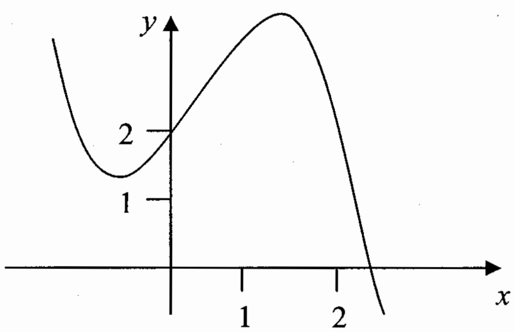
Girraween High School
Mathematics (*Extension 1*)

Year 12 - Task4 (2006)
Time allowed – 90 minutes

DIRECTIONS TO CANDIDATES
- All necessary working should be shown.
- Marks may be deducted for careless or badly arranged work
- Start each question on a *new* sheet of paper.

Question 1 (10 marks) *Marks*

(a) A student is using Newton's method to calculate an approximate value for the single zero of the polynomial $f(x) = -x^3 + x^2 + 2x + 2$ (as below).



(i) Use Newton's method with $x_0 = 2$, to approximate the zero. Correct to 2 decimal places. 3

(ii) Use the graph above and use it to show why $x_0 = 1$ would be a bad initial approximation to the root (no calculations needed). 1

(b) A particle's displacement is $x = 5 - 3\sin\left(2t + \frac{\pi}{4}\right)$, in units of centimeters and seconds

(i) In what interval is the particle moving? 2

(ii) Write down the period of the motion. 1

(iii) Find the first two times after time zero when the particle is closest to the origin. 3

Question 2 (13 marks) *Marks*

(a) The velocity v m/s of a particle moving in simple harmonic motion along the x axis is given by $v^2 = -5 + 6x - x^2$, where x is in metres.

(i) Between which two points is the particle oscillating? 2

(ii) Find the center of motion of the particle. 1

(iii) Find the maximum speed of the particle. 2

(iv) Find the acceleration of the particle in terms of x. 3

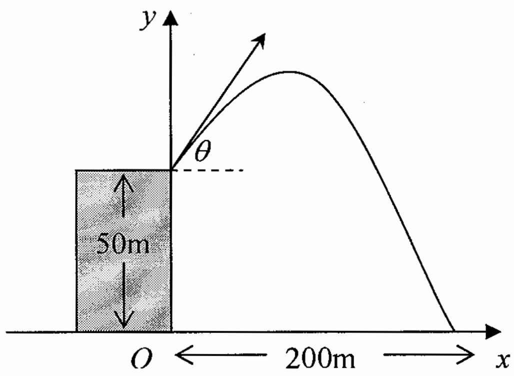
(b) At time t (in minutes) the temperature T (in degrees) of a body in a room of constant temperature 20°C is decreasing according to the equation $\frac{dT}{dt} = -k(T-20)$ for some constant k, where $k > 0$.

(i) Verify that $T = 20 + Ae^{-kt}$, is a solution to the above equation. 1

(ii) The initial temperature of the body is 90C and it falls to 70C after 10 minutes. Find the temperature of the body after a further 5 minutes. 4

Question 3 **(15 marks)** ***Marks***

(a) The diagram shows the path of a projectile launched at an angle of θ, from the top of a building 50 m high with an initial velocity of 40 m/s. The acceleration due to gravity is assumed to be 10 m/s^2. Take the origin to be the base of the tower.



(i) Given that $\ddot{x} = 0$ and $\ddot{y} = -10$,
show that, $x = 40t\cos\theta$ and $y = -5t^2 + 40t\sin\theta + 50$ **4**

(ii) The projectile lands on the ground 200 metres from the base of the building. Find the two possible angles for θ.
Give your answers to the nearest degree. **4**

(b) In a certain chemical process, the amount y grams of a certain substance at time t is given by the formula $y = 3 + e^{-kt}$.

