TIME: 80 Mins

Instructions
1. Each question is to be answered on a separate sheet of paper
2. Show all necessary working
3. Sketches must be large ,have ruled axes and be neatly drawn
4. Marks will be deducted for careless and badly arranged work.

QUESTION 1 (25 Marks)

(a) Find the exact value of 5

(i) $\sin^{-1}(1)$ (ii) $\cos^{-1}\left(\frac{-1}{\sqrt{2}}\right)$ (iii) $\sin\left(\tan^{-1}\left(\frac{5}{12}\right)\right)$

(b) Find the derivatives of 2

(i) $y = \sin^{-1}\frac{x}{2}$ (ii) $y = \sin^{-1} 2x$

(iii) $y = x\tan^{-1} x$ (iv) $y = \tan^{-1}(\sin x)$ 4

(c) For $f(x) = 3\cos^{-1} 2x$

(i) Evaluate $f\left(\frac{1}{2}\right)$ 1

(ii) Find $f'(x)$ 2

(iii) State the domain and range of f(x) 2

(d) Sketch $y = \tan^{-1} x$ clearly stating domain and range 4

(e) (i) Write down the formula for sin(x+y) 1

(ii) If $x = \sin^{-1}\left(\frac{3}{5}\right)$ and $y = \sin^{-1}\left(\frac{5}{13}\right)$. Find sin(x+y) without using a calculator. 4

QUESTION 2 (23 Marks)

Evaluate

(a) (i) $\int \frac{dx}{x^2+3}$ (ii) $\int \frac{dx}{\sqrt{1-4x^2}}$ 4

(iii) $\int_0^1 \frac{1}{\sqrt{4-x^2}}dx$ (iv) $\int_0^{\frac{1}{\sqrt{2}}} \frac{1}{2t^2+1}dt$ 6

(b) Use the substitution $u = e^x$. to evaluate 4

$$\int \frac{e^x}{1+e^{2x}}dx$$

(c) Use the substitution u=1-x to find exact value 5

$$\int_0^1 x\sqrt{1-x}\,dx$$

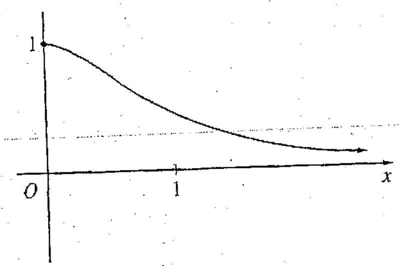
(d) Write down the general solution to 2

$$\cos 2x = \frac{1}{2}$$

(e) Without evaluating the integral explain why $\int_{-1}^{1} \tan^{-1} x\, dx = 0$ 2

QUESTION 3 (20 Marks)

(a) The diagram below shows a sketch of the graph of y=f(x)



(i) Explain why the function y=f(x) has an inverse function $y = f^{-1}(x)$ 1

(ii) Copy this diagram onto your answer sheet.
On the same set of axes sketch the graph of $y = f^{-1}(x)$ 2

(iii) State the domain and range of $y = f^{-1}(x)$ 2

(b) At what points on the curve $y = \cos^{-1} x$ is the gradient $\frac{-2}{\sqrt{3}}$. 4

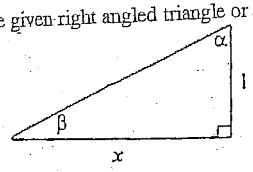
EXTENSION 1 GIRRAWEEN HIGH SCHOOL TASK 3

QUESTION 3 (cont)

(c) (i)Differentiate f(x) = $\tan^{-1} x + \tan^{-1}\frac{1}{x}$ for $x \neq 0$ and show that $f'(x) = 0$. 2

(ii) What does the result in (i) imply about f(x). 1

(iii)By considering the given right angled triangle or otherwise find the value of f(x) 2



QUESTION THREE : (23 MARKS) 2001

a) Show that $\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{4}{5} = \frac{\pi}{2}$ 4

b) Sketch the graph of $y = 2\sin^{-1}\frac{x}{3}$ and state its domain and range. 4

c) Evaluate, giving exact values.

