Locus and Complex Numbers

\( \omega = f(z) \), find the locus of \( \omega \) or \( z \)
given some condition for \( \omega \) or \( z \)

(Make the condition the subject)

\( \omega \) is purely real \( \Rightarrow \) \( \text{Im}(\omega) = 0, \arg \omega = 0 \) or \( \pi \)

\( \omega \) is purely imaginary \( \Rightarrow \) \( \text{Re}(\omega) = 0, \arg \omega = \pm \frac{\pi}{2} \)

\( \arg \left( \frac{\text{linear function}}{\text{linear function}} \right) = \theta \Rightarrow \text{locus is an arc of a circle} \)

* minor arc if \( \theta > \frac{\pi}{2} \)

* major arc if \( \theta < \frac{\pi}{2} \)

* semicircle if \( \theta = \frac{\pi}{2} \)
e.g. (i) Find the locus of \( w \) if \( w = \frac{z+2}{2}, |z| = 4 \)

\[
w = \frac{z+2}{z}
\]

\[
zw = z + 2
\]

\[
z(w-1) = 2
\]

\[
z = \frac{2}{(w-1)}
\]

\[
\therefore \left| \frac{2}{w-1} \right| = 4
\]

\[
\frac{2}{|w-1|} = 4
\]

\[
|w-1| = \frac{1}{2}
\]

\[
\therefore \text{locus is a circle, centre}(1,0)\text{ and radius }\frac{1}{2}
\]

i.e. \((x-1)^2 + y^2 = \frac{1}{4}\)
(ii) Find the locus of $z$ if $w = \frac{z+1}{z-1}$ and $w$ is purely real

\[
w = \frac{(x+1) + iy}{(x-1) + iy} \times \frac{(x-1) - iy}{(x-1) - iy}
\]

\[
= \frac{(x^2 - 1) - i(x+1)y + i(x-1)y + y^2}{(x-1)^2 + y^2}
\]

If $w$ is purely real then $\text{Im}(w) = 0$

\[i.e. -(x+1)y + (x-1)y = 0\]

\[- xy - y + xy - y = 0\]

\[- 2y = 0\]

\[y = 0\]

\[\therefore \text{locus is } y = 0, \text{excluding } (1,0)\]

\[(z-1 \neq 0, \text{bottom of fraction } \neq 0)\]

\[i.e. \text{answer the original question}\]

Note: locus is $y = 0$, excluding $(1,0)$ only
(iii) Find the locus of \( z \) if \( \arg \left( \frac{z}{z-4} \right) = \frac{\pi}{6} \)

\[
\arg \left( \frac{z}{z-4} \right) = \frac{\pi}{6}
\]

\[
\arg z - \arg(z - 4) = \frac{\pi}{6}
\]

\[
\frac{y}{2} = \tan 60
\]

\[
y = 2 \tan 60 = 2\sqrt{3}
\]

\[
r^2 = 2^2 + (2\sqrt{3})^2
\]

\[
r^2 = 16
\]

\[
r = 4
\]

\[
\therefore \text{centre is } (2, -2\sqrt{3})
\]

\[
\therefore \text{locus is the major arc of the circle }
\]

\[
(x - 2)^2 + (y + 2\sqrt{3})^2 = 16 \text{ formed by the chord joining (0,0) and (4,0) but not including these points.}
\]

\[
\text{Exercise 4N; 5, 6}
\]

\[
\text{Exercise 4O; 3 to 10, 12, 13a, 14a}
\]

\[
\text{HSC Geometrical Complex Numbers Questions}
\]

\[
\text{NOTE: } \arg z > \arg(z-4)
\]

\[
\therefore \text{below axis}
\]