JAMES RUSE A H S
GESTA NON VERBA

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION 2007

MATHEMATICS EXTENSION 2

Time Allowed – 3 Hours
(Plus 5 minutes Reading Time)

All questions may be attempted

All questions are of equal value

Department of Education approved calculators are permitted

In every question, show all necessary working

Marks may not be awarded for careless or badly arranged work

No grid paper is to be used unless provided with the examination paper

The answers to all questions are to be returned in separate bundles clearly labeled Question 1, Question 2, etc. Each question must show your Candidate Number.

Question 1 **Marks**

(a) (i) Find the real numbers a, b and c such that $\frac{1}{x(4+x^2)} = \frac{a}{x} + \frac{bx+c}{4+x^2}$. **2**

(ii) Find $\int \frac{1}{x(4+x^2)} dx$. **2**

(b) Evaluate $\int_0^2 x\sqrt{2-x}\,dx$, leaving your answer in exact form. **3**

(c) Find the zeros of $P(x) = x^4 - 5x^3 + 7x^2 + 3x - 10$ over the complex field if $2 - i$ is a zero. **3**

(d) Given that $I_{2n+1} = \int_0^1 x^{2n+1} e^{x^2} dx$ where n is a positive integer, show that **2**

$$I_{2n+1} = \frac{1}{2}e - nI_{2n-1}.$$

Hence, or otherwise, evaluate $\int_0^1 x^5 e^{x^2} dx$. **3**

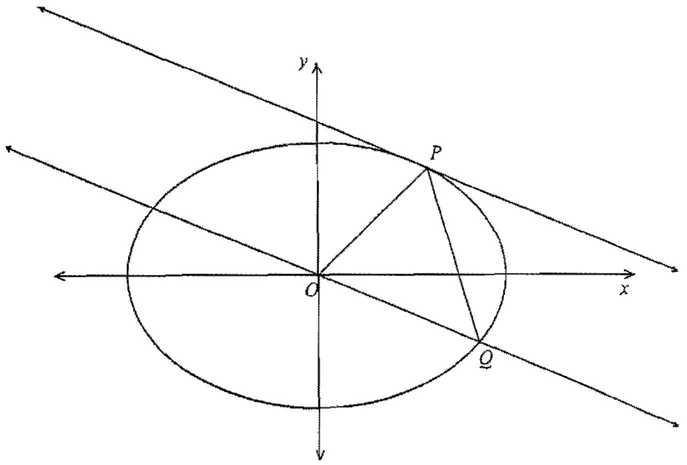
Question 2 (15 Marks) [START A NEW PAGE]

Marks

(a)(i) Given that $z^2 = -3 - 4i$, find z. 4

(ii) Solve the equation $x^2 - 3x + 3 + i = 0$ over the complex field. 3

(b)



In the diagram above, $P(a\cos\theta, b\sin\theta)$ is a point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where P lies in the first quadrant.

A straight line through the origin parallel to the tangent at P meets the ellipse at the point Q, where P and Q both lie on the same side of the y-axis.

(i) Prove that the equation of the line OQ is $xb\cos\theta + ya\sin\theta = 0$. 2

(ii) Find the coordinates of the point Q given that Q lies in the fourth quadrant. 3

(iii) Prove that the area of ΔOPQ is independent of the position of P. 3

Question 3 (15 Marks) [START A NEW PAGE]

Marks

(a) A particle is projected from the origin with a speed V and an angle of elevation α on level ground. 3

A vertical wall of "unlimited" height is a distance d from the origin, and the plane of the wall is perpendicular to the plane of the particle's trajectory.

If $d < \frac{V^2}{g}$, show that the particle will strike the wall before it hits the ground provided that $\beta < \alpha < \frac{\pi}{2} - \beta$ where $\beta = \frac{1}{2}\sin^{-1}\left[\frac{gd}{V^2}\right]$.

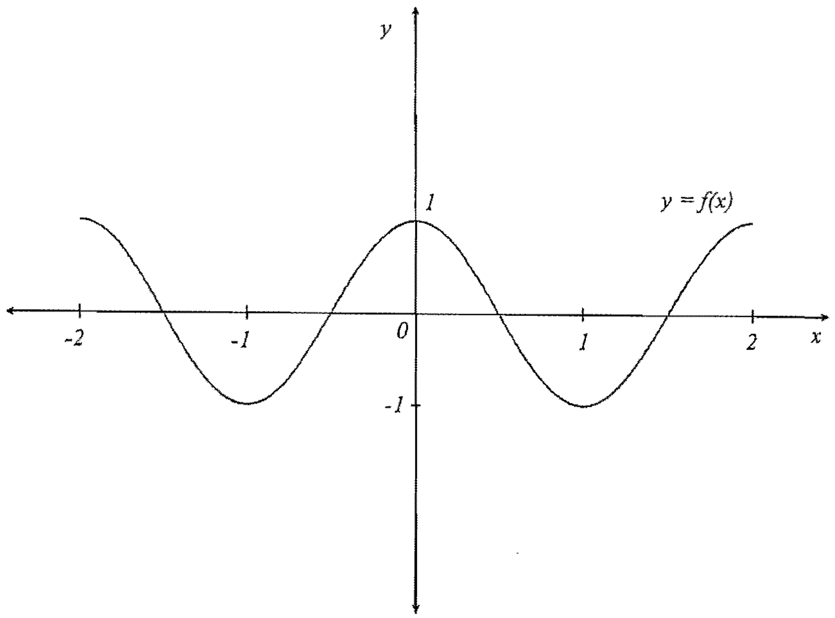
You may assume that the range on the horizontal plane from the point of projection is $\frac{V^2 \sin 2\alpha}{g}$.