(i) Show that $\frac{dy}{dt} = -k(y-3)$. **1**

(ii) If initially y decreases at the rate of 0.08 grams/hour.
Find the value of k. **3**

(iii) Find the rate of change when $y = 3.5$. **1**

(iv) What values can y take? **2**

Question 4 (11 marks)

(a) A stone is thrown horizontally with a velocity of 30 m/s from a point O on the top of a tower 45 m high. ($g = 10$ m/s^2)

(i) Write the acceleration, velocity and displacement equations of the motion. 4

(ii) What time elapses before the stone strikes the ground? 2

(iii) How far from the base of the tower does the stone strike the ground? 2

(iv) Find the acute angle θ between the path of the stone and the horizontal at the moment the stone hits the ground. 3

Question 5 (8 marks)

(a) Using the fact that $(1+x)^4(1+x)^9 = (1+x)^{13}$, show that

$${}^4C_0\,{}^9C_4 + {}^4C_1\,{}^9C_3 + {}^4C_2\,{}^9C_2 + {}^4C_3\,{}^9C_1 + {}^4C_4\,{}^9C_0 = {}^{13}C_4$$ 3

(b) Consider the binomial expansion

$$(1+x)^n = 1 + {}^nC_1x + {}^nC_2x^2 + \ldots + {}^nC_rx^r + \ldots + {}^nC_nx^n$$

Prove the following,

(i) $1 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n = 2^n$ 1

(ii) $1 - \frac{1}{2}{}^nC_1 + \frac{1}{3}{}^nC_2 - \frac{1}{4}{}^nC_3 + \ldots + \frac{(-1)^n}{n+1}{}^nC_n = \frac{1}{n+1}$ 4

Girraween High - Year 12 (2006)

TASK 4 — (Ext 1) Mathematics

(Q1)

a) (i) $f(x) = -x^3+x^2+2x+2$

$f'(x) = -3x^2+2x+2$

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$x_{n+1} = x_n - \frac{(-x_n^3+x_n^2+2x_n+2)}{-3x_n^2+2x_n+2}$

If $x_0 = 2$.

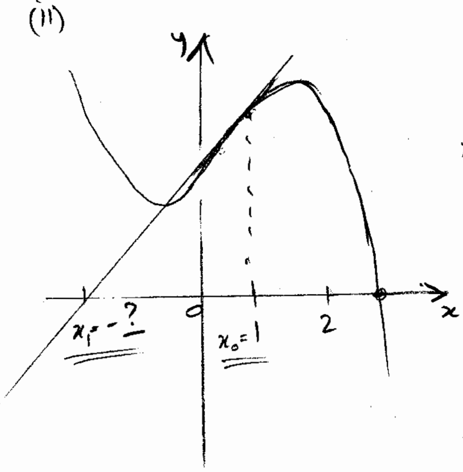
$x_1 = 2.\dot{3}$

$x_2 = 2.272\ldots$

$x_3 = 2.269\ldots$

∴ To 2 dec. pls. the zero is $\underline{2.27}$

(ii)



Since the tangent at $x=1$, intersects the x-axis, even further away from the zero than $x=1$. (Due to negative gradient) It is a bad initial approx.

b) $x = 5 - 3\sin\left(2t+\frac{\pi}{4}\right)$

i) Motion is between $x = 2$ and 8

ii) $T = \frac{2\pi}{n}$

(Period) $T = \frac{2\pi}{2} = \pi$ seconds.

iii) When $x=2$

$2 = 5 - 3\sin\left(2t+\frac{\pi}{4}\right)$

$-3 = -3\sin\left(2t+\frac{\pi}{4}\right)$

$\sin\left(2t+\frac{\pi}{4}\right) = 1$

$2t+\frac{\pi}{4} = \frac{\pi}{2}, \frac{5\pi}{2}$

$2t = \frac{\pi}{4}, \frac{9\pi}{4}$

$t = \frac{\pi}{8}, \frac{9\pi}{8}$ seconds.

