Girraween High School **Yr 12 – Task 3 (2007)**
Mathematics (Extension 1) *Time allowed – 90 minutes*

DIRECTIONS TO CANDIDATES
- All necessary working should be shown.
- Marks may be deducted for careless or badly arranged work
- Start each question on a *new* sheet of paper.

Question 1 **(17 marks)** **Marks**

(a) Consider the function $y = 2\sin^{-1}\left(\frac{x}{3}\right)$. ~~Sketch the function.~~

(i) State the domain and range 3

(ii) Evaluate $f(0)$ 1

(iii) Draw the graph of $y = f(x)$ 3

(b) Draw a sketch of the curve, $y = (x-3)^2 - 3$.

(i) Find the largest positive domain such that the graph defines a function $f(x)$ which has an inverse 2

(ii) Find the inverse function, stating its domain and range. 4

(iii) Sketch the graphs of $f(x)$ and $f^{-1}(x)$ on the same axes. 4

Question 2 **(11 marks)**

(a) Evaluate in terms of π.

(i) $\cos^{-1}\left(-\frac{1}{2}\right) - \sin^{-1}\left(-\frac{1}{2}\right)$ 2

(ii) $\cos^{-1}\left(\sin^2\frac{\pi}{4}\right)$ 2

(b) Evaluate as a rational number.

$\tan\left(2\tan^{-1}\left(-\frac{1}{2}\right)\right)$ 3

Marks

(b) Using $\sin(A+B)=\sin A\cos B+\cos A\sin B$, prove that

$$\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{5}{13}=\sin^{-1}\frac{56}{65}.$$ 4

Question 3 **(17 marks)**

(a) Differentiate the following

(i) $y=\cos^{-1}2x$ 2

(ii) $y=\tan^{-1}\sqrt{x}$ 3

(iii) $y=\dfrac{1}{\sin^{-1}x}$ 3

(iv) $y=x\sin^{-1}x+\sqrt{1-x^2}$ 4

(b) Show that the curves $y=\cos^{-1}x$ and $y=2\tan^{-1}(1-x)$ intersect the y-axis at the same point, and have a common tangent at this point. 5

Question 4 **(17 marks)**

(a) Find the following integrals.

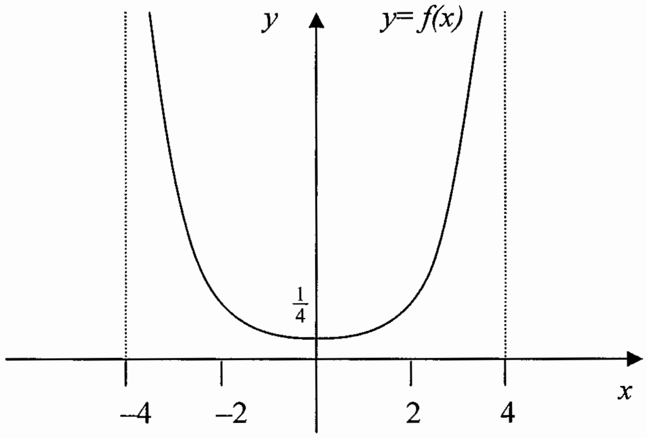
(i) $\displaystyle\int\frac{dx}{1+9x^2}$ 3

(ii) $\displaystyle\int\frac{5dx}{x^2+2}$ 3

(iii) $\displaystyle\int_0^{\frac{3}{2}}\frac{dx}{\sqrt{9-2x^2}}$ 3

(iv) $\displaystyle\int_1^{\sqrt{3}}\frac{dx}{\sqrt{4-x^2}}$ 3

(b) **Marks**



Let $f(x)=\dfrac{1}{\sqrt{16-x^2}}$. The graph of $y=f(x)$ is sketched above.

(i) Show that $f(x)$ is an even function. 2

(ii) Find the area enclosed by $y=f(x)$, the x-axis, $x=2$, and $x=-2$. 3

Question 5 (19 marks)

(a) Find the following integrals using the given substitution.

