JAMES RUSE A.H.S.

GESTA NON VERBA

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION 2007

MATHEMATICS EXTENSION 1

Time Allowed – 2 Hours
(Plus 5 minutes Reading Time)

All questions may be attempted

All questions are of equal value

Department of Education approved calculators are permitted

In every question, show all necessary working

Marks may not be awarded for careless or badly arranged work

No grid paper is to be used unless provided with the examination paper

The answers to all questions are to be returned in separate bundles clearly labeled Question 1, Question 2, etc. Each question must show your Candidate Number.

QUESTION 1. (Start on a new sheet of paper) **MARKS**

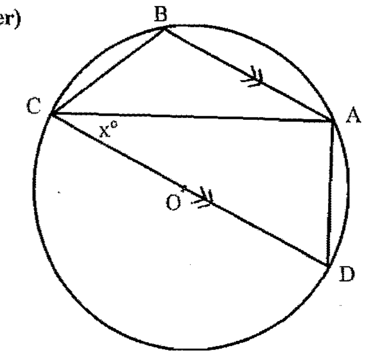
a) If $y = x\tan^{-1}x$, find $\dfrac{dy}{dx}$. **2**

b) If $f(x) = \sin^{-1}(1-2x)$, show that $f'(x) = \dfrac{-1}{\sqrt{x-x^2}}$. **3**

c) *P(x)* is an odd polynomial of degree 3. It has $(x+4)$ as a factor and, when it is divided by $(x-3)$, the remainder is 21. Find *P(x)*. **3**

d) By making the substitution $u = x-2$, evaluate $\displaystyle\int_4^5 \frac{x(x-4)}{(x-2)}\,dx$. **4**

QUESTION 2. (Start on a new sheet of paper)



a) The points *A*, *B*, *C* and *D* lie on the circumference of a circle centred at *O*. *CD* is a diameter of the circle and *AB* is parallel to *CD*. $\angle ACD = x°$. Find an expression for $\angle ACB$ in terms of *x*. **3**

b) Use the method of mathematical induction to show that the expression $9^n - 8n - 1$ is divisible by 64 for all integers $n \geq 2$. **5**

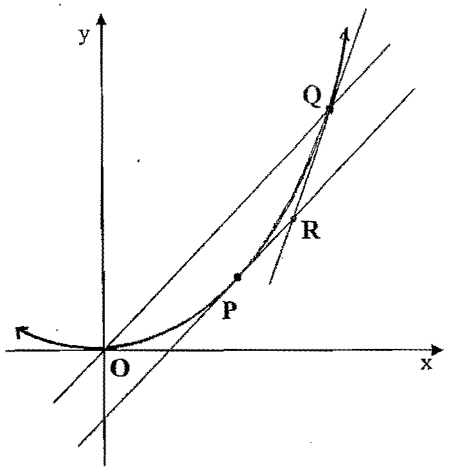
c) i) Given that ${}^nC_r = \dfrac{n!}{r!(n-r)!}$, show that $\dfrac{r \times {}^nC_r}{{}^nC_{r-1}} = n - r + 1$. **1**

ii) Hence show that:
$$\frac{{}^nC_1}{{}^nC_0} + \frac{2\times{}^nC_2}{{}^nC_1} + \frac{3\times{}^nC_3}{{}^nC_2} + \ldots + \frac{n\times{}^nC_n}{{}^nC_{n-1}} = \frac{n}{2}(n+1).$$ **3**

QUESTION 3. (Start on a new sheet of paper) MARKS

a) Evaluate the definite integral $\int_0^{\frac{1}{\sqrt{2}}} \frac{x}{\sqrt{1-x^4}}\,dx$ by using the substitution $u = x^2$. **4**

b) The point $P(2ap, ap^2)$ is on the parabola $x^2 = 4ay$ and a straight line OQ is drawn through the vertex parallel to the tangent at P. This line meets the parabola again at Q and the tangent to the parabola at Q meets the tangent at P in R, as shown in the diagram.



