GIRRAWEEN HIGH SCHOOL
YEAR 12 – Task 3 - 2004
MATHEMATICS (Extension)
Time allowed – 90 minutes

DIRECTIONS TO CANDIDATES
- Attempt ALL questions.
- All necessary working should be shown in every question.
  Marks may be deducted for careless or badly arranged work.
- Board-approved calculators may be used.
- Start each question on a *new* sheet of paper.

Question 1 (21 marks) — Marks

(a) Differentiate with respect to x

(i) $\sin^{-1}(2x)$ — **3**

(ii) $\cos^{-1}(x^2)$ — **3**

(iii) $\tan^{-1}\left(\frac{x}{2}\right)$ — **3**

(b) Find

(i) $\int_0^1 \frac{dx}{1+x^2}$ — **3**

(ii) $\int_1^2 \frac{dx}{\sqrt{4-x^2}}$ — **3**

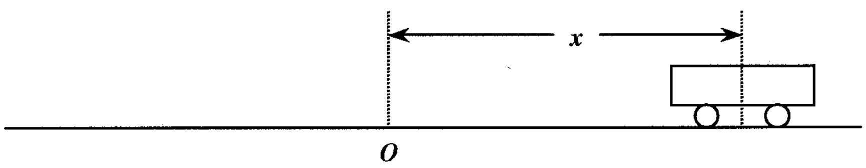
(iii) $\int \frac{dx}{1+7x^2}$ — **3**

(iv) $\int \frac{1}{\sqrt{9-4x^2}}dx$ — **3**

Question 2 (~~11~~ 13 marks) — Marks

(a) A trolley is moving about the origin O. The displacement, x metres, of the trolley from the O at time t seconds is given by,

$$x = 6\sin\left(2t + \frac{\pi}{4}\right).$$



(i) Show that it is simple harmonic motion. — ~~2~~ 3

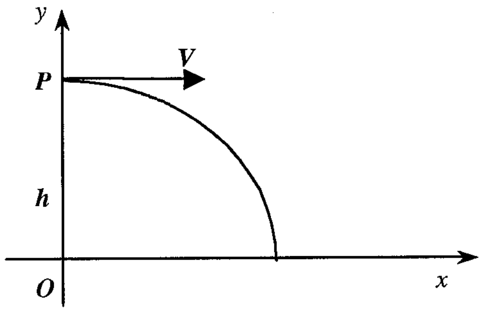
(ii) State the period and amplitude of the motion. — 2

(iii) Find the velocity of the trolley when $t = 0$. — 2

(iv) Find the first time after $t = 0$ when the center of the trolley is at $x = 3$. — ~~2~~ 3

(b) A particle is moving according to $\ddot{x} = 2x$. Initially it is stationary at $x = 1$. Find v^2 as a function of x. — 3

Question 3 **(~~7~~ 9 marks)** **Marks**



(a) A particle is projected horizontally from a point P, h metres above O, with A velocity of V metres per second. The equation of motion of the particle are, $\ddot{x} = 0$ & $\ddot{y} = -g$

Using calculus, show that the position of the particle at time t is given by. $x = Vt$, $y = h - \frac{1}{2}gt^2$ ~~2~~4

(b) A canister containing a life raft is dropped from a plane to a stranded sailor. The plane is traveling at a constant velocity of 216 km/h, at a height of 120 metres above sea level, along a path that passes above the sailor.

(i) Convert 216 km/h to m/s. **2**

(ii) How long will the canister take to hit the water? (Take $g = 10$ m/s^2.) **3**

Question 4 **(10 marks)**

(a) A particle is moving in SHM along the x-axis and its velocity v m/s at the position x metres is given by $v^2 = 16 + 4x - 2x^2$.

(i) Show that the acceleration at any time t is given by, $\ddot{x} = -2(x-1)$. **2**

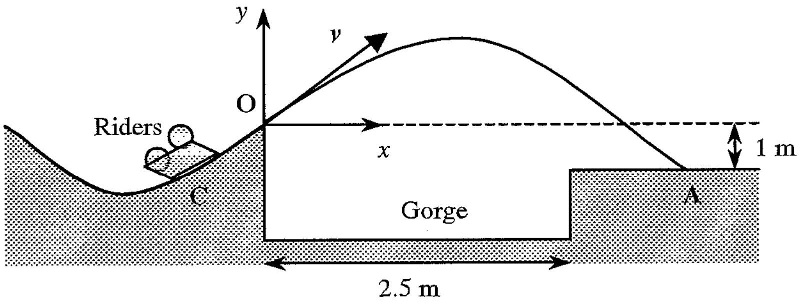
(ii) Find the center and period of the motion. **2**