(b) Express $z = \frac{\sqrt{2}}{1-i}$ in the modulus-argument form and hence find z^5 in the form of $x + yi$. 4

Question 3 continues on page 4

Question 3 cont'd **Marks**

(c)



The diagram shows the graph of the continuous function $y = f(x)$. Critical points occur at $x = -2, -1, 0, 1, 2$.

On the sheets provided draw separate sketches of the graphs of the following :

(i) $y = |f(x)|$ **1**

(ii) $y = \dfrac{1}{f(x)}$ **2**

(iii) $y = \sqrt{f(x)}$ **2**

(iv) $y = xf(x)$ **3**

Question 4 (15 Marks) [START A NEW PAGE] **Marks**

(a) Find $\int \frac{1}{x(\ln x)^2}\,dx$. **2**

(b) Five letters are chosen from the letters of the word MOBILITY. These five letters are then placed alongside each other to form a five-letter arrangement. **4**

Find the number of different arrangements which are possible.

(c) $P\left(3p, \frac{3}{p}\right)$ and $Q\left(3q, \frac{3}{q}\right)$ are points on different branches of the hyperbola $xy = 9$.

(i) Find the equation of the tangent at P. **2**

(ii) Find the point of intersection T, of the tangents at P and Q. **3**

(iii) If the chord PQ passes through the point (0, 2), find the locus of T. **3**

(iv) Find the restriction on the locus of T. **1**

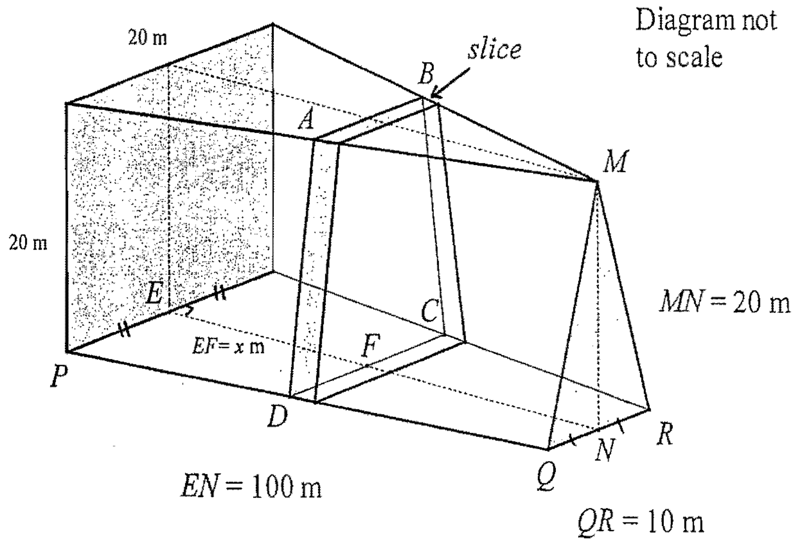
Question 5 (15 Marks) [START A NEW PAGE]

(a) (i) Find the volume generated when the area bounded by $y = \sin x$ and the x-axis, for $0 \le x \le \pi$, is rotated about the x-axis. **3**

(ii) The area described in (i) is now rotated about the line $x = 2\pi$. Find the volume of the solid formed. **4**

Question 5 continues on page 6

(b) A boat showroom is built on level ground. The length of the showroom is 100m. At one end of the showroom the shape is a square measuring 20m by 20m and at the other end an isosceles triangle of height 20m and base 10m.



(i) If EF is x m in length, show that the length of DC is $\left[20-\frac{x}{10}\right]$ m. 2

(ii) By considering trapezoidal slices parallel to the ends of the showroom, find the volume enclosed by the showroom in m^3. 6

(a) Find $\int \frac{dx}{x^2-6x+13}$. 2

(b) A food parcel of 1 kg is dropped from a helicopter which is hovering 800 metres above a group of stranded bushwalkers. After 10 seconds a parachute opens automatically. Air resistance is neglected for the first 10 seconds but the effect of the open parachute is to cause a retardation of $2v$ newtons where v ms^{-1} is the velocity of the parcel after t seconds, $(t \geq 10)$.

Take the position of the helicopter as the origin, the downwards direction as positive and the value of g, the acceleration due to gravity as 10m s^{-2}.