(i) $\int x\sqrt{x^2+2}dx$ $\quad u=x^2+2$ 3

(ii) $\int \dfrac{x^2dx}{\sqrt{1-x^3}}$ $\quad u=1-x^3$ 4

(iii) $\int x\sqrt{x+1}dx$ $\quad x=u^2-1$ 4

(b) Evaluate following integrals using the given substitution.

(i) $\int_0^1 x\left(1+x^2\right)^2 dx$ $\quad u=x^2+1$ 4

(ii) $\int_0^{\frac{\pi}{6}} \dfrac{2\cos x dx}{1+4\sin^2 x}$ $\quad u=2\sin x$ 4

Question 6 (14 marks) **Marks**

(a) Find the general solution (in terms of π) of the following functions.

(i) $\sin\theta = \frac{1}{2}$ 3

(ii) $\cos\theta = -\frac{1}{\sqrt{2}}$ 3

(b) Given that, $x^2 + 4x + 5 = (x+a)^2 + b$

(i) Find a and b. 2

(ii) Hence evaluate $\int \frac{dx}{x^2 + 4x + 5}$ 2

(c) The function $f(x) = \sec x$ for $0 \le x < \frac{\pi}{2}$, and is not defined for other values of x.

(i) State the domain and range of the inverse function $f^{-1}(x)$ 1

(ii) Show that, $f^{-1}(x) = \cos^{-1}\left(\frac{1}{x}\right)$. 1

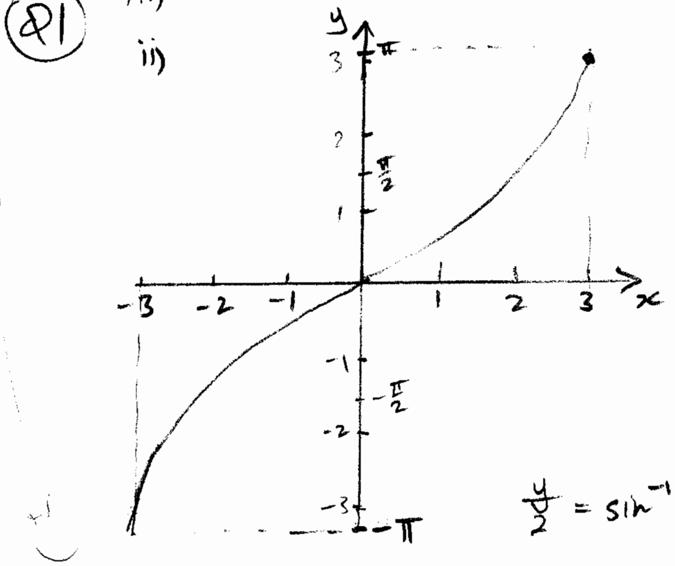
(iii) Hence find, $\frac{d}{dx} f^{-1}(x)$. 2

Yr 12 Mathematics (Ext)

TASK 3 (2007) SOLUTIONS

Q1 a) i) $f(0) = 2\sin^{-1}\left(\frac{0}{3}\right) = 0$ (1)

ii)



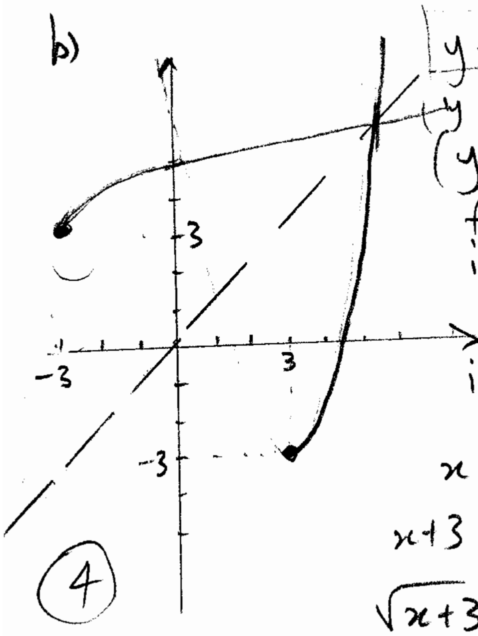
$\frac{y}{2} = \sin^{-1}\left(\frac{x}{3}\right)$ (3)