You are given that the tangent at P has equation $y = px - ap^2$.

i) Write down the equation of the line OQ. **1**

ii) Find the coordinates of Q in terms of a and p. **2**

iii) Show that the equation of the tangent at Q is $y = 2px - 4ap^2$. **1**

iv) Find the coordinates of R. **2**

v) Show that, as P varies on the parabola, R moves on another parabola whose equation is $x^2 = \frac{9}{2}ay$. **2**

QUESTION 4. (Start on a new sheet of paper) MARKS

a) Consider the function $f(x) = \frac{e^x}{(1+e^x)}$.

i) Find $f'(x)$ and deduce that $f(x)$ is increasing for all x. **2**

ii) State the range of $f(x)$. **1**

iii) Find the inverse function $f^{-1}(x)$. **2**

iv) Draw $y = f(x)$ and $y = f^{-1}(x)$ on the same diagram. **2**

b) A particle moves in a straight line on the x-axis. At time t its velocity is v and its acceleration is a.

i) If $a = 4x - 4$ and initially $x = 6$ and $v^2 = 64$, show that $v^2 = 4x^2 - 8x - 32$. **2**

ii) Use this expression for v^2 to find the possible values of x. **1**

iii) Describe the motion of the particle if $v = -8$ initially. **2**

QUESTION 5. (Start on a new sheet of paper)

a) A particle moves in a straight line with displacement in centimetres from the point $x = 0$ at time t seconds given by $x = \sin 3t + 2\cos 3t$ for $t \ge 0$.

i) Express x in the form $R\sin(3t + \alpha)$ where $R>0$ and $0 \le \alpha \le \frac{\pi}{2}$. **2**

ii) Show that the motion is simple harmonic. **1**

iii) Write down the period of motion. **1**

iv) Find at what time, to the nearest tenth of a second, the particle first reaches $x = 1$. **2**

b) Fred deposited \$20,000 at the beginning of January into an account which paid interest at the rate of 0.5% per month compounded monthly. He withdrew \$50 each month from the account each month, immediately after the interest was paid.

i) How much money was in the account immediately after the first withdrawal? **1**

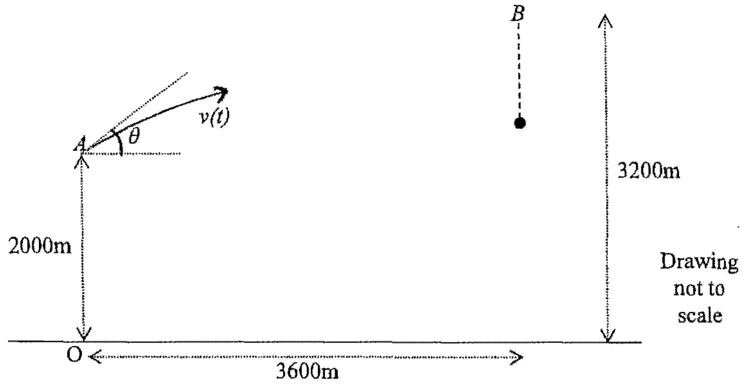
ii) Show that, after making the n^{th} withdrawal, his account balance is given by the expression $\$(10{,}000 \times 1.005^n + 10{,}000)$. **3**

iii) Find the number of months it will take for his account balance to be \$50,000. **2**

QUESTION 6. (Start on a new sheet of paper) MARKS

a) Find the term independent of x in the expansion of $\left(\frac{2x^2}{3}-\frac{3}{2x}\right)^9$. **3**

b) An aeroplane, A, flying at a height of 2000m observes a stationary blimp, B, at a height of 3200m drop an object. As the object is dropped, the plane fires a projectile towards it at a speed of 240m/s and at an angle θ to the horizontal. The horizontal distance between the plane and the blimp is 3600m at the time that the projectile is fired.



The origin of coordinates, O, is taken to be the point on the ground below A.
The particle's coordinates at time t (secs) are given by : $x = 240t\cos\theta$,

$$y = 2000 + 240t\sin\theta - \frac{gt^2}{2}$$

The coordinates of the dropped object at time t are : $x = 3600$,

$$y = 3200 - \frac{gt^2}{2}$$

(You may use g=10m/s^2)

i) What is the angle θ at which the projectile must be fired to intercept the object, and how long does it take to reach it? **3**

ii) At what height does the projectile intercept the object? **1**

c) A man notices two towers, one due North and one in a direction NθE (ie at an angle θ east of north). The angle of elevation β of both towers is the same but the height of one tower is twice the height of the other. Show that

$$\cos\theta = \frac{5\cot^2\beta - \cot^2\alpha}{4\cot^2\beta}$$ **5**

where α is the angle of elevation of the top of the taller tower from the top of the shorter.