(i) Write down the equation of motion of the parcel before the parachute opens. 1

(ii) Determine the velocity and the distance fallen by the parcel at the end of the 10 seconds. 4

(iii) Write down the equation of motion for $t \geq 10$. 1

(iv) What is the terminal velocity of the parcel? 1

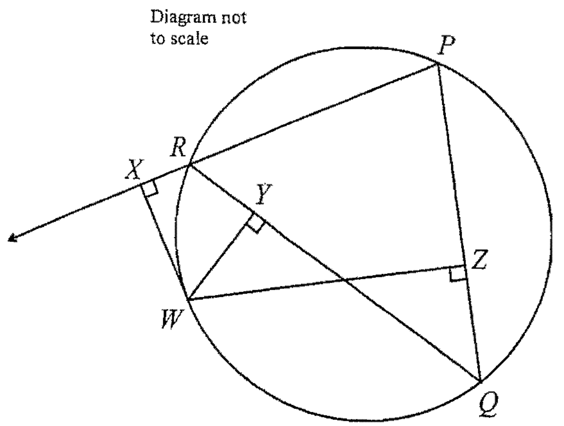
(v) Show that the velocity of the parcel after the parachute has opened is given by: 3

$$v = 5 + 95e^{-2(t-10)}, \quad t \geq 10.$$

(vi) Find the distance fallen, x, as a function of t and hence find the distance the parcel has fallen 2 minutes after it leaves the helicopter. 3

Question 7 (15 Marks) [START A NEW PAGE] **Marks**

(a)



PQR is a triangle inscribed in a circle. W is a point on the arc QR.
From W, perpendiculars are drawn to PR (produced), QR and PQ, meeting them at X, Y and Z respectively.

Copy the diagram.

(i) Explain why $WXRY$ and $WYZQ$ are cyclic quadrilaterals. **2**

(ii) Prove that the points X, Y and Z are collinear. **4**

(b)(i) By considering the expansion of $(\cos\theta + i\sin\theta)^5$ and by using De Moivre's Theorem show that **2**

$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta.$$

(ii) Hence finds all the four roots of the equation **2**

$$16x^4 - 20x^2 + 5 = 0.$$

(iii) Hence or otherwise, show that $\cos\frac{\pi}{10}\cos\frac{3\pi}{10} = \frac{\sqrt{5}}{4}$. **3**

(iv) Find the exact value of $\sin\frac{3\pi}{5}\sin\frac{6\pi}{5}$. **2**

Question 8 (15 Marks) [START A NEW PAGE] **Marks**

(a) The region R in the Argand diagram is defined by:

$$|z-1| \le |z-i| \text{ and } |z-2-2i| \le 1.$$

(i) Sketch the region R. **3**

(ii) If z describes the boundary of the region R, find **2**

(α) the value of $|z|$ when $\arg(z)$ has the smallest value.

(β) z in the form of $a+ib$ when $\arg(z-1) = \frac{\pi}{4}$. **3**

(b) A certain type of merry go-round consists of seats hung from pivots attached to the rim of a horizontal circular disc. The disc is rotated by a motor attached to the vertical axle. As the angular velocity increases, the seats swing out and move up. The seat is represented by a point with mass m kg suspended by a rod of length h metres below the pivot, which is a metres from the axle of rotation.

Neglecting the mass of the rod, assume that when the disc rotates with constant angular velocity w radians per second, there is an equilibrium position such that the rod makes an angle θ with the vertical as shown in the diagram on the following page.

Question 8 continues on page 10

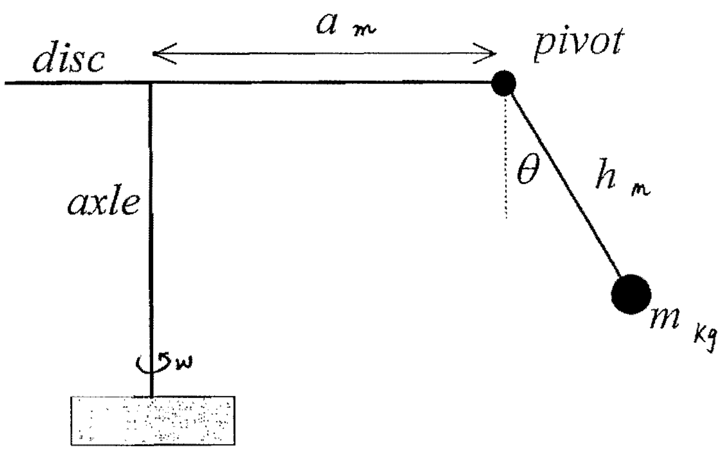


Diagram not to scale

(i) Show that w and θ satisfy the equation **3**

$$(a + h\sin\theta)w^2 = g\tan\theta$$

where g is the acceleration due to gravity.

