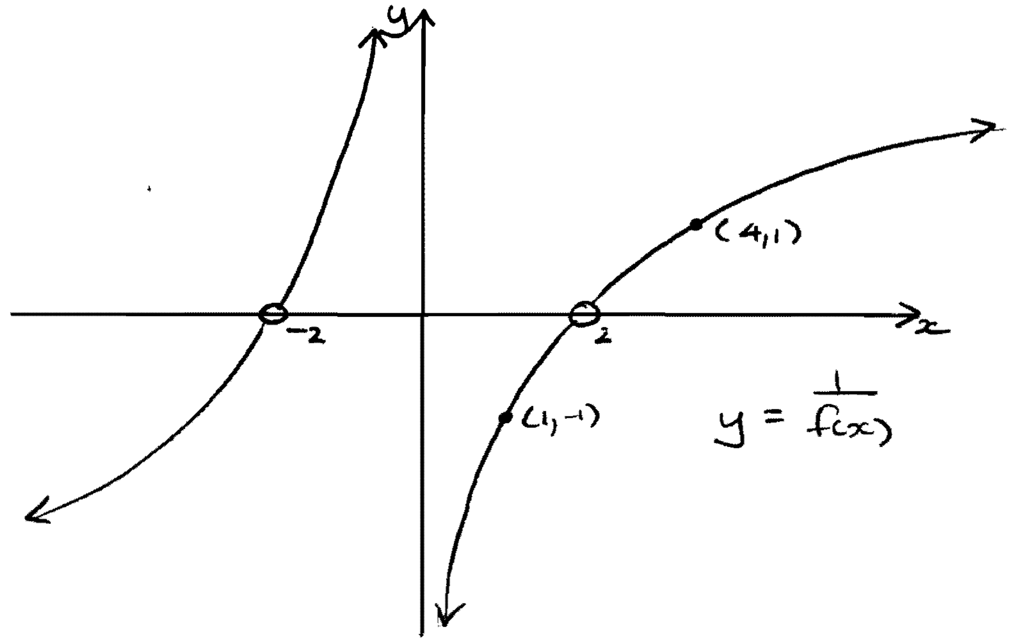
$= 1 + 8e - 4 - 4\int_1^e dx$

$= 8e - 3 - 4(e-1)$

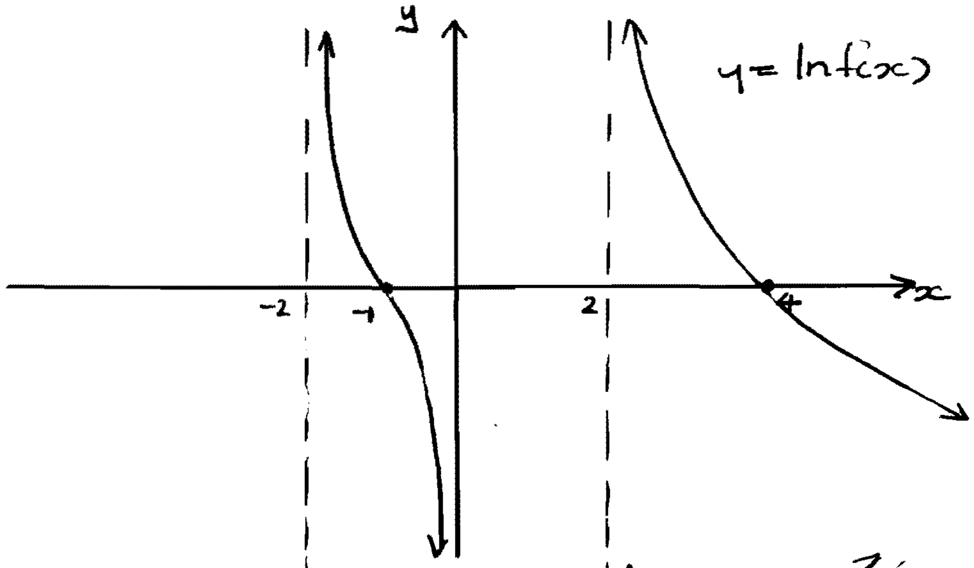
$= \underline{\underline{4e + 1}}$

y

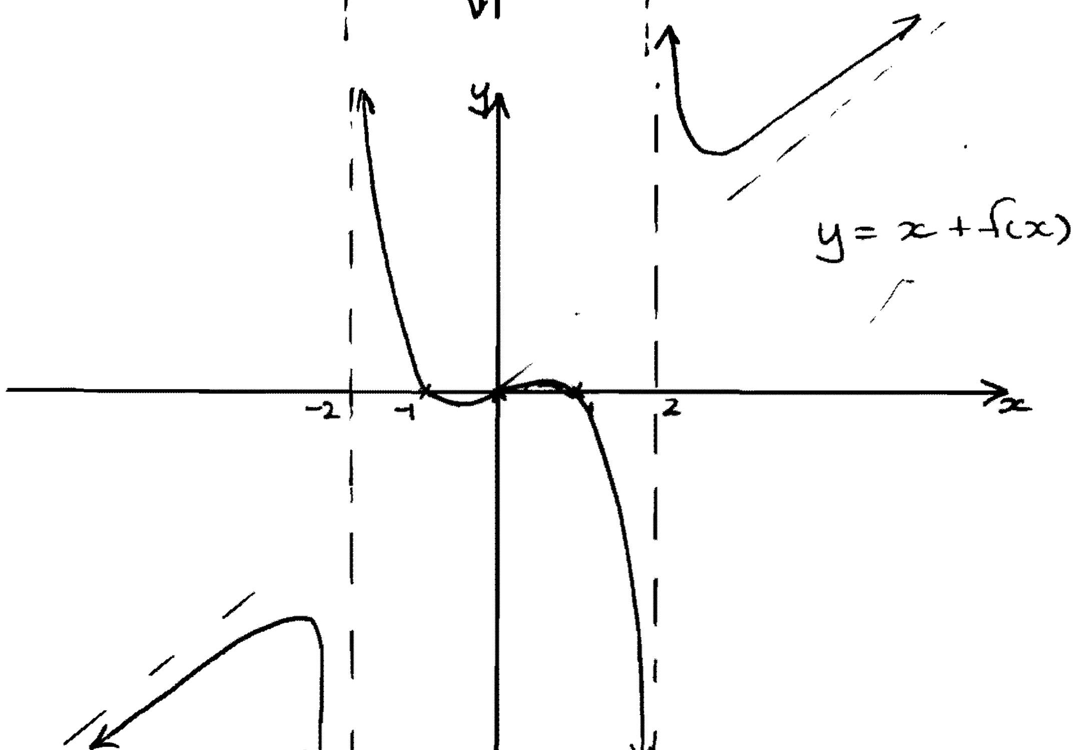
c)(i)



(ii)



(iii)



Question 15

a)(i) $\angle APB = 90°$

$\therefore$ APB is concyclic with AB diameter ($\angle$ in semicircle $= 90$)

M is midpoint of AB $\therefore$ centre of circle APB

Thus $AM = PM$ $(= BM)$ ($=$ radii)

(ii) $\triangle AMP$ is isosceles (2 $=$ sides)

$\therefore$ ~~$\angle PMA = \alpha$~~

$\angle MAP = \angle MPA = \alpha$ (base $\angle$'s isosceles $\triangle$)

$\angle CPQ = \angle MPA = \alpha$ (vertically opposite $\angle$'s

$\angle BDC = \angle BAC = \alpha$ ($\angle$'s in same segment $=$)

In $\triangle PDC$; $\angle DPC = 90°$ (given)

$\angle DPC + \angle PDC + \angle PCD = 180$ ($\angle$ sum $\triangle = 180$)

$90 + \alpha + \angle PCD = 180$

$\angle PCD = 90 - \alpha$

In $\triangle PQC$; $\angle PQC + \angle PCD + \angle CPQ = 180$ ($\angle$ sum $\triangle = 180$)

$\angle PQC + 90 - \alpha + \alpha = 180$

$\angle PQC = 90$

$\therefore MQ \perp DC$

b) When multiplying complex numbers, modulii are multiplied and arguments are added.

$|w_1| = |w_2|$ (sides in a rhombus)

$\arg w_2 = \arg w_1 + \frac{\pi}{3}$ ($\angle AOC = \frac{\pi}{3}$)