(Q2) a) i) $v^2 = -5+6x-x^2$

$v^2 = -(5-6x+x^2)$

$v^2 = -(x-1)(x-5)$

when $v=0$: $0 = (x-1)(x-5)$

$x = 1, 5$ ∴ End Points of Motion.

ii) ∴ Centre of motion is $x = 3$

iii) Max speed at centre of motion:

$v^2 = -(3-1)(3-5)$

$v^2 = 4$

$v = 2$ m/s

iv) $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}(-5+6x-x^2)\right)$

$\ddot{x} = 3-x$.

b) i) $T = 20 + Ae^{-kt}$, but $Ae^{-kt} = T-20$

$\frac{dT}{dt} = -kAe^{-kt}$

$\frac{dT}{dt} = -k(T-20)$ ∴ is a solution

ii) $T = 20 + Ae^{-kt}$

when $t=0$, $T = 90°C$

$90 = 20 + Ae^{-k(0)}$

∴ $A = 70$

∴ $T = 20+70e^{-kt}$

when $t=10$, $T = 70°C$

$70 = 20+70e^{-k(10)}$

$50 = 70e^{-10k}$

$\frac{5}{7} = e^{-10k}$

$\log_e\frac{5}{7} = -10k$

$k = -\frac{1}{10}\log_e\frac{5}{7}$ ≑

when $t = 15$

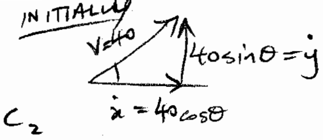
$T = 20+70e^{-\left(-\frac{1}{10}\log_e\frac{5}{7}\right)(15)}$

$T = 62.25\ °C$

$T ≑ 62°C$ after another 5 minutes

Q3

i) $\ddot{x} = 0$ $\qquad$ $\ddot{y} = -10$



$\dot{x} = C_1$ $\qquad$ $\dot{y} = -10t + C_2$

when $t=0$, $\dot{x} = 40\cos\theta$, $\dot{y} = 40\sin\theta$.

$\therefore C_1 = 40\cos\theta$ $\qquad$ $\therefore 40\sin\theta = -10(0) + C_2$

$C_2 = 40\sin\theta$

$\dot{x} = 40\cos\theta$ $\qquad$ $\dot{y} = -10t + 40\sin\theta$

$x = 40\cos\theta\, t + C_3$ $\qquad$ $y = -5t^2 + 40\sin\theta t + C_4$

when $t=0$, $x=0$, $y=50$.

$0 = 40\cos\theta(0) + C_3$ $\qquad$ $50 = -5(0)^2 + 40\sin\theta(0) + C_4$

$C_3 = 0$ $\qquad$ $\therefore C_4 = 50$

$\therefore x = 40\cos\theta\, t$ $\qquad$ $y = -5t^2 + 40\sin\theta\, t + 50$.

ii) If; $x = 200$, $y = 0$.

$200 = 40t\cos\theta$ — ① $\qquad$ $0 = -5t^2 + 40\sin\theta\, t + 50$ — ②

$\therefore t = \frac{5}{\cos\theta}$ — ③

sub ③ into ②

$$0 = -5\left(\frac{5}{\cos\theta}\right)^2 + 40\sin\theta\left(\frac{5}{\cos\theta}\right) + 50$$

$$0 = -125\sec^2\theta + 200\tan\theta + 50$$

$$5\sec^2\theta - 8\tan\theta - 2 = 0$$

$$5(1+\tan^2\theta) - 8\tan\theta - 2 = 0$$

$$5\tan^2\theta - 8\tan\theta + 3 = 0$$

$$(5\tan\theta - 3)(\tan\theta - 1) = 0$$

$5\tan\theta \quad -3$
$\tan\theta \quad -1$

$\therefore \tan\theta = 3/5$, $\quad \tan\theta = 1$

$\theta = 30°58'$ $\qquad$ $\theta = 45°$

$\theta \doteqdot 31°$ $\qquad$ $\theta = 45°$

b) i) $y = 3 + e^{-kt}$ $\Rightarrow \therefore e^{-kt} = y - 3$

$$\frac{dy}{dt} = -ke^{-kt}$$

$$\frac{dy}{dt} = -k(y-3)$$

ii) When $t=0$,

$y = 3 + e^{-k(0)}$

$y = 4$.

$\therefore$ If $\frac{dy}{dt} = 0.08$

$-0.08 = -k(4-3)$

$k = 0.08$

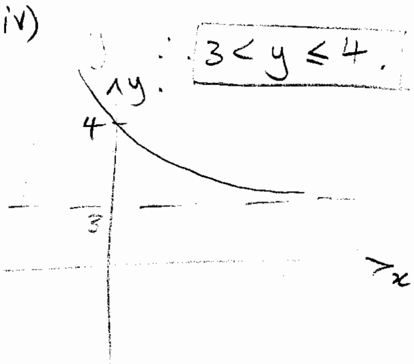
iii) $\therefore \frac{dy}{dt} = -0.08(y-3)$

$\therefore$ When $y = 3.5$.