(ii) Use graphical method to show that for a given w, there is only one value of θ in the domain $0 \le \theta \le \frac{\pi}{2}$, which satisfies the above equation. **3**

(iii) Given $a = 4$, $h = 2.5$, $\theta = 30^0$ and using $g = 10\text{ms}^{-2}$, find the angular velocity w correct to 3 significant figures when the merry-go-round is in equilibrium. **1**

END

(a)(i) $\frac{a}{x} + \frac{bx+c}{4+x^2} = \frac{4a + ax^2 + bx^2 + cx}{x(4+x^2)}$ 1

$\therefore 4a = 1 \Rightarrow a = \frac{1}{4}$ #

$a + b = 0 \quad b = -\frac{1}{4}$ # 1

$c = 0$ #

(ii) $\int \frac{dx}{x(4+x^2)} = \int \frac{dx}{4x} + \int \frac{-\frac{1}{4}x\,dx}{4+x^2} + c$

$= \frac{1}{4}\ln|x| - \frac{1}{8}\int \frac{2x}{x^2+4}$

$= \frac{1}{4}\ln|x| - \frac{1}{8}\ln|x^2+4| + c$ # 2

b) $\int_0^2 x\sqrt{2-x}\,dx \qquad u = x \quad dV = \sqrt{2-x} \qquad du = dx \quad v = -\frac{2}{3}(2-x)^{3/2}$

$= \left[-\frac{2}{3}x(2-x)^{3/2}\right]_0^2 + \frac{2}{3}\int_0^2 (2-x)^{3/2}dx$ 1

$= 0 + \frac{2}{3}\cdot\left[(2-x)^{5/2}\left(-\frac{2}{5}\right)\right]_0^2$ 1

$= \frac{4}{15} 2^{5/2}$ # or $\frac{16\sqrt{2}}{15}$ # 1

(c) Since all coeff are real and $2-i$ a zero, $2+i$ is also a zero 1

$(x-2+i)(x-2-i) = x^2 - 4x + 5$

$$\begin{array}{r} x^2 - x - 2 \\ x^2-4x+5 \overline{) x^4 - 5x^3 + 7x^2 + 3x - 10} \\ x^4 - 4x^3 + 5x^2 \\ \hline -x^3 + 2x^2 + 3x \\ -x^3 + 4x^2 - 5x \\ \hline -2x^2 + 8x - 10 \\ -2x^2 + 8x - 10 \end{array}$$ 1

$x^2 - x - 2 = (x-2)(x+1)$

Zeros are $2, -1, 2+i, 2-i$ # 1

d)i) $I_{2n+1} = \int_0^1 x^{2n+1}e^{x^2} \qquad u = x^{2n} \quad v' = xe^{x^2} \qquad \frac{du}{dx} = x^{2n-1}\cdot 2n \quad v = \frac{e^{x^2}}{2}$

$= \int_0^1 x^{2n}\cdot x e^{x^2}dx$

$= \left[\frac{x^{2n}e^{x^2}}{2}\right]_0^1 - \int_0^1 \frac{1}{2}e^{x^2} 2n x^{2n-1} dx$ 1

$= \frac{e}{2} - n\int_0^1 x^{2n-1}e^{x^2}dx$

$= \frac{1}{2}e - n I_{2n-1}$ # 1

ii) $(n=2)$ $I_5 = \frac{e}{2} - 2I_3 = \frac{e}{2} - 2\left[\frac{e}{2} - I_1\right]$

$I_5 = \frac{e}{2} - e + 2I_1 = -\frac{e}{2} + 2I_1$ 1

$(n=0)$ $I_1 = \int_0^1 x e^{x^2}dx = \frac{1}{2}\int_0^1 2xe^{x^2}dx = \left[\frac{e^{x^2}}{2}\right]_0^1 = \frac{1}{2}(e-1)$ 1

$\therefore I_5 = -\frac{e}{2} + \frac{2}{2}(e-1) = -\frac{e}{2} + e - 1 = \frac{e}{2} - 1$ # 1

Question 2

a(i) Let $z = x + yi$

$z^2 = x^2 + 2xyi - y^2 = -3 - 4i$ 1

$x^2 - y^2 = -3$ ①

$2xy = -4 \therefore xy = -2 \Rightarrow y = -\frac{2}{x}$ ② 1

$x^2 - \frac{4}{x^2} = -3$

$x^4 + 3x^2 - 4 = 0$

$(x^2+4)(x^2-1) = 0 \quad \therefore x = \pm 2i$ or ± 1 1

But $x \in R \therefore x = \pm 1$.

when $x = 1, \quad y = -2$

$x = -1, \quad y = 2$

$\therefore z = 1 - 2i$ or $-1 + 2i$ # 1

b(i) $x = a\cos\theta \quad \frac{dx}{d\theta} = -a\sin\theta$

$y = b\sin\theta \quad \frac{dy}{d\theta} = b\cos\theta$ $\Big\}$ $\frac{dy}{dx} = \frac{-b\cos\theta}{a\sin\theta}$ 1

Eq of OQ passing thru (0,0).