(iii) Find the extreme points of the motion and hence find the amplitude of the motion. **2**

(b) A spherical balloon is being inflated so that its rate of increase of volume $\frac{dV}{dt}$ is 5 cm^3/s. Find the rate of increase $\frac{dS}{dt}$ of its surface area S when the radius of the balloon is 8 cm. **4**

Question 5 (~~13~~ 19 marks) **Marks**

(a) Two riders are in a snow-mobile C, which is about to make a jump across a gorge of width 2.5m. At the edge of the gorge, at the point O, the speed v of the car is 5 m/s and the angle of projection $\theta = 30°$ above the horizontal.



Taking the co-ordinate axes at O and $g = 10\text{ms}^{-2}$,

(i) Show that the car C will be able to make the jump across the gorge and find the horizontal distance of the point A of landing from the point of projection O. ~~3~~ 6

(ii) Find the angle of the velocity with the horizontal at the point A. ~~3~~ 4

(b) In a colony of 2000 birds, the number N infected with a disease at time t is given by $N = \dfrac{2000}{1 + ke^{-2000t}}$, where k is a constant and t is in years.

(i) Show that eventually all the birds will be infected. 2

(ii) If at $t = 0$, one bird was infected, after how many days will 1000 birds be infected? Give your answer to the nearest hour (assume 365.25 days in a year). ~~3~~ 4

(iii) Show that $\dfrac{dN}{dt} = N(2000 - N)$. ~~2~~ 3

Year 12 · TASK 3 (2004)

Mathematics (Extension)

Q1 a) (i) $= \frac{1}{\sqrt{1-(2x)^2}} \cdot 2$

$= \frac{2}{\sqrt{1-4x^2}}$ (3)

(ii) $= -\frac{1}{\sqrt{1-(x^2)^2}} \cdot 2x$

$\frac{-2x}{\sqrt{1-x^4}}$ (3)

(iii) $= \frac{2}{2^2+x^2}$

$\frac{2}{4+x^2}$ (3)

b) (i) $= \left[\tan^{-1} x\right]_0^1$

$= \frac{\pi}{4} - 0$

$= \frac{\pi}{4}$ (3)

(ii) $= \left[\sin^{-1}\frac{x}{2}\right]_1^2$

$= \sin^{-1}\frac{2}{2} - \sin^{-1}\frac{1}{2}$

$= \frac{\pi}{2} - \frac{\pi}{6}$

$= \frac{\pi}{3}$ (3)

(iii) $= \int \frac{dx}{1+(\sqrt{7}x)^2}$

$= \frac{1}{\sqrt{7}} \tan^{-1}(\sqrt{7}x) + C$ (3)

(iv) $= \int \frac{dx}{\sqrt{3^2-(2x)^2}}$

$\frac{1}{2} \sin^{-1}\left(\frac{2x}{3}\right) + C$ (3)

Q2 a) (i) $x = 6\sin\left(2t+\frac{\pi}{4}\right)$

$\dot{x} = 12\cos\left(2t+\frac{\pi}{4}\right)$

$\ddot{x} = -24\sin\left(2t+\frac{\pi}{4}\right)$

$\ddot{x} = -4 \times 6\sin\left(2t+\frac{\pi}{4}\right)$

$\ddot{x} = -2^2 x$ (3)

which is in the form $\ddot{x} = -n^2 x$ for a particle to move in SHM.

(ii) $T = \frac{2\pi}{n} = \frac{2\pi}{2} = \pi$ seconds

amplitude $= 6$ m (2)

(iii) when $t=0$

$\dot{x} = 12\cos\left(2(0)+\frac{\pi}{4}\right)$

$= 12\left(\frac{1}{\sqrt{2}}\right)$

$= \frac{12}{\sqrt{2}} = 6\sqrt{2}$ m/s. (2)

(iv) when $x=3$

$3 = 6\sin\left(2t+\frac{\pi}{4}\right)$

$\frac{1}{2} = \sin\left(2t+\frac{\pi}{4}\right)$

$2t+\frac{\pi}{4} = \frac{\pi}{6}, \frac{5\pi}{6}, \ldots$

$2t = -\frac{\pi}{12}, \frac{7\pi}{12}, \ldots$

$t = -\frac{\pi}{24}, \frac{7\pi}{24}, \ldots$

(3)

$t = \frac{7\pi}{24}$ is first time particle is at $x=3$

b) $\ddot{x} = 2x \rightarrow a = 2x$

$v^2 = 2\int a\,dx$

$v^2 = 2\int 2x\,dx$

$v^2 = 2x^2 + C$

when $x=1$, $v=0$.