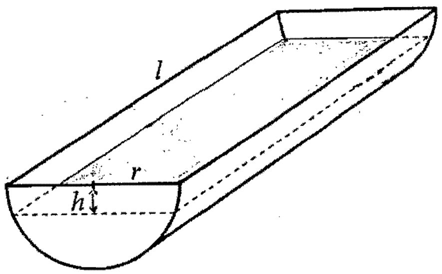
QUESTION 7. (Start on a new sheet of paper) MARKS

a) A group of four contestants for a quiz game are to be selected at random from a class of eight girls and five boys.

i) What is the probability that the team comprises three girls and one boy? **2**

ii) Find the probability that there are more girls chosen than boys. **2**

b) A water trough takes the shape of a hollow semi-circular prism with length l and radius r. It is placed on horizontal ground and filled with water. The surface of the water is at a distance h below the top of the trough, as shown in the diagram.



i) Show that the area A of the flat surface area of water is given by

$$A = 2l\sqrt{r^2 - h^2}$$ **2**

ii) Show that the volume V of water in the trough is given by

$$V = l\left(r^2\cos^{-1}\left(\frac{h}{r}\right) - h\sqrt{r^2-h^2}\right)$$ **2**

iii) If the water level is falling, show that $\frac{dV}{dt} = -2l\sqrt{r^2-h^2}\,\frac{dh}{dt} = -A\frac{dh}{dt}$. **3**

iv) On a sunny day, the rate of evaporation at any time (and hence $-\frac{dV}{dt}$) is proportional to A. Show that the water level falls at a constant rate. **1**

END OF THE PAPER

1) a) $y = x\tan^{-1}x$

$\frac{dy}{dx} = \tan^{-1}x + \frac{x}{1+x^2}$ (Product Rule) 2

b) $f(x) = \sin^{-1}(1-2x)$

Let $u = 1-2x$, $\frac{du}{dx} = -2$,

$\therefore f'(x) = \frac{1}{\sqrt{1-(1-2x)^2}} \times -2$ 2

$= \frac{-2}{\sqrt{1-(1+4x^2-4x)}}$

$= \frac{-2}{\sqrt{4x-4x^2}} = \frac{-1}{\sqrt{x-x^2}}$ 1

c) $P(x) = Ax(x+4)(x-4)$ 2

$P(3) = 3A \times 7 \times -1 = 21 \quad \therefore A = 1$ 1

$\therefore P(x) = -x(x+4)(x-4)$ 1

$(= 16x - x^3)$

d) Let $I = \int_4^5 \frac{x(x-4)}{(x-2)}\,dx$

Let $u = x-2$ $\quad x=5 \Rightarrow u=3$, $x=4 \Rightarrow u=2$

$\frac{du}{dx} = 1 \quad \therefore$ "$dx = du$" 1

$\therefore I = \int_2^3 \frac{(u+2)(u-2)}{u}\,du$

$= \int_2^3 \frac{u^2-4}{u}\,du$

$= \int_2^3 u - \frac{4}{u}\,du$ 1

$= \left[\frac{u^2}{2} - 4\ln u\right]_2^3$ 1

$= \left(\frac{9}{2} - 4\ln 3\right) - (2 - 4\ln 2)$

$= \frac{5}{2} - 4\ln\left(\frac{3}{2}\right)$ 1

(2) a) ABCD is a cyclic quadrilateral

$\angle CAD = 90°$ (Angle at circumference in semi circle) 1

$\angle BAC = x°$ (Alternate angles are equal $AB \parallel CD$) 1

$\therefore \angle BAD = x° + 90°$

$\therefore \angle BCD = 180° - (x°+90°)$ (Opposite angles in cyclic quad. supp^y)

$= 90° - x°$

$\therefore \angle BCA = \angle BCD - \angle ACD$ 1

$= 90° - 2x°$

b) When $n=2$,

$T_2 = 9^2 - 16 - 1$

$= 64$

which is divisible by 64 1

So induction starts.