$y = -\frac{b\cos\theta}{a\sin\theta}x$

$b\cos\theta x + a\sin\theta y = 0$ is the eq of OQ 1

P is in 1st Quad $\therefore a > 0, \ b > 0$

For point Q

$\frac{x^2}{a^2} + \frac{b^2\cos^2\theta}{b^2 a^2 \sin^2\theta}x^2 = 1$ 1

$\frac{x^2}{a^2} + \frac{\cos^2\theta}{a^2\sin^2\theta}x^2 = 1$

$x^2(\sin^2\theta + \cos^2\theta) = a^2\sin^2\theta$

$x^2 = a^2\sin^2\theta$

But Q is in 4th Quad $\therefore x_Q > 0$ 1

$x_Q = a\sin\theta$

$y_Q = -\frac{b\cos\theta}{a\sin\theta}(a\sin\theta)$

$y_Q = -b\cos\theta$

$\therefore Q = (a\sin\theta, -b\cos\theta)$ 1

(iii) Perpendicular distance from Q to OP

$d = \frac{|ab\cos^2\theta + ab\sin^2\theta|}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}$ 1

$d = \frac{|ab|}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}$

Area $\triangle OQP = \frac{1}{2}\times\frac{|ab|}{\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}}\times OQ$

$= \frac{|ab|}{2\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}}\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}$ 1

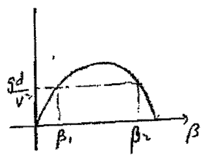
$= \frac{|ab|}{2}$ independent of position of P 1 #

Q.3

a) For the particle to hit the wall, the wall must be at a distance d less than the range for angle β

$d < \frac{V^2\sin 2\beta}{g}$ 1

$\sin 2\beta > \frac{gd}{V^2}$



at β_1, β_2 $\sin 2\beta = \frac{gd}{V^2}$

$2\beta = n\pi + (-1)^n\sin^{-1}\left(\frac{gd}{V^2}\right)$ 1

$\beta = \frac{n\pi}{2} + \frac{1}{2}(-1)^n\sin^{-1}\left(\frac{gd}{V^2}\right)$

$n=0$ $\beta_1 = \frac{1}{2}\sin^{-1}\left(\frac{gd}{V^2}\right)$

$n=1$ $\beta_2 = \frac{\pi}{2} - \frac{1}{2}\sin^{-1}\left(\frac{gd}{V^2}\right) = \frac{\pi}{2} - \beta_1$

$\therefore$ the particle will hit wall at an angle α such that

$\frac{1}{2}\sin^{-1}\left(\frac{gd}{V^2}\right) < \alpha < \frac{\pi}{2} - \frac{1}{2}\sin^{-1}\left(\frac{gd}{V^2}\right)$ 1

b) $z = \frac{\sqrt{2}}{1-i}\cdot\frac{1+i}{1+i} = \frac{\sqrt{2}(1+i)}{2} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ 1

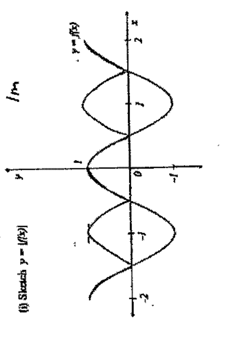
$z = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2}\ \text{cis}\frac{\pi}{4}$

$z = 1\cdot\text{cis}\frac{\pi}{4}$ 1

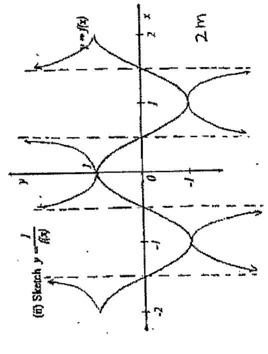
$z^5 = 1^5\ \text{cis}\frac{5\pi}{4} = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}$ 1

$z^5 = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$ # 1

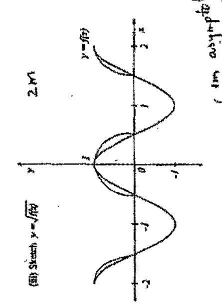
1m



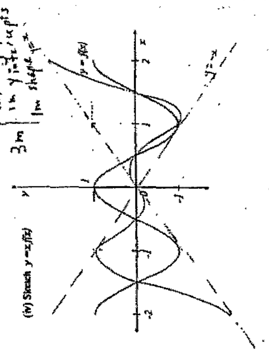
2m



2m



3m { 1m asymptotes, 1m y intercepts, 1m shape



(a) $u = \ln x \quad du = \frac{1}{x}dx$

$\int \frac{du}{u^2} = -u^{-1} + c = -\frac{1}{\ln x} + c$ #

(b) No. of ways to choose the 5 letters:

No "I" MOBLTY $^6C_5 = 6$

1 "I" MOBILTY $^6C_4 = 15$

2 "I" MOBILITY $\frac{^6C_3}{2!} = 10$

To arrange the 5 letters $= 5!$

∴ Total No. of different arrangements

$= (6+15+10) \times 5! = 3720$ #

c(i) $\frac{dy}{dx} = \frac{dy}{dp} \times \frac{dp}{dx} = \frac{-3}{p^2} \times \frac{1}{3} = -\frac{1}{p^2}$

