Girraween High School

Year 12 Mathematics (*Extension 1)*

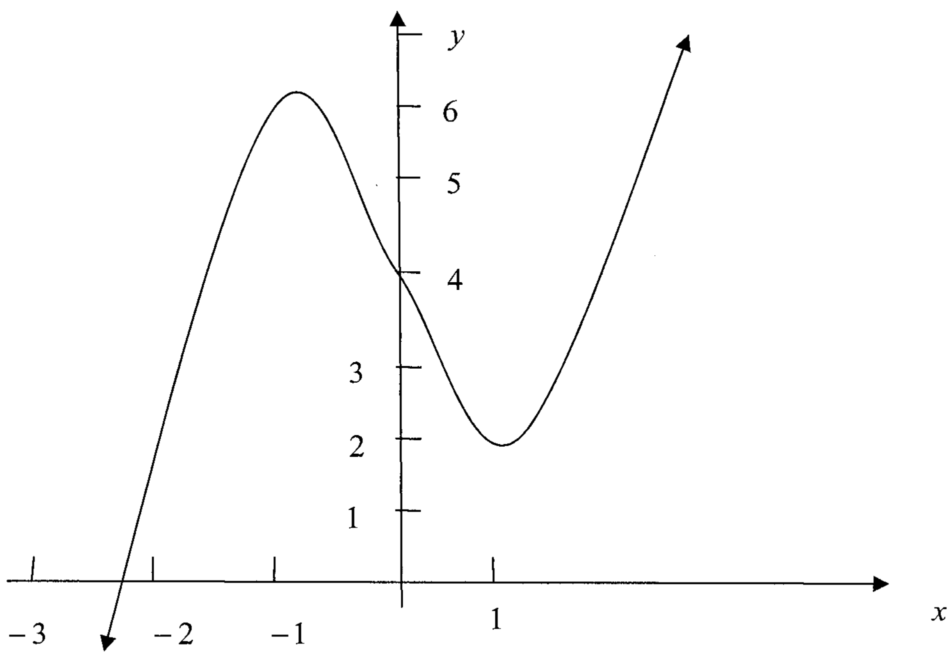
Task 4 2007

Time allowed – 90 minutes

DIRECTIONS TO CANDIDATES
*All necessary working should be shown.
*Marks will be deducted for careless or badly arranged work.
*Start each question on a *new* sheet of paper.
*Approved calculators may be used. A list of integrals will be provided.

Question 1 (*13 marks)* *Marks*

(a) Part of the graph of $f(x) = x^3 - 3x + 4$ is sketched below:



(i)Use Newton's method with $x_o = -2$ to approximate the zero of this graph which lies between $x = -2$ and $x = -3$ correct to 2 decimal places. 3

(ii) Use the graph above to explain why $x_0 = 0$ would be a bad initial approximation to the root between $x = -3$ and $x = -2$.
(no calculations are needed) 1

Question (1) (continued)

(b) According to Newton's law of cooling the temperature of my coffee is decreasing at a rate given by $\frac{dT}{dt} = -k(T-25)$ where T is the temperature in degrees celsius at time t minutes and $k>0$.

(i) Verify that $T = 25 + Ae^{-kt}$ is a solution to this equation. 1

(ii) If my coffee is initially at $80^0 C$ but cools to $70^0 C$ in 10 minutes find A and k. Hence find the temperature of the coffee after 20 minutes. 5

(iii) I like to drink my coffee once its temperature gets *below* $50^0 C$. After how many minutes will this happen? *(Answer to the nearest minute.)* 3

Question 2 *(21 marks)*

(a) A particle is moving in simple harmonic motion with equation $x = 3 + 6\sin(4t + \frac{\pi}{3})$ where x is the distance from a point 0 in metres and t is the time in seconds.

(i) Find the initial position of the particle. 1

(ii) Find the period of the motion and where the particle is oscillating between. 3.

(iii) Differentiate to find the velocity v and the acceleration a of the particle and verify that $a = -16(x-3)$. 3

(iv) Find the first time that the particle is at the origin and its acceleration at this time. 5

(b) The population of feral goats on an island is given by $P = 2000 - 500e^{-kt}$ where P is the population and t is the time in years from time $t = 0$.

(i) Verify that $\frac{dP}{dt} = k(2000 - P)$. 2

(ii) Find the initial population. 2

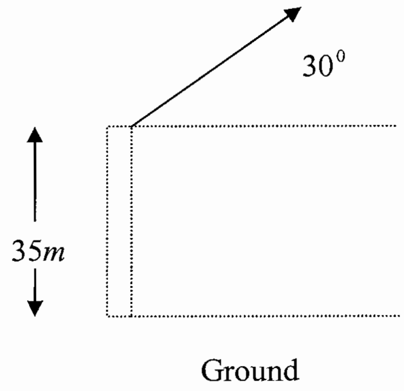
(iii) If the population is initially increasing at 100 goats per year find the value of k. 2

(iv) Find the rate of change in the population after 5 years. 2

(v) What is the upper limit to the population? 1

Question 3 *(23 marks)*

(a) A projectile is launched at $60m/s$ at an angle of 30^0 above the horizontal from a catapult which is located on top of a tower $35m$ high.