i. $\sin^{-1}(1)$ 1

ii. $\cos^{-1}(0)$ 1

iii. $\tan^{-1}(\frac{1}{\sqrt{3}})$ 1

iv. $\tan\left[\sin^{-1}(-\frac{\sqrt{3}}{2})\right]$ 3

v. $\cos\left[\sin^{-1}(\frac{4}{5})\right]$ 4

vi. $\tan^{-1}(\cos\pi)$ 3

d) Find the general solution for $\sin\theta = \frac{1}{\sqrt{2}}$ 2

QUESTION FOUR : (29 MARKS) 2001

a) Differentiate the following 11

i) $y = \cos^{-1}(3x)$

ii) $y = 3\sin^{-1}(\frac{x}{6})$

iii) $y = \left[\tan^{-1}(x+1)\right]^5$

iv) $y = \tan^{-1}(e^x)$

v) $y = e^{\cos^{-1}x}$

b) Integrate the following 18

i) $\int \frac{1}{9+x^2}dx$

ii) $\int \frac{3}{2\sqrt{1-x^2}}dx$

iii) $\int \frac{dx}{1+9x^2}$

iv) $\int \frac{2}{\sqrt{25-4t^2}}dt$

v) $\int_{-1}^{1} \frac{dx}{\sqrt{4-x^2}}$

vi) $\int_{-1/3}^{1/\sqrt{3}} (4-9x^2)^{-1/2}dx$

Solutions to Task 3 EXT 1 YEAR 12 2005

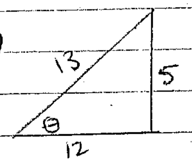
QUESTION 1

(a)(i) $\sin^{-1}(1) = \frac{\pi}{2}$ (1)

(b)(i) $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = \pi - \frac{\pi}{4}$

$= \frac{3\pi}{4}$ (2)

(ii)



$\sin\theta = \frac{5}{13}$ (2)

(b)(i) $\frac{dy}{dx} = \frac{1}{\sqrt{4-x^2}}$ (1)

(ii) $\frac{dy}{dx} = \frac{2}{\sqrt{1-4x^2}}$ (1)

(iii) $\frac{x}{1+x^2} + \tan^{-1}x$ (2)

(iv) $\frac{\cos x}{1+\sin^2 x}$ (2).

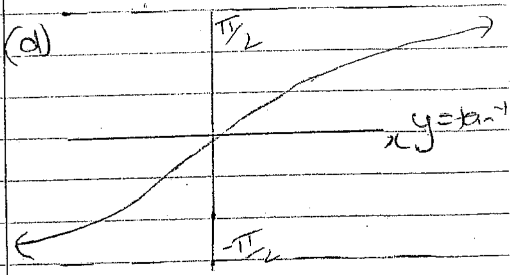
(c)(i) $3\cos^{-1}1 = 0$ (1)

(ii) $\frac{-6}{\sqrt{1-4x^2}}$ (2)

(ii) Domain $-\frac{1}{2} \le x \le \frac{1}{2}$

Range $0 \le y \le 3\pi$ (2)

(d)



2 for graph
1 for asymptotes
1 for correct shape.

x all reals

$-\frac{\pi}{2} < y < \frac{\pi}{2}$ (2)

(e)(i) $\sin x\cos y + \cos x\sin y$ (1)

(ii) $\sin x = \frac{3}{5}$ $\sin y = \frac{5}{13}$

$\cos x = \frac{4}{5}$ $\cos y = \frac{12}{13}$

$\therefore \sin(x+y) = \frac{3}{5}\times\frac{12}{13} + \frac{4}{5}\times\frac{5}{13}$

$= \frac{56}{65}$ (4)

QUESTION 2

a (i) $\frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} + c$ (2)

1 mark for $+c$

(ii) $\frac{1}{2}\sin^{-1}2x + c$. (2)

(iii) $\left[\sin^{-1}\frac{x}{2}\right]_0^1$

$= \sin^{-1}\frac{1}{2} - \sin^{-1}0$

$= \frac{\pi}{6}$ (3)

(v) $\left[\frac{1}{\sqrt{2}}\tan^{-1}\sqrt{2}x\right]_0^{\frac{1}{\sqrt{2}}}$

$= \frac{1}{\sqrt{2}}\left[\tan^{-1}1 - \tan^{-1}0\right]$

$= \frac{\pi}{4\sqrt{2}} = \frac{\pi\sqrt{2}}{8}$ (3)

(b) $u = e^x$ $du = e^x\,dx$

$\int \frac{du}{1+u^2} = \tan^{-1}u + c$

$= \tan^{-1}e^x + c$ (4)

c) $\int x\sqrt{1-x}\,dx$ $u = 1-x$

$\frac{du}{dx} = -1$

when $x=0$ $u=1$

$x=1$ $u=0$

and $x = 1-u$

$-\int_1^0 (1-u)\sqrt{u}\,du$

$= \int_0^1 u^{1/2} - u^{3/2}\,du$

$= \left[\frac{2}{3}u^{3/2} - \frac{2}{5}u^{5/2}\right]_0^1$

$= \frac{2}{3} - \frac{2}{5}$

$= \frac{4}{15}$ (5)