Eq of tangent at P: $y - \frac{3}{p} = -\frac{1}{p^2}(x - 3p)$

$y - \frac{3}{p} = \frac{-x}{p^2} + \frac{3}{p}$

$p^2y + x = 6p$ # ①

ii) Similarly eq. of tangent at Q:

$q^2y + x = 6q$ ②

solving ① & ② simultaneously

$y(p^2 - q^2) = 6(p - q) \therefore y = \frac{6}{p+q}$

$x = 6p - p^2y = 6p - \frac{6p^2}{p+q} = \frac{6p^2 + 6pq - 6p^2}{p+q}$

$x = \frac{6pq}{p+q}$

$\therefore T = \left(\frac{6pq}{p+q}, \frac{6}{p+q}\right)$

(iii) slope of chord $PQ = \frac{3(\frac{1}{p} - \frac{1}{q})}{3(p-q)}$

$= \frac{\frac{q-p}{pq}}{p-q} = -\frac{1}{pq}$

Eq of PQ: $y - \frac{3}{p} = -\frac{1}{pq}(x - 3p)$

$y - \frac{3}{p} = -\frac{x}{pq} + \frac{3}{q}$

$pqy + x = 3(p+q)$

PQ passes thro' (0,2)

$pq(2) + 0 = 3(p+q)$

$2pq = 3(p+q)$

$\therefore \frac{6pq}{p+q} = 6 \times \frac{3}{2} = 9$

$\frac{6}{p+q} = 6 \times \frac{3}{2pq} = \frac{9}{pq}$

∴ Locus of T: $x = 9$

iv) Since P, Q are on different branches of the rectangular hyperbola

$p \cdot q < 0$

Restriction on the locus of T: $x = 9$ and $y < 0$

∴ T must be in 4th Quad. #

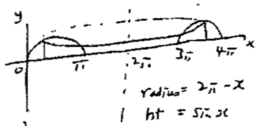
Question 5:

a) i) $V = \pi \int_0^{\pi} (\sin x)^2 dx$

$= \pi \int_0^{\pi} \left(\frac{1 - \cos 2x}{2}\right) dx$

$= \frac{\pi}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi} = \frac{\pi}{2}(\pi - 0) = \frac{\pi^2}{2}$ unit³ #

ii) $V = \int_0^{\pi} 2\pi(2\pi - x)\sin x \, dx$



$= \int_0^{\pi} (4\pi^2 \sin x - 2\pi x \sin x) dx$

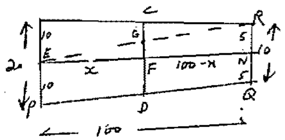
$= 4\pi^2(-\cos x)\Big]_0^{\pi} - 2\pi \int_0^{\pi} x \sin x \, dx$

$= -4\pi^2(-1-1) - 2\pi\left[-x\cos x\right]_0^{\pi} - \int_0^{\pi} \cos x \, dx$

$= 8\pi^2 - 2\pi(\pi) - [\sin x]_0^{\pi}$

$= 6\pi^2$ #

b) Join ER cutting CF at G



Base Plan

$\frac{FG}{5} = \frac{x}{100}$

$FG = \frac{x}{20}$

$\frac{CG}{10} = \frac{100-x}{100}$

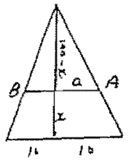
$CG = \frac{100-x}{10}$

$CF = CG + FG = \frac{x}{20} + \frac{100-x}{10} = \frac{x + 200 - 2x}{20}$

$= \frac{200-x}{20}$

$CD = 2 \times CF = \frac{200-x}{20} \times 2 = \frac{200-x}{10} = 20 - \frac{x}{10}$ m #

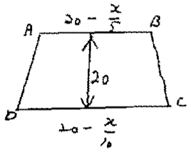
$\frac{a}{10} = \frac{100-x}{100}$



$a = \frac{100-x}{10} = 10 - \frac{x}{10}$

$AB = 2a = 20 - \frac{x}{5}$

Each trapezoidal slice:



Area $= \frac{20}{2}\left[20 - \frac{x}{5} + 20 - \frac{x}{10}\right]$

$= 10\left(40 - \frac{3x}{10}\right)$

$= 400 - 3x$

ΔV (volume slice) $= (400 - 3x)\Delta x$

volume of the showroom $= \lim_{\Delta x \to 0} \sum \Delta V$

$= \int_0^{100} (400 - 3x) dx = \left[400x - \frac{3x^2}{2}\right]_0^{100}$

$= 25000 \text{ m}^3$ #

Question 6

(a) $\int \frac{dx}{(x^2 - 6x + 9) + 4} = \int \frac{dx}{(x-3)^2 + 4}$ let $u = x - 3$, $du = dx$

$= \int \frac{du}{u^2 + 2^2} = \frac{1}{2}\tan^{-1}\left(\frac{u}{2}\right) + c = \frac{1}{2}\tan^{-1}\left(\frac{x-3}{2}\right) + c$ #