(i)

D: $-1 \leq \frac{x}{3} \leq 1$ R: $-\frac{\pi}{2} \leq \frac{y}{2} \leq \frac{\pi}{2}$ (3)

D: $-3 \leq x \leq 3$ R: $-\pi \leq y \leq \pi$

b)



$y = (x-3)^2 - 3$

$(y = x^2 - 6x + 9 - 3)$

$(y = x^2 - 6x + 6)$

$f(x)$

i) D: $x \geq 3$ (2)

R: $y \geq -3$.

ii) $f^{-1}(x)$:

$x = (y-3)^2 - 3$

$x + 3 = (y-3)^2$

$\sqrt{x+3} = y - 3$

$f^{-1}(x): y = \sqrt{x+3} + 3$

D: $x \geq -3$ (4)

R: $y \geq 3$

(4)

Q2 a) i) $\cos^{-1}\left(-\frac{1}{2}\right) - \sin^{-1}\left(-\frac{1}{2}\right)$

$= \frac{2\pi}{3} - \left(-\frac{\pi}{6}\right)$

$= \frac{5\pi}{6}$ (2)

ii) $\cos^{-1}\left(\sin^2 \frac{\pi}{4}\right)$

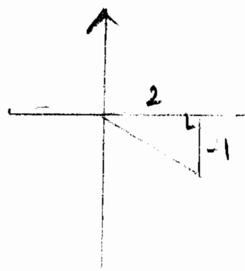
$= \cos^{-1}\left(\left(\frac{1}{\sqrt{2}}\right)^2\right)$

$= \cos^{-1}\left(\frac{1}{2}\right)$

$= \frac{\pi}{3}$. (2)

iii) $\tan\left(2\tan^{-1}\left(-\frac{1}{2}\right)\right)$

$= \tan(2\theta)$



$= \frac{2\tan\theta}{1 - \tan^2\theta}$

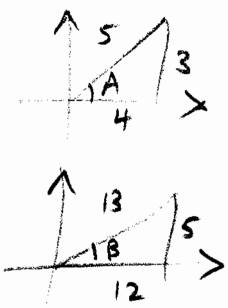
$= \frac{2\left(-\frac{1}{2}\right)}{1 - \left(-\frac{1}{2}\right)^2}$

$= \frac{-1}{1 - \frac{1}{4}}$ (3)

$= -\frac{4}{3}$

b) If $\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{5}{13}$

(A) + (B)



$\sin(A+B) = \sin A\cos B + \cos A \sin B$

$= \frac{3}{5} \cdot \frac{12}{13} + \frac{4}{5} \cdot \frac{5}{13}$

$= \frac{36}{65} + \frac{20}{65}$

$\sin(A+B) = \frac{56}{65}$ (4)

$\therefore A + B = \sin^{-1}\frac{56}{65}$

Q3

a) i) $y = \cos^{-1} 2x$

$$y' = \frac{-1}{\sqrt{1-(2x)^2}} \times 2$$

$$y' = \frac{-2}{\sqrt{1-4x^2}}$$

2

ii) $y = \tan^{-1}\sqrt{x}$

$$y' = \frac{1}{1+(\sqrt{x})^2} \times \frac{1}{2} x^{-\frac{1}{2}}$$

$$= \frac{1}{1+x} \times \frac{1}{2\sqrt{x}}$$

$$y' = \frac{1}{2\sqrt{x}(1+x)}$$

3

iii) $y = \dfrac{1}{\sin^{-1} x}$

$$y = (\sin^{-1} x)^{-1}$$

$$y' = -(\sin^{-1} x)^{-2} \times \frac{1}{\sqrt{1-x^2}}$$

$$y' = \frac{-1}{(\sin^{-1} x)^2 \sqrt{1-x^2}}$$

3

iv) $y = x \sin^{-1} x + \sqrt{1-x^2}$

$$= x \cdot \frac{1}{\sqrt{1-x^2}} + \sin^{-1} x \cdot 1 + \frac{1}{2}(1-x^2)^{-\frac{1}{2}} \times -2x$$