$\therefore$ $w_2 = w_1 \operatorname{cis}\frac{\pi}{3}$

(ii) $\vec{OB} = \vec{OA} + \vec{OC}$

$= w_1 + w_1 \operatorname{cis}\frac{\pi}{3}$

$= w_1 + w_1\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$

$= \left(\frac{3}{2} + \frac{\sqrt{3}}{2}i\right) w_1$

$\vec{AC} = w_2 - w_1$

$= w_1\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) - w_1$

$= \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) w_1$

(iii) $i(w_1 + w_2) = i\left(\frac{3}{2} + \frac{\sqrt{3}}{2}i\right) w_1$

$= \left(-\frac{\sqrt{3}}{2} + \frac{3}{2}i\right) w_1$

$= \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\sqrt{3}\, w_1$

$= \sqrt{3} \times \vec{AC}$

$\therefore$ $\vec{AC} = \frac{1}{\sqrt{3}} i \times \vec{OB}$

ie vector is rotated $90°$

Thus diagonals are $\perp$

c) (i) $x = a\sec\theta$ $\qquad$ $y = b\tan\theta$

$\frac{dx}{d\theta} = a\sec\theta\tan\theta$ $\qquad$ $\frac{dy}{d\theta} = b\sec^2\theta$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = \frac{b\sec^2\theta}{a\sec\theta\tan\theta} = \frac{b\sec\theta}{a\tan\theta}$$

$$y - b\tan\theta = \frac{b\sec\theta}{a\tan\theta}(x - a\sec\theta)$$

$$ay\tan\theta - ab\tan^2\theta = bx\sec\theta - ab\sec^2\theta$$

$$bx\sec\theta - ay\tan\theta = ab(\sec^2\theta - \tan^2\theta)$$

$$\underline{\underline{bx\sec\theta - ay\tan\theta = ab}}$$

(ii) $y = \pm\frac{b}{a}x$

$$bx\sec\theta \mp bx\tan\theta = ab$$

$$x = \frac{a}{\sec\theta \mp \tan\theta} \qquad \therefore \cancel{y = \frac{\pm b}{\sec\theta \mp \tan\theta}}$$

A is $\left(\frac{a}{\sec\theta - \tan\theta}, \frac{b}{\sec\theta - \tan\theta}\right)$ $\quad$ B is $\left(\frac{a}{\sec\theta + \tan\theta}, \frac{-b}{\sec\theta + \tan\theta}\right)$

$OA \times OB$

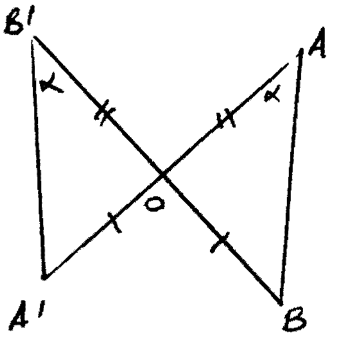
$$= \sqrt{\frac{a^2}{(\sec\theta - \tan\theta)^2} + \frac{b^2}{(\sec\theta - \tan\theta)^2}} \times \sqrt{\frac{a^2}{(\sec\theta + \tan\theta)^2} + \frac{b^2}{(\sec\theta + \tan\theta)^2}}$$

$$= \frac{\sqrt{a^2 + b^2}}{(\sec\theta - \tan\theta)} \times \frac{\sqrt{a^2 + b^2}}{(\sec\theta + \tan\theta)}$$

$$= \frac{a^2 + b^2}{\sec^2\theta - \tan^2\theta}$$

$$= \underline{\underline{a^2 + b^2}}$$

(iii)



$\angle AOB = \angle A'OB'$ (vertically opposite ∠'s =)

$\frac{OB'}{OA} = \frac{OA'}{OB'}$ (given)

$\triangle AOB \parallel\!\parallel \triangle B'OA'$ (SAS, sides in ratio)

$\angle BAO = A'B'O$ (matching ∠'s in ||| △'s)

$\therefore$ <u>ABA'B' are concyclic</u> (∠'s in same segment are =)

(iv) $AO \times OA' = AO \times OB$ $\quad$ ($OA' = OB$, given)

$= a^2 + b^2$

$$OS \times OS' = ae \times ae$$
$$= a^2e^2$$
$$= a^2 + b^2$$
$$= AO \times OB$$
$$= AO \times OA'$$

$\therefore$ A, S, A' and S' are concyclic (ratio of intercepts of intersecting chords =)

<u>Thus A, S, B and S' all lie on same circle</u>

Question 16

a)(i) $z^n + z^{-n} = \cos n\theta + i\sin n\theta + \cos(-n\theta) + i\sin(-n\theta)$

but $\cos\theta$ is even and $\sin\theta$ is odd

$\therefore z^n + z^{-n} = \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$

$= \underline{\underline{2\cos n\theta}}$

(ii) $(z^{2n} + z^{2n-2} + z^{2n-4} + \ldots + z^{-2n})\sin\theta$

$= (z^{2n} + z^{-2n} + z^{2n-2} + z^{2-2n} + z^{2n-4} + z^{4-2n} + \ldots + z^2 + z^{-2} + z^0)\sin\theta$