Assume that

$9^k - 8k - 1 = 64A$ for $k \geq 2$ 1

and $A \in \mathbb{Z}$.

Then

$9^{k+1} - 8(k+1) - 1 = 9.9^k - 8k - 9$

$= 9(9^k - 1) - 8k$

$= 9(9^k - 8k - 1) + 64k$

$= 9 \times 64A + 64k$

$= 64(9A + k)$ 2

Thus if true for $n=k$, also true for $n=k+1$.

$\therefore$ By principle of M.I.,

$9^n - 8n - 1$ is divisible by 64, for $n \geq 2$ 1

c) i) $\frac{r\,^nC_r}{^nC_{r-1}} = \frac{r\,n!}{(n-r)!\,r!} \cdot \frac{(r-1)!\,(n-(r-1))!}{n!}$

$= \frac{r(r-1)!}{r!} \cdot \frac{(n-r+1)!}{(n-r)!}$

$= n-r+1$ 1

2 c ii) Using result from (i)

$n + (n-1) + (n-2) \quad + 2 + 1$ 1

This an AP with $a=n, d=-1$

Sum $= \frac{n}{2}(2n + (n-1)(-1))$

$= \frac{n(n+1)}{2}$ 2

(3) a) Let $I = \int_0^{1/\sqrt{2}} \frac{x\,dx}{\sqrt{1-x^4}}$

Let $u = x^2$ $\quad x = 1/\sqrt{2} \Rightarrow u = 1/2$

"$du = 2x\,dx$" $\quad x = 0 \Rightarrow u = 0$ 1

$\therefore I = \frac{1}{2}\int_0^{1/2} \frac{du}{\sqrt{1-u^2}}$ 1

$= \left[\frac{1}{2}\sin^{-1}u\right]_0^{1/2}$ 1

$= \frac{1}{2} \cdot \frac{\pi}{6} = \frac{\pi}{12}$ 1

b) i) $y = px$ 1

ii) Crosses parabola where

$x^2 = 4apx$ 1

$x = 4ap \quad (x \neq 0)$

$\therefore y = 4ap^2$

Q is $(4ap, 4ap^2)$ 1

iii) Q is point with parameter "$2p$"

$\therefore$ Tangent is $y = (2p)x - a(2p)^2$

$y = 2px - 4ap^2$ 1

iv) Two tangents cross at R

$px - ap^2 = 2px - 4ap^2$

$\Rightarrow x = 3ap \quad (p \neq 0)$ 1

$\therefore y = 2ap^2$

R is $(3ap, 2ap^2)$ 1

v) Eliminate p from $\left.\begin{matrix} x = 3ap \\ y = 2ap^2 \end{matrix}\right\}$ 1