$$= \frac{x}{\sqrt{1-x^2}} + \sin^{-1} x - \frac{x}{\sqrt{1-x^2}}$$

$$= \sin^{-1} x$$

4

b) $y = \cos^{-1} x$ and $y = 2\tan^{-1}(1-x)$

when $x = 0$

$y = \cos^{-1}(0)$

$y = \frac{\pi}{2}$

$y = 2\tan^{-1}(1-0)$

$y = 2\left(\frac{\pi}{4}\right)$

$y = \frac{\pi}{2}$

∴ Both intersect the y-axis at the point $\left(0, \frac{\pi}{2}\right)$

$$y' = \frac{-1}{\sqrt{1-x^2}}$$

$$y' = \frac{2}{1+(1-x)^2} \times -1$$

when $x = 0$

$$y' = \frac{-1}{\sqrt{1-(0)}}$$

$$y' = -1$$

$$y' = \frac{-2}{1+(1-0)^2}$$

$$y' = -1$$

∴ Both curves have common tangent at the point $\left(0, \frac{\pi}{2}\right)$

5

Q4 a) i) $\int \frac{dx}{1+9x^2}$

$= \frac{1}{9}\int \frac{1}{\left(\frac{1}{3}\right)^2 + x^2}\,dx$

$= \frac{1}{9} \times 3 \tan^{-1} 3x + C$

$= \frac{1}{3}\tan^{-1}(3x) + C$ (3)

ii) $\int \frac{5}{x^2+2}\,dx$

$= 5\int \frac{1}{(\sqrt{2})^2 + x^2}\,dx$

$= 5 \times \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C$

$= \frac{5}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C$ (3)

iii) $\int_0^{3/2} \frac{dx}{\sqrt{9-2x^2}}$

$= \frac{1}{\sqrt{2}}\int_0^{3/2} \frac{1}{\sqrt{\frac{9}{2} - x}}\,dx$

$= \frac{1}{\sqrt{2}}\left[\sin^{-1}\left(\frac{\sqrt{2}x}{3}\right)\right]_0^{3/2}$

$= \frac{1}{\sqrt{2}}\left(\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) - \sin^{-1}(0)\right)$

$= \frac{1}{\sqrt{2}}\left(\frac{\pi}{4}\right)$

$= \frac{\pi}{4\sqrt{2}}$ (3)

iv) $\int_1^{\sqrt{3}} \frac{dx}{\sqrt{4-x^2}}$

$= \left[\sin^{-1}\left(\frac{x}{2}\right)\right]_1^{\sqrt{3}}$

$= \sin^{-1}\frac{\sqrt{3}}{2} - \sin^{-1}\frac{1}{2}$

$= \frac{\pi}{3} - \frac{\pi}{6}$

$= \pi/6.$ (3)

b) i) $f(x) = \frac{1}{\sqrt{16-x^2}}$

$f(-x) = \frac{1}{\sqrt{16-(-x)^2}} = \frac{1}{\sqrt{16-x^2}}$

$\therefore f(x) = f(-x)$ $\therefore$ an EVEN FUNCTION (2)

ii) $A = 2 \times \int_0^2 \frac{1}{\sqrt{16-x^2}}\,dx$

$= 2 \times \left[\sin^{-1}\left(\frac{x}{4}\right)\right]_0^2$

$= 2 \times \left(\sin^{-1}\frac{1}{2} - \sin^{-1} 0\right)$

$= 2\left(\frac{\pi}{6} - 0\right)$

$= \pi/3$ (3)

Q5

a) i) $\int x\sqrt{x^2+2}\,dx$ $\qquad u = x^2+2$, $du = 2x\,dx$, $\frac{1}{2}du = x\,dx$

$= \frac{1}{2}\int \sqrt{u}\,du$

$= \frac{1}{2}\times\frac{2}{3}u^{3/2} + C$

$= \frac{1}{3}(x^2+2)^{3/2} + C \left(= \frac{\sqrt{(x^2+2)^3}}{3} + C\right)$

3

ii) $\int \frac{x^2\,dx}{\sqrt{1-x^3}}$ $\qquad u = 1-x^3$, $du = -3x^2\,dx$, $-\frac{1}{3}du = x^2\,dx$