(b) i) $\ddot{y} = 10$

ii) $\dot{y} = 10t + C_1 \quad t = 0, \dot{y} = 0, \therefore C_1 = 0$

$\dot{y} = 10t$

$y = 5t^2 + C_2 \quad t = 0, y = 0, \therefore C_2 = 0$

$y = 5t^2$

$t = 10$, $\dot{y} = 10 \times 10 = 100$ m/s #

$y = 5 \times 10^2 = 500$ m #

iii) For $t \geq 10$ $\downarrow 10$ $\uparrow 2v$ $\ddot{y} = \frac{dv}{dt} = 1(10 - 2v) = 10 - 2v$

$\ddot{y} = 10 - 2v$ #

iv) terminal velocity when $\ddot{y} = 0$

$10 - 2V = 0 \quad V = 5$ m/s #

$dt = \ldots$

$\int \frac{dv}{10 - 2v} = \int dt$

$\int \frac{dv}{2v - 10} = -\int dt$

$\ln|v - 5| = -2t + K$

$t = 10, V = 100$

$\ln|100 - 5| = -20 + K$

$K = 20 + \ln 95$

$\ln|v - 5| = -2t + 20 + \ln 95$

$\frac{v-5}{95} = e^{-2(t-10)}$

$v = 5 + 95e^{-2(t-10)}, \quad t \geq 10$

vi) $\frac{dx}{dt} = 5 + 95e^{-2(t-10)}$

$x = 5t + \frac{95e^{-2(t-10)}}{-2} + C_2$

$t = 10, \ x = 500$

$500 = 50 - \frac{95}{2}e^0 + C_2$

$497.5 = C_2$

$\therefore x = 5t - \frac{95}{2}e^{-2(t-10)} + 497.5$

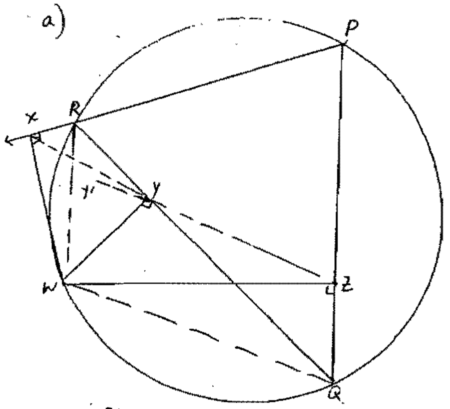
After 1 minute ($t = 60$)

$x = 5 \times 60 - \frac{95}{2}e^{-2(50)} + 497.5$

$x = 797.5$ m #

∴ The particle has fallen 797.5 m after 1 min.

a)



$\angle RXW = \angle QYW = 90°$ (given)

(i) $WXRY$ is a cyclic quad.
(exterior angle equals interior opposite angle) #

$\angle WYQ = \angle WZQ = 90°$ (given)
$WYZQ$ is a cyclic quad.
(line interval WQ subtends equal angles at Y, Z, on the same side of the line interval, the 4 end points $WQZY$ are then concyclic) #

b) $(\cos\theta + i\sin\theta)^5$

$= \cos^5\theta + 5i\cos^4\theta\sin\theta + 10i^2\cos^3\theta\sin^2\theta + 10i^3\cos^2\theta\sin^3\theta + 5i^4\cos\theta\sin^4\theta + i^5\sin^5\theta$

$= \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta(1-\cos^2\theta) - 10i\cos^2\theta\sin^3\theta + 5\cos\theta(1-\cos^2\theta)^2 + i\sin^5\theta$

$= \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta + 10\cos^5\theta - 10i\cos^2\theta\sin^3\theta + 5\cos\theta(1-2\cos^2\theta+\cos^4\theta) + i\sin^5\theta$

By De Moivre's Theorem,
$(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$

$\therefore \cos 5\theta = \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta - 10\cos^3\theta + 5\cos^5\theta$

$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

ii) Let $x = \cos\theta$. Then $\cos 5\theta = 0$ means $x(16x^4 - 20x^2 + 5) = 0$

Hence the roots of $16x^4 - 20x^2 + 5 = 0$ are the non-zero values of $\cos\theta$, where θ is a solution of $\cos 5\theta$

$\cos 5\theta = 0$ when $5\theta = 2n\pi \pm \frac{\pi}{2}$, $n \in \mathbb{Z}$

There are 4 distinct non-zero values of $\cos\theta$, namely $\frac{\pi}{10}, \frac{3\pi}{10}, \frac{7\pi}{10}, \frac{9\pi}{10}$

$\therefore$ the four roots are $\cos\frac{\pi}{10}, \cos\frac{3\pi}{10}, \cos\frac{7\pi}{10}, \cos\frac{9\pi}{10}$ #

iii) Product of all roots $= \left(\cos\frac{\pi}{10}\cos\frac{3\pi}{10}\cos\frac{7\pi}{10}\cos\frac{9\pi}{10}\right) = \frac{e}{a} = \frac{5}{16}$

But $\cos\frac{9\pi}{10} = -\cos\frac{\pi}{10}$, $\cos\frac{7\pi}{10} = -\cos\frac{3\pi}{10}$