$p = x/3a$; $y = 2a\left(\frac{x}{3a}\right)^2$

$x^2 = \frac{9ay}{2}$ 1

(4) i) $f(x) = 1 - \frac{1}{1+e^x} = 1 - (1+e^x)^{-1}$

$f'(x) = \frac{e^x}{(1+e^x)^2}$

$e^x > 0 \quad (1+e^x)^2 > 0 \quad \therefore f'(x) > 0$

$\therefore f(x)$ increasing for all x

ii) Range $\{y : 0 < y < 1\}$

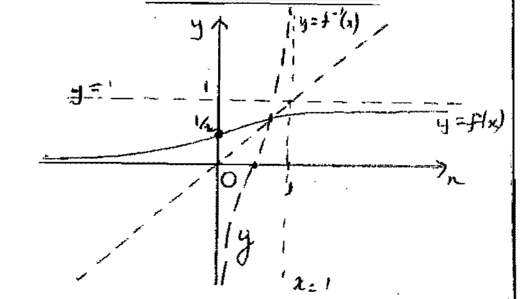
iii) Let $y = 1 - \frac{1}{1+e^x}$

$\frac{1}{1+e^x} = 1 - y$

$1 + e^x = \frac{1}{1-y}$

$e^x = \frac{1}{1-y} - 1 = \frac{y}{1-y}$

$f^{-1}(x) = \ln\left(\frac{x}{1-x}\right)$ 2



b) i) $\frac{d}{dx}\left(\frac{v^2}{2}\right) = 4x - 4$

$\therefore \frac{v^2}{2} = 2x^2 - 4x + k$ 1

When $x = 6$, $v^2 = 64$

$\therefore 32 = 72 - 24 + k \Rightarrow k = -16$

$\therefore v^2 = 4x^2 - 8x - 32$ 1

ii) $v^2 = 4(x^2 - 2x - 8)$

$= 4(x-4)(x+2)$

$v^2 \geq 0 \quad \therefore \quad -2 \leq x \leq 4$ 1

iii) Particle moving to left from $x=6$

Stops at $x=4$ and immediately moves off to right, accelerating indefinitely 2

$\ldots R\sin(3t+\alpha) = R\sin 3t\cos\alpha + R\cos 3t\sin\alpha$

Equating coefficients:

$R\sin\alpha = 2$, $R\cos\alpha = 1$ $\quad R=\sqrt{5}$, $\tan\alpha = 2$ **1**

$\therefore x = \sqrt{5}\sin(3t + \tan^{-1}(2))$ **1**

ii) $\dot{x} = 3\sqrt{5}\cos(3t+\tan^{-1}(2))$

$\ddot{x} = -9\sqrt{5}\sin(3t+\tan^{-1}(2))$

$\ddot{x} = -9x$

which is of the form $\ddot{x} = -n^2x$ which defines S.H.M. **1**

iii) Period $\left(=\frac{2\pi}{n}\right) = \frac{2\pi}{3}$ secs. **1**

iv) When $x = 1$,

$\sin(3t+\tan^{-1}(2)) = 1/\sqrt{5}$

$3t + \tan^{-1}(2) = n\pi + (-1)^n\sin^{-1}\left(\frac{1}{\sqrt{5}}\right)$ **1**

$n=1$ gives 1st +ve sol^n:

$3t = \pi - \sin^{-1}\left(\frac{1}{\sqrt{5}}\right) - \tan^{-1}(2)$

$t = \frac{1}{3}\left(\pi - \sin^{-1}\left(\frac{1}{\sqrt{5}}\right) - \tan^{-1}(2)\right)$ **1**

$= 0.5$ secs (to 1D)

b) i) After 1 withdrawal

$\$(20000 \times 1.005 - 50) = \$20050$ **1**

ii) Let $P = \$20000$, $D = \$50$, $r = 0.005$

After 1 month $P(1+r) - D$

.. 2 .. $P(1+r)^2 - D(1+r) - D$

.. n .. $P(1+r)^n - D(1+r)^{n-1} - D(1+r)^{n-2} \ldots - D$ **1**

$= P(1+r)^n - \frac{D((1+r)^n - 1)}{r}$ **1**

$= 20000(1.005)^n - 10000(1.005)^n + 10000$

$= 10000(1.005)^n + 10000$ **1**

$10000 \times (1.005)^n \geq 40000$

$(1.005)^n \geq 4$

$n\log(1.005) \geq \log 4$

$n \geq \frac{\log 4}{\log 1.005}$ **1**

$\therefore n = 278$ (months) **1**

⑥ a) r^{th} term is ${}^9C_r\left(\frac{2x^2}{3}\right)^r\left(\frac{-3}{2x}\right)^{9-r}$ **1**

$= {}^9C_r\left(\frac{2}{3}\right)^r\left(\frac{-3}{2}\right)^{9-r}x^{3r-9}$

Required term has $3r - 9 = 0$, $r = 3$ **1**

$\therefore$ Coefficient is ${}^9C_3(-1)^6\left(\frac{3}{2}\right)^3$

$= {}^9C_3\left(\frac{3}{2}\right)^3 \left(= \frac{567}{2} = 283\tfrac{1}{2}\right)$ **1**