$0 = 2(1)^2 + C$ (3)

$C = -2$

$v^2 = 2x^2 - 2$ (or) $v^2 = 2(x^2-1)$

Q3 a)

$\ddot{x} = 0$

$\dot{x} = c_1$

when $t=0,\ \dot{x} = V$

$\therefore c_1 = V$

$\therefore \dot{x} = V$

$x = Vt + c_2$

when $t=0,\ x=0$

$\therefore c_2 = 0$

$\therefore x = Vt$

$\ddot{y} = -g$

$\dot{y} = -gt + c_3$

when $t=0,\ \dot{y}=0$

$\therefore c_3 = 0$

$\therefore \dot{y} = -gt$

$y = -\frac{1}{2}gt^2 + c_4$

when $t=0,\ y=h$

$\therefore c_4 = h$

$y = h - \frac{1}{2}gt^2$ (4)

b) (i) 216 km/h $(\div 3.6)$ $(216) \times \frac{1000}{3600}$

$= 60\text{ m/s}.$ (2)

(ii) from (a) $y = 120 - \frac{1}{2}(10)t^2$

when $y=0$

$0 = 120 - 5t^2$

$5t^2 = 120$

$t^2 = 24$

$t = \pm\sqrt{24} = \pm 2\sqrt{6}$ (3)

only $+2\sqrt{6}$ secs.

Q4 $v^2 = 16 + 4x - 2x^2$ $\therefore v^2 = 2(8 + 2x - x^2)$

(i) $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$a = \frac{d}{dx}\left(\frac{1}{2}(16 + 4x - 2x^2)\right)$

$= \frac{d}{dx}(8 + 2x - x^2)$

$a = 2 - 2x$

$a = -2(x-1)$ or $\ddot{x} = -2(x-1)$ (2)

(ii) $\therefore n = \sqrt{2}$

$T = \frac{2\pi}{\sqrt{2}}$

$T = \sqrt{2}\pi$ seconds

Centre of Motion is $x = 1\text{m}$ (2)

as $\ddot{x} = 0$ when $x=1$

(iii) when $v=0$

$0 = 2(8 + 2x - x^2)$

$= 2(4-x)(2+x)$

$\therefore$ ... $x = -2, 4$

$\therefore$ Amplitude is 3m (2)

b) $\frac{dV}{dt} = 5\text{ cm}^3/\text{s}$ $V = \frac{4}{3}\pi r^3$

$\therefore \frac{dV}{dr} = 4\pi r^2$

$\therefore \frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$

$5 = 4\pi r^2 \times \frac{dr}{dt}$

$\therefore \frac{dr}{dt} = \frac{5}{4\pi r^2}$

then $\frac{dS}{dt} = \frac{dS}{dr} \times \frac{dr}{dt}$ $S = 4\pi r^2$ $\frac{dS}{dr} = 8\pi r$

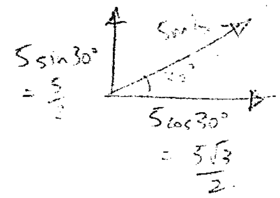
$= 8\pi r \times \frac{5}{4\pi r^2}$

$\frac{dS}{dt} = \frac{10}{r}$; when $r = 8$; $\frac{dS}{dt} = \frac{10}{8}$ (4)

$\frac{dS}{dt} = 1\frac{1}{4}\text{ m}^2/\text{s}$

Q5

INITIAL Vel Diagram



$\ddot{x} = 0$

$\dot{x} = \frac{5\sqrt{3}}{2}$

$x = \frac{5\sqrt{3}}{2}t$

$\ddot{y} = -10$

$\dot{y} = -10t + \frac{5}{2}$

$y = -5t^2 + \frac{5}{2}t$

when $y = -1\text{ m}$

$-1 = -5t^2 + \frac{5}{2}t$

$10t^2 - 5t - 2 = 0$

$\therefore t = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(10)(-2)}}{2(10)}$