$= -\frac{1}{3}\int \frac{du}{\sqrt{u}}$

$= -\frac{1}{3}\int u^{-\frac{1}{2}}\,du$

$= -\frac{1}{3}\times 2u^{\frac{1}{2}} + C$

$= -\frac{2}{3}(1-x^3)^{\frac{1}{2}} + C \left(= -\frac{2\sqrt{1-x^3}}{3} + C\right)$

4

iii) $\int x\sqrt{x+1}\,dx$ $\qquad x = u^2-1$, $dx = 2u\,du$, also $u = \sqrt{x+1}$

$= \int (u^2-1)\,u\,.\,2u\,du$

$= \int 2u^4 - 2u^2\,du$

$= \frac{2u^5}{5} - \frac{2u^3}{3} + C$

$\frac{2}{5}(\sqrt{x+1})^5 - \frac{2}{3}(\sqrt{x+1})^3 + C$

4

b) i) $\int_0^1 x(1+x^2)^2\,dx$ $\qquad u = x^2+1$; $x=1$, $u=2$; $x=0$, $u=1$; $du = 2x\,.\,dx$, $\frac{1}{2}du = x\,.\,dx$

$= \frac{1}{2}\int_1^2 u^2\,.\,du$

$= \frac{1}{2}\left[\frac{u^3}{3}\right]_1^2$

$= \frac{1}{2}\left(\frac{8}{3} - \frac{1}{3}\right)$

$= \frac{7}{6}$

4

ii) $\int_0^{\pi/6} \frac{2\cos x}{1+4\sin^2 x}\,dx$ $\qquad u = 2\sin x$, $du = 2\cos x\,dx$; $x = \pi/6$, $u = 1$; $x = 0$, $u = 0$

$= \int_0^1 \frac{du}{1+u^2}$

$= \left[\tan^{-1}u\right]_0^1$

$= \tan^{-1}(1) - \tan^{-1}(0)$

$= \frac{\pi}{4} - 0$

$= \pi/4.$

4

Q6 c) $f(x) = \sec x$ $\quad D: 0 \le x < \frac{\pi}{2}$

$\therefore R: y \geq 1$

i) $\therefore f^{-1}(x)$ $\quad D: x \geq 1$

$R: 0 \le y < \pi/2$ (1)

ii) $f^{-1}(x)$: $x = \sec y$

$$x = \frac{1}{\cos y}$$

$$\cos y = \frac{1}{x}$$

$f^{-1}(x)$: $y = \cos^{-1}\left(\frac{1}{x}\right)$ (1)

iii) $\frac{d}{dx} f^{-1}(x)$

$$= \frac{-1}{\sqrt{1 - \left(\frac{1}{x}\right)^2}} \times -1x^{-2}$$

$$= \frac{1}{x^2\sqrt{\frac{x^2-1}{x^2}}}$$ (2)

$$= \frac{1}{x\sqrt{x^2-1}}$$

b) i) $x^2 + 4x + 5 = x^2 + 4x + 4 + 1$

$= (x+2)^2 + 1$

$\therefore$ $\underline{a = 2}$, $\underline{b = 1}$ (2)

ii) $\int \frac{1}{x^2+4x+5}\,dx$

$$= \int \frac{1}{1 + (x+2)^2}\,dx$$

$$= \tan^{-1}(x+2) + C$$ (2)

a) i) $\sin\theta = \frac{1}{2}$

$\theta = \sin^{-1}\left(\frac{1}{2}\right) + 2n\pi$ $\underline{\underline{\text{or}}}$ $\left(\pi - \sin^{-1}\left(\frac{1}{2}\right)\right) + 2n\pi$

$\theta = \frac{\pi}{6} + 2n\pi$; $\frac{5\pi}{6} + 2n\pi$

where n is an integer. (3)

ii) $\cos\theta = -\frac{1}{\sqrt{2}}$

$\theta = 2n\pi \pm \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

$\theta = 2n\pi \pm \frac{3\pi}{4}$ where n is an integer. (3)