$= (2\cos 2n\theta + 2\cos(2n-2)\theta + 2\cos(2n-4)\theta + \ldots + 2\cos 2\theta + 1)\sin\theta$

$= \sin(2n+1)\theta - \sin(2n-1)\theta + \sin(2n-1)\theta - \sin(2n-3)\theta + \ldots$

$+ \sin 3\theta - \sin\theta + \sin\theta$

$= \underline{\underline{\sin(2n+1)\theta}}$

(iii) $8\cos^3 2\theta + 4\cos^2 2\theta - 4\cos 2\theta - 1$

$= (8\cos^3 2\theta - 6\cos 2\theta) + (4\cos^2 2\theta - 2) + 2\cos 2\theta + 1$

$= 2\cos 6\theta + 2\cos 4\theta + 2\cos 2\theta + 1$

$= z^6 + z^{-6} + z^4 + z^{-4} + z^2 + z^{-2} + 1$

$= \dfrac{(z^6 + z^{-6} + z^4 + z^{-4} + z^2 + z^{-2} + 1)\sin\theta}{\sin\theta}$

$= \underline{\underline{\dfrac{\sin 7\theta}{\sin\theta}}}$

(iv) $8x^3 + 4x^2 - 4x - 1 = 0$, let $x = \cos 2\theta$

$\dfrac{\sin 7\theta}{\sin\theta} = 0$

$\sin 7\theta = 0$, $\sin\theta \neq 0$

$7\theta = \pi, 2\pi, 3\pi, 4\pi, \ldots$

$\theta = \frac{\pi}{7}, \frac{2\pi}{7}, \frac{3\pi}{7}, \frac{4\pi}{7}, \ldots$

$x = \cos\frac{2\pi}{7}, \cos\frac{4\pi}{7}, \cos\frac{6\pi}{7}, \cos\frac{8\pi}{7}, \ldots$

<u>Hence $\cos\frac{2\pi}{7}$ is a root of the equation</u>

b) $\tan\frac{\pi}{4} = \tan(2\times\frac{\pi}{8})$

$$1 = \frac{2\tan\frac{\pi}{8}}{1-\tan^2\frac{\pi}{8}}$$

$$1-\tan^2\frac{\pi}{8} = 2\tan\frac{\pi}{8}$$

$$\tan^2\frac{\pi}{8} + 2\tan\frac{\pi}{8} - 1 = 0$$

$$\tan\frac{\pi}{8} = \frac{-2+\sqrt{8}}{2} \qquad (\tan\tfrac{\pi}{8} > 0,\ \text{acute})$$

$$= \frac{-1+2\sqrt{2}}{2}$$

$$\underline{\underline{= \sqrt{2}-1}}$$

(ii) $$(2\cos\theta)^4 = (z+\tfrac{1}{z})^4$$

$$16\cos^4\theta = z^4 + 4z^2 + 6 + \frac{4}{z^2} + \frac{1}{z^4}$$

$$= z^4 + \frac{1}{z^4} + 4(z^2 + \frac{1}{z^2}) + 6$$

$$= 2\cos 4\theta + 8\cos 2\theta + 6$$

$$2\cos 4\theta = 16\cos^4\theta - 16\cos^2\theta + 8 - 6$$

$$\underline{\underline{\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1}}$$

(iii) $$I = \int_{-1}^{1} \frac{1}{\sqrt{1+x}+\sqrt{1-x}+2}\,dx$$

$x = \sin 4\theta$
$dx = 4\cos 4\theta\, d\theta$

$$= 2\int_0^{\frac{\pi}{8}} \frac{4\cos 4\theta\, d\theta}{\sqrt{1+\sin 4\theta}+\sqrt{1-\sin 4\theta}+2}$$

$$= 8\int_0^{\frac{\pi}{8}} \frac{\cos(\frac{\pi}{2}-4\theta)\, d\theta}{\sqrt{1+\sin(\frac{\pi}{2}-4\theta)}+\sqrt{1-\sin(\frac{\pi}{2}-4\theta)}+2}$$

$$= 8\int_0^{\frac{\pi}{8}} \frac{\sin 4\theta\, d\theta}{\sqrt{1+\cos 4\theta}+\sqrt{1-\cos 4\theta}+2}$$

$$= 8\int_0^{\frac{\pi}{8}} \frac{\sin 4\theta\, d\theta}{\sqrt{2\cos^2 2\theta}+\sqrt{2\sin^2 2\theta}+2}$$

$$= 8\int_0^{\frac{\pi}{8}} \frac{\sin 4\theta\, d\theta}{\sqrt{2}(\cos 2\theta + \sin 2\theta)+2}$$