$\therefore \left(\cos\frac{\pi}{10}\cos\frac{3\pi}{10}\right)^2 = \frac{5}{16}$

$\cos\frac{\pi}{10}\cos\frac{3\pi}{10} = +\frac{\sqrt5}{4}$ # (Since $\cos\frac{\pi}{10} > 0$, $\cos\frac{3\pi}{10} > 0$ $\therefore \cos\frac{\pi}{10}\cos\frac{3\pi}{10} > 0$)

iv) $\sin\frac{3\pi}{5} = \cos\left(\frac{\pi}{2} - \frac{3\pi}{5}\right) = \cos\left(-\frac{\pi}{10}\right) = \cos\frac{\pi}{10}$

$\sin\frac{6\pi}{5} = \cos\left(\frac{\pi}{2} - \frac{6\pi}{5}\right) = \cos\left(-\frac{7\pi}{10}\right) = \cos\frac{7\pi}{10} = -\cos\frac{3\pi}{10}$

$\therefore \sin\frac{3\pi}{5}\cdot\sin\frac{6\pi}{5} = \cos\frac{\pi}{10}\left(-\cos\frac{3\pi}{10}\right) = -\frac{\sqrt5}{4}$ # (from part iii)

Q7a ii)

Construction: Join XY, WQ, RW. Join ZY and extend it to Y'

Proof: $\angle XRW = \angle XYW$ (angles at circumference in same segment)

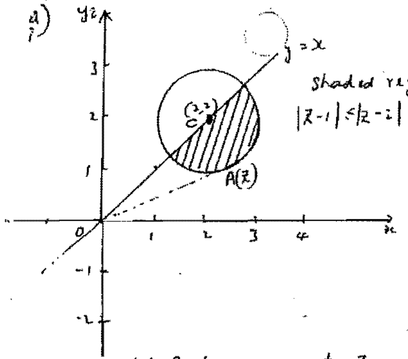
$\angle XRW = \angle ZQW$ (exterior angle equal to interior opposite angle in cyclic quadrilateral $RPQW$)

Similarly, $\angle Y'YW = \angle ZQW$ since in part (i) we have proved $WYZQ$ is a cyclic quadrilateral

$\therefore \angle XRW = \angle XYW = \angle Y'YW$

Since $\angle XYW = \angle Y'YW$, X, Y, Z must be collinear.

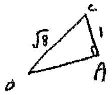
a) i)



Shaded region (R): $|z-1| \le |z-i|$ and $|z-2-2i| \le 1$

1m $y=x$
1m circle
1m shaded area

ii) α) Let Point A represents z.
When $\arg(z)$ has the smallest value, OA is tangent to the circle

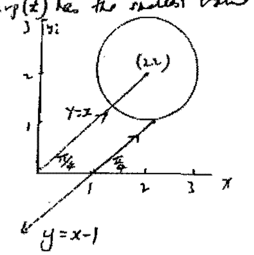


$OC = \sqrt{2^2+2^2} = \sqrt8$

$\therefore OA = \sqrt{8-1} = \sqrt7$

$\therefore |z| = \sqrt7$ when $\arg(z)$ has the smallest value.

β) $\text{Arg}(z-1) = \frac{\pi}{4}$ is the line $y = x-1$ ①



Pt of intersection with the circle $(x-2)^2+(y-2)^2=1$

$(x-2)^2 + (x-3)^2 = 1$

$x^2 - 4x + 4 + x^2 - 6x + 9 = 1$

$2x^2 - 10x + 13 = 1$

$x^2 - 5x + 6 = 0$

$(x-3)(x-2) = 0$

$\therefore x = 2$ or 3

When $x = 2$, $y = 1$
$x = 3$, $y = 2$

$z = 2+i$ or $3+2i$

b) i) $T\cos\theta = mg$

$T = \frac{mg}{\cos\theta}$ — ①

$m r w^2 = T\sin\theta$

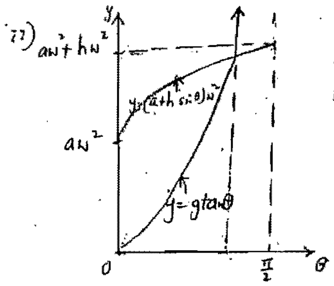
$m(a + h\sin\theta)w^2 = T\sin\theta$ — ②

Sub ① into ②

$m(a+h\sin\theta)w^2 = \frac{mg}{\cos\theta}\sin\theta$

$(a + h\sin\theta)w^2 = g\tan\theta$ #

ii)



1m $y=(a+h\sin\theta)w^2$
1m $y = g\tan\theta$
1m for pt of intersection

As there is only 1 pt of intersection there is only one value of θ that satisfies $(a+h\sin\theta)w^2 = g\tan\theta$

iii) $(a+h\sin\theta)w^2 = g\tan\theta$

$(4 + 2.5\sin 30°)w^2 = 10\tan 30°$

$5.25w^2 = \frac{10\sqrt3}{3}$

$w^2 = 1.0997$

$w = 1.0487$

$w = 1.05$ radian/second