(d) $2x = 2n\pi \pm \cos^{-1}\frac{1}{2}$

$x = n\pi \pm \frac{\pi}{6}$ (2)

(e) $\tan^{-1}(-x) = -\tan^{-1}x$

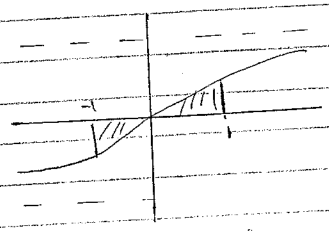
so $\tan^{-1}x$ is an odd function

$\int_{-a}^{a} f(x)\,dx$ of an odd function

$= 0$.

OR

From graph

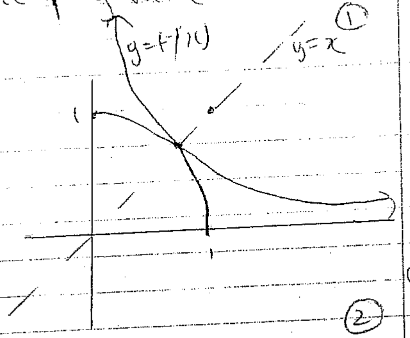


(2)

since symmetric above and below axis $\int_{-1}^{0} f(x)\,dx = -\int_0^1 f(x)\,dx$ so cancel.

QUESTION 3

a) (i) It satisfies the horizontal lines test meaning there is only one x value for every y value ①



②

1 for drawing in $y=x$
1 for graph of $y=f^{-1}(x)$

(iii) $x \not< 0 < x \leq 1$ ②

$y \geq 0$

b) $\frac{-1}{\sqrt{1-x^2}} = \frac{-2}{\sqrt{3}}$

$\therefore -2\sqrt{1-x^2} = -\sqrt{3}$

$\sqrt{1-x^2} = \frac{\sqrt{3}}{2}$

$\therefore 1-x^2 = \frac{3}{4}$

$x^2 = \frac{1}{4}$

$x = \pm\frac{1}{2}$

$x=\frac{1}{2}$ $y=\frac{\pi}{3}$ or $x=-\frac{1}{2}$ $y=\frac{2\pi}{3}$ ④

c) (i) $\frac{1}{1+x^2} + \frac{-1/x^2}{1+\frac{1}{x^2}}$

$= \frac{1}{1+x^2} + \frac{-1}{x^2+1}$ ②

$= 0$

(ii) $f(x)$ is a constant or $f(x)$ has 0 gradient so parallel to x-axis ①

(iii) $\tan\beta = \frac{1}{x}$ $\tan\alpha = x$

$\tan^{-1}\frac{1}{x} + \tan^{-1}x = \alpha+\beta$

but $\alpha+\beta = \frac{\pi}{2}$

so $\tan^{-1}\frac{1}{x} + \tan^{-1}x = \frac{\pi}{2}$

or If $x=1$

$\tan^{-1}1 + \tan^{-1}1 = \frac{\pi}{4}+\frac{\pi}{4}$

$= \pi/2$ ②

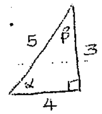
2001

Q3)

a) $\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{4}{5} = \frac{\pi}{2}$

Let $\alpha = \sin^{-1}\frac{3}{5}$ $\beta = \sin^{-1}\frac{4}{5}$

$\therefore \sin\alpha = \frac{3}{5}$ $\sin\beta = \frac{4}{5}$



$\therefore \alpha+\beta = \frac{\pi}{2}$ ($\angle$ sum of $\triangle$) ④

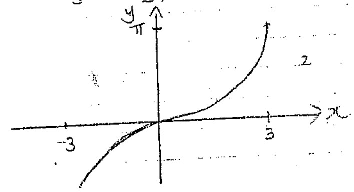
$\therefore \sin^{-1}\frac{3}{5} + \sin^{-1}\frac{4}{5} = \frac{\pi}{2}$

b) D: $-1 \leq \frac{x}{3} \leq 1$

$\therefore$ D: $-3 \leq x \leq 3$

R: $-\frac{\pi}{2} \leq \frac{y}{2} \leq \frac{\pi}{2}$

$\therefore$ R: $-\pi \leq y \leq \pi$



④

Q3)