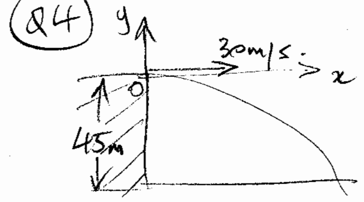
$\frac{dy}{dt} = -0.08(3.5-3)$

$= -0.04$ grams/hour.

iv) $\therefore 3 < y \leq 4$.



Q4



∴ INITIALLY

$\dot{x} = 30, \ \dot{y} = 0$

i) $\ddot{x} = 0$ $\qquad$ $\ddot{y} = -10$

$\dot{x} = C_1$ $\qquad$ $\dot{y} = -10t + C_2$

when $t=0$, $\dot{x} = 30$, $\dot{y} = 0$

$C_1 = 30$ $\qquad$ $0 = -10(0) + C_2$

$C_2 = 0$

$\dot{x} = 30$ $\qquad$ $\dot{y} = -10t$

$x = 30t + C_3$ $\qquad$ $y = -5t^2 + C_4$

when $t=0$, $x=0$, $y=0$

$0 = 30(0) + C_3$ $\qquad$ $0 = -5(0)^2 + C_4$

$C_3 = 0$ $\qquad$ $C_4 = 0$

$x = 30t$ $\qquad$ $y = -5t^2$

ii) when $y = -45$

$-45 = -5t^2$

$t^2 = 9$

$t = 3$ seconds

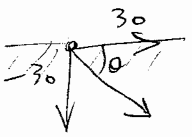
iii) When $t = 3$.

$x = 30(3)$

$x = 90$ m

iv) when $t = 3$

$\dot{x} = 30$ $\qquad$ $\dot{y} = -10(3)$



$\tan\theta = \frac{30}{30}$

$\theta = 45°$

Q5 $(1+x)^4(1+x)^9 = (1+x)^{13}$

a) $({}^4C_0 + {}^4C_1 x + {}^4C_2 x^2 + {}^4C_3 x^3 + {}^4C_4)$

$\times ({}^9C_0 + {}^9C_1 x + {}^9C_2 x^2 + {}^9C_3 x^3 + {}^9C_4 x^4 + \dots)$

$= ({}^{13}C_0 + \dots + {}^{13}C_4 x^4 + \dots {}^{13}C_{13} x^{13})$

By equating co-efficients of x^4.

${}^4C_0\,{}^9C_4 + {}^4C_1\,{}^9C_3 + {}^4C_2\,{}^9C_2 + {}^4C_3\,{}^9C_1 + {}^4C_4\,{}^9C_0 = {}^{13}C_4$

b)(i) $(1+x)^n = 1 + {}^nC_1 x + {}^nC_2 x^2 + \dots + {}^nC_n x^n$ (A)

sub $x=1$ $\quad (1+1)^n = 1 + {}^nC_1(1) + {}^nC_2(1)^2 + \dots + {}^nC_n(1)^n$

$2^n = 1 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_n$

(ii) By integrating (A)

$$\frac{(1+x)^{n+1}}{n+1} = x + {}^nC_1\frac{x^2}{2} + {}^nC_2\frac{x^3}{3} + \dots + {}^nC_n\frac{x^{n+1}}{n+1}$$

By Integrating (A) between $x=0$ & $x=-1$

sub $(x=0)$

$$\frac{1}{n+1} = 0 + 0 + 0 + 0 + \dots + 0. \quad (B)$$

sub $(x=-1)$

$$\frac{(0)^{n+1}}{(n+1)} = -1 + \frac{1}{2}{}^nC_1 - \frac{1}{3}{}^nC_2 + \dots + \frac{{}^nC_n(-1)^{n+1}}{n+1} \quad (C)$$

(B) − (C)

$$\frac{1}{n+1} = 1 - \frac{1}{2}{}^nC_1 + \frac{1}{3}{}^nC_2 - \frac{1}{4}{}^nC_3 + \dots + \frac{{}^nC_n(-1)^{n}}{n+1}$$