$$= 8\int_0^{\frac{\pi}{8}} \frac{\sin 4\theta\, d\theta}{2\cos(2\theta-\frac{\pi}{4})+2}$$

$$= 4\int_0^{\frac{\pi}{8}} \frac{\sin(\frac{\pi}{2}-4\theta)\, d\theta}{\cos(\frac{\pi}{4}-2\theta-\frac{\pi}{4})+1}$$

$$= 4\int_0^{\frac{\pi}{8}} \frac{\cos 4\theta\, d\theta}{\cos(-2\theta)+1}$$

$$= 4\int_0^{\frac{\pi}{8}} \frac{\cos 4\theta\, d\theta}{\cos 2\theta + 1}$$

$$= 4\int_0^{\frac{\pi}{8}} \frac{\cos 4\theta\, d\theta}{2\cos^2\theta} = \underline{\underline{\int_0^{\frac{\pi}{8}} \frac{2\cos 4\theta}{\cos^2\theta}\, d\theta}}$$

(β) $I = \int_0^{\frac{\pi}{8}} \frac{16\cos^4\theta - 16\cos^2\theta + 2}{\cos^2\theta}\, d\theta$

$= \int_0^{\frac{\pi}{8}} (16\cos^2\theta - 16 + 2\sec^2\theta)\, d\theta$

$= \int_0^{\frac{\pi}{8}} (8 + 8\cos 2\theta - 16 + 2\sec^2\theta)\, d\theta$

$= \left[4\sin 2\theta - 8\theta + 2\tan\theta\right]_0^{\frac{\pi}{8}}$

$= 4\sin\frac{\pi}{4} - \pi + 2\tan\frac{\pi}{8} - 0$

$= \frac{4}{\sqrt{2}} - \pi + 2\sqrt{2} - 2$

$= \underline{\underline{4\sqrt{2} - \pi - 2}}$

c) On LHS

$T_k = \frac{1}{2k-1} - \frac{1}{2k}$

$= \frac{2k - 2k + 1}{2k(2k-1)}$

$= \frac{1}{2k(2k-1)}$

as k increases, T_k decreases

$\therefore \quad S_k > kT_k$

LHS $> \frac{k}{2k(2k-1)}$

$> \underline{\underline{\frac{k}{(2k+1)(2k+2)}}}$ (denominator is larger $\therefore$ term must be smaller)

(ii) $n\left(1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{2n-1}\right) > (n+1)\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \dots + \frac{1}{2n}\right)$

Prove true for $n=2$

LHS $= 2(1 + \frac{1}{3})$ RHS $= 3(\frac{1}{2} + \frac{1}{4})$

$= \frac{8}{3}$ $= \frac{9}{4}$

$= 2.\dot{6}$ $= 2.25$

LHS > RHS

Hence the result is true for $n=2$

Assume the result is true for $n=k$, $k \geq 2$

ie $k\left(1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{2k-1}\right) > (k+1)\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \dots + \frac{1}{2k}\right)$

Prove the result is true for $n=k+1$

ie $(k+1)\left(1+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2k+1}\right) > (k+2)\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots+\frac{1}{2k+2}\right)$

Proof

$$(k+1)\left(1+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2k+1}\right)$$

$$= k\left(1+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2k-1}\right)+1+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2k-1}+\frac{k+1}{2k+1}$$

$$> (k+1)\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots+\frac{1}{2k}\right)+\left(1+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2k-1}\right)+\frac{k+1}{2k+1}$$

$$> (k+1)\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots\frac{1}{2k}\right)+\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots\frac{1}{2k}+\frac{k}{(2k+1)(2k+2)}+\frac{k+1}{2k+1}$$

$$= (k+2)\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots+\frac{1}{2k}\right)+\frac{k+(k+1)(2k+2)}{(2k+1)(2k+2)}$$

$$= (k+2)\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots+\frac{1}{2k}\right)+\frac{2k^2+5k+2}{(2k+1)(2k+2)}$$

$$= (k+2)\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots+\frac{1}{2k}\right)+\frac{(2k+1)(k+2)}{(2k+1)(2k+2)}$$

$$= (k+2)\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots+\frac{1}{2k}\right)+\frac{k+2}{2k+2}$$

$$= (k+2)\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots+\frac{1}{2k}+\frac{1}{2k+2}\right)$$

Hence the result is true for $n=k+1$ if it is also true for $n=k$.

Since the result is true for $n=2$, then it is true for all integral values of $n \geq 2$ by induction.