$= \frac{5 \pm \sqrt{25 + 80}}{20}$

$t = \frac{5 \pm \sqrt{105}}{20}$

$t = \frac{5 + \sqrt{105}}{20}$ only

sub $t = \frac{5 + \sqrt{105}}{20}$ into $x = \frac{5\sqrt{3}}{2}t$

$x = \frac{5\sqrt{3}}{2}\left(\frac{5 + \sqrt{105}}{20}\right)$

$= \frac{25\sqrt{3} + 5\sqrt{315}}{40}$

$= \frac{5\sqrt{3} + \sqrt{315}}{8}$

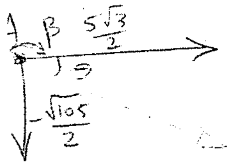
$\therefore$ Greater than $x = 2.5\text{m}$ (6)

i.e. ... will ...

(a) cont. (iii) when $\underline{y = -1}$; $t = \frac{5+\sqrt{105}}{20}$ from (i)

$\dot{x} = \frac{5\sqrt{3}}{2}$; $\dot{y} = -10\left(\frac{5+\sqrt{105}}{20}\right) + \frac{5}{2}$

$\dot{y} = -\frac{\sqrt{105}}{2}$



$\therefore \tan\theta = \frac{\sqrt{105}/2}{5\sqrt{3}/2}$

or

$\beta = 180 - 49°48'$ $\quad \tan\theta = \frac{\sqrt{105}}{5\sqrt{3}}$

$\beta = 130°12'$ $\quad \theta = 49°48'$

(4)

b) $N = \frac{2000}{1+ke^{-2000t}}$

(i) when $t \to \infty$, $e^{-2000t} \to 0$.

$\therefore N = \frac{2000}{1+0}$

$N \to 2000$

(2)

(ii) when $t=0$; $N=1$

$1 = \frac{2000}{1+ke^{-2000(0)}}$

$1 = \frac{2000}{1+k}$

$1+k = 2000$

$\therefore k = 1999$

$N = \frac{2000}{1+1999e^{-2000t}}$

when $N=1000$

$1000 = \frac{2000}{1+1999e^{-2000t}}$

$1+1999e^{-2000t} = 2$

$1999e^{-2000t} = 1$

$e^{-2000t} = \frac{1}{1999}$ $\quad \therefore t = \frac{\ln\left(\frac{1}{1999}\right)}{-2000}$

$\ln e^{-2000t} = \ln\frac{1}{1999}$ $\quad t = 0.0038$ years

$-2000t = \ln\frac{1}{1999}$ $\quad t = 1.388$ days $\therefore t \doteq 33.3$ hrs.

(iii) If $N = 2000\left(1+ke^{-2000t}\right)^{-1}$

$$\frac{dN}{dt} = -2000\left(1+ke^{-2000t}\right)^{-2} \cdot -2000ke^{-2000t}$$

$$= \frac{2000^2 ke^{-2000t}}{\left(1+ke^{-2000t}\right)^2} \quad (1)$$

but,

$N + Nke^{-2000t} = 2000$

$Nke^{-2000t} = 2000 - N$

$ke^{-2000t} = \frac{2000-N}{N} \quad (2)$

Sub (2) into (1)

$$\frac{dN}{dt} = \frac{2000^2\left(\frac{2000-N}{N}\right)}{\left(1+\frac{2000-N}{N}\right)^2}$$

$$= \frac{2000^2\left(\frac{2000-N}{N}\right)}{\frac{2000^2}{N^2}}$$

(3)

$$\boxed{\frac{dN}{dt} = N(2000-N)}$$

Q5 ([illegible])

$\dot{x} = $ [illegible] $\qquad \ddot{y} = -10$

$\dot{x} = \frac{5\sqrt{3}}{2} \qquad \dot{y} = -10t + \frac{5}{2}$

$x = \frac{5\sqrt{3}}{2}t$ ① $\qquad y = -5t^2 + \frac{5}{2}t$ ②

$t = \frac{2x}{5\sqrt{3}}$ ③

sub ③ into ②

$y = -\left(\frac{2x}{\sqrt{3}}\right)^2 + \frac{5}{2}\left(\frac{2x}{5\sqrt{3}}\right)$ Cartesian Eqn.

$y = -\frac{2}{15}x^2 + \frac{1}{\sqrt{3}}x$