c) i) $\sin^{-1}(1) = \frac{\pi}{2}$ ① ii) $\cos^{-1}(0) = \frac{\pi}{2}$ ① iii) $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$ ①

iv) $\tan\left[\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right]$

$= \tan\left[-\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right]$

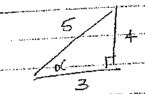
$= \tan\left(-\frac{\pi}{3}\right)$ ③

$= -\sqrt{3}$

v) $\cos\left[\sin^{-1}\left(\frac{4}{5}\right)\right]$

Let $\alpha = \sin^{-1}\frac{4}{5}$

$\therefore \sin\alpha = \frac{4}{5}$



$\cos\left[\sin^{-1}\left(\frac{4}{5}\right)\right]$

$= \cos\alpha$

$= \frac{3}{5}$ ④

vi) $\tan^{-1}(\cos\pi)$

$= \tan^{-1}(-1)$

$= -\tan^{-1}(1)$

$= -\frac{\pi}{4}$ ③

d) $\sin\theta = \frac{1}{\sqrt{2}}$

$\therefore \theta = \pi k + (-1)^k \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$

where k is an integer ②

$\therefore \theta = \pi k + (-1)^k \cdot \frac{\pi}{4}$

Q4) 2001

a) i) $y = \cos^{-1}(3x)$

$\frac{dy}{dx} = \frac{-3}{\sqrt{1-9x^2}}$ (2)

ii) $y = 3\sin^{-1}\left(\frac{x}{6}\right)$

$\frac{dy}{dx} = 3 \cdot \frac{1}{\sqrt{36-x^2}}$

$= \frac{3}{\sqrt{36-x^2}}$ (2)

iii) $y = \left[\tan^{-1}(x+1)\right]^5$

$\frac{dy}{dx} = 5\left[\tan^{-1}(x+1)\right]^4 \cdot \frac{1}{1+(x+1)^2}$

$= \frac{5\left[\tan^{-1}(x+1)\right]^4}{x^2+2x+2}$ (3)

iv) $y = \tan^{-1}(e^x)$

$\frac{dy}{dx} = \frac{e^x}{1+e^{2x}}$ (2)

v) $y = e^{\cos^{-1}x}$

$\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}} \cdot e^{\cos^{-1}x}$

$= \frac{-e^{\cos^{-1}x}}{\sqrt{1-x^2}}$ (2)

b) i) $\int \frac{1}{9+x^2}\,dx$

$= \frac{1}{3}\tan^{-1}\frac{x}{3} + C$ (2)

ii) $\int \frac{3}{2\sqrt{1-x^2}}\,dx$

$= \frac{3}{2}\int \frac{1}{\sqrt{1-x^2}}\,dx$

$= \frac{3}{2}\sin^{-1}x + C$ (2)

iii) $\int \frac{dx}{1+9x^2}$

$= \frac{1}{3}\int \frac{3\,dx}{1+(3x)^2}$ (3)

$= \frac{1}{3}\tan^{-1}3x + C$

iv) $\int \frac{2}{\sqrt{25-4t^2}}\,dt$

$= \int \frac{2}{\sqrt{5^2-(2t)^2}}\,dt$ (3)

$= 2\sin^{-1}\frac{2t}{5} + C$

v) $\int_{-1}^{1} \frac{dx}{\sqrt{4-x^2}}$

$= \left[\sin^{-1}\frac{x}{2}\right]_{-1}^{1}$

$= \sin^{-1}\frac{1}{2} - \sin^{-1}\left(\frac{-1}{2}\right)$

$= \sin^{-1}\left(\frac{1}{2}\right) + \sin^{-1}\left(\frac{1}{2}\right)$

$= 2\sin^{-1}\left(\frac{1}{2}\right)$

$= \frac{2\pi}{6}$

$= \frac{\pi}{3}$ (4)

vi) $\int_{-1/3}^{1/\sqrt{3}} \frac{1}{\sqrt{4-9x^2}}\,dx$

$= \frac{1}{3}\int_{1/3}^{1/\sqrt{3}} \frac{3}{\sqrt{2^2-(3x)^2}}\,dx$

$= \left[\frac{1}{3}\sin^{-1}\frac{3x}{2}\right]_{1/3}^{1/\sqrt{3}}$

$= \frac{1}{3}\left[\sin^{-1}\frac{\sqrt{3}}{2} - \sin^{-1}\frac{1}{2}\right]$

$= \frac{1}{3}\left(\frac{\pi}{3} - \frac{\pi}{6}\right)$

$= \frac{\pi}{18}$ (4)