b) i) Intercept when x and y equal simultaneously

$240t\cos\theta = 3600$

$t\cos\theta = 15$

$t = 15\sec\theta$

$2000 + 240t\sin\theta - \frac{gt^2}{2} = 3200 - \frac{gt^2}{2}$ **1**

$240t\sin\theta = 1200$

$t\sin\theta = 5$

Solving $\tan\theta = 1/3$

$\theta = \tan^{-1}(1/3)$ $(= 18^\circ 26')$ **1**

At time $t = 15\sec\theta$

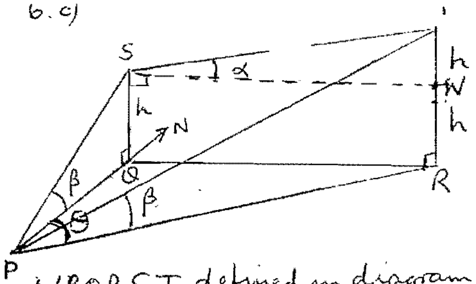
$= 5\sqrt{10}$ secs. $(= 15.81\text{ s})$ **1**

ii) Putting $t = 5\sqrt{10}$ and $g = 10$ into last equation

height $= 3200 - \frac{10.250}{2}$

$= 1950$ m

6. c)



WPQRST defined in diagram.

$\Delta PSQ \ ||| \ \Delta PTR$ (Equiangular)

$\therefore \frac{PR}{PQ} = \frac{RT}{QS} = 2$

But $PQ = h\cot\beta$ **1**

$\therefore PR = 2h\cot\beta$. **1**

$QR = SW = h\cot\alpha$ **1**

Apply cosine rule to ΔPQR **1**

$\cos\theta = \frac{PQ^2 + PR^2 - QR^2}{2.PQ.PR}$

$= \frac{h^2\cot^2\beta + 4h^2\cot^2\beta - h^2\cot^2\alpha}{2.h\cot\beta.2h\cot\beta}$ **1**

$\cos\theta = \frac{5\cot^2\beta - \cot^2\alpha}{4\cot^2\beta}$

⑦ a) i) Ways of choosing team of 4 $= {}^{13}C_4$ $(= 715)$

Ways of choosing 3 boys and 1 girl $= {}^8C_3{}^5C_1 = 280$

$\therefore$ Probability $= \frac{{}^8C_3{}^5C_1}{{}^{13}C_4}$ $(\doteqdot 0.39)$ **1** **1**

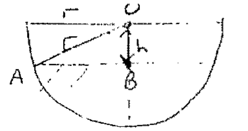
ii) Ways of choosing 4 girls $= {}^8C_4 = 70$ **1**

Prob all girls ${}^8C_4/{}^{13}C_4$

Prob 3 or 4 girls

$= \frac{{}^8C_4 + {}^8C_3{}^5C_1}{{}^{13}C_4}$ $(\doteqdot 0.49)$ **1**

b)



i) Consider cross section of trough.

$OA = r$ $\therefore AB = \sqrt{r^2 - h^2}$ (Pythag)

$\therefore$ Surface area $= l \times 2AB$

$= 2l\sqrt{r^2-h^2}$ **2**

ii) Let $\angle AOB = \theta$ $(= \cos^{-1}(h/r))$

Area of shaded segment is

$\frac{r^2}{2}(2\theta - \sin 2\theta)$

$= r^2(\theta - \sin\theta\cos\theta)$

$= r^2\left(\cos^{-1}(h/r) - \frac{\sqrt{r^2-h^2}}{r}\cdot\frac{h}{r}\right)$

$\therefore V = l\left(r^2\cos^{-1}(h/r) - h\sqrt{r^2-h^2}\right)$ **2**

iii) $\frac{dV}{dt} = l\left(\frac{-r^2/r}{\sqrt{1-h^2/r^2}} - \sqrt{r^2-h^2} + \frac{h^2}{\sqrt{r^2-h^2}}\right)\frac{dh}{dt}$ **2**

$= \left(\frac{-r^2 - (r^2-h^2) + h^2}{\sqrt{r^2-h^2}}\right)l\frac{dh}{dt}$

$= \frac{-2l(r^2-h^2)}{\sqrt{r^2-h^2}}\frac{dh}{dt}$

$= -2l\sqrt{r^2-h^2}\frac{dh}{dt} = -A\frac{dh}{dt}$ **1**

iv) $-\frac{dV}{dt} \propto A$ $\therefore -\frac{dV}{dt} = kA$

where k is constant of proportionality

$\therefore kA = A\frac{dh}{dt} \Rightarrow \frac{dh}{dt} = k$

$\therefore$ h increasing at constant rate (i.e. water falling at constant rate) **1**