Assuming no air resistance and the acceleration due to gravity is $10m/s^2$ downwards:

(i) Show that the displacement equations at time t are given by

$$x = 60t\cos 30^0$$
$$y = -5t^2 + 60t\sin 30^0 + 35$$ 4

(ii) Find the greatest height achieved by the projectile. 3

(iii) Find when and where the projectile hits the ground. 3

(iv) Find the velocity and angle at which the projectile hits the ground. 4

Question 3 *(continued)*

(b) A particle is moving in simple harmonic motion with a period of $\frac{2\pi}{3}$ seconds.

(i) Find the expression for the acceleration in terms of x where x is the displacement in metres. 2

(ii) If the velocity of the particle is 12m/s when the particle is 3m from the centre of motion find an expression for v^2 in terms of x. 3

(iii) Find the amplitude of the motion and the greatest speed the particle reaches. 4

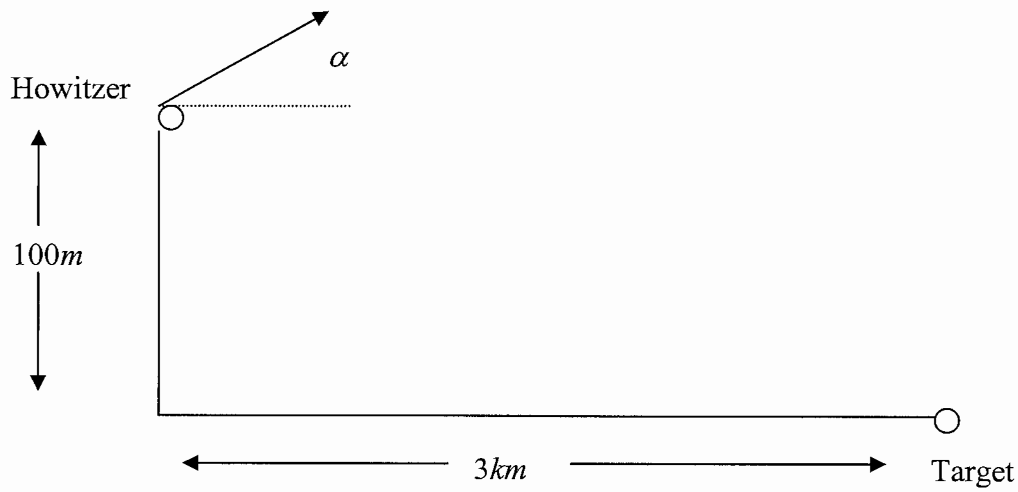
Question 4 *(6 marks)*

A howitzer located at the top of a cliff $100m$ high is aiming to hit a target located $3km$ from the base of the cliff. Given that the muzzle velocity (the speed at which the howitzer fires its shell) is $200m/s$, acceleration due to gravity is $10m/s^2$ downwards and it is aiming at an angle of α above the horizontal its position at time t is given by

$$x = 200t\cos\alpha$$

$$y = -5t^2 + 200t\sin\alpha + 100$$

Do *not* prove these!



(a) Show that $45\tan^2\alpha - 120\tan\alpha + 41 = 0$ 4

(b) Find the two possible angles α to the horizontal which the howitzer could fire its shell at in order to hit its target. 2

Question 5 ***(12 marks)***

(a) Use $(1+x)^3(1+x)^5 = (1+x)^8$ to prove that ${}^3C_1.{}^5C_5 + {}^3C_2.{}^5C_4 + {}^3C_3.{}^5C_3 = {}^8C_6$ 3

(b) (i) Prove that ${}^nC_0 + {}^nC_1 + {}^nC_2 + ... + {}^nC_n = 2^n$ 3

(ii) Prove that $\frac{{}^nC_0}{1} + \frac{{}^nC_1}{2} + \frac{{}^nC_2}{3} + ... + \frac{{}^nC_n}{n+1} = \frac{2^{n+1}-1}{n+1}$ 4

Solutions:

Q.(1)(a)(i) $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$ $\qquad f(x) = x^3 - 3x + 4$, $\quad f'(x) = 3x^2 - 3$

$= -2 - \frac{(-2)^3 - 3\times -2 + 4}{3\times(-2)^2 - 3}$

$= -2\frac{2}{9}$ or $-2.\dot{2}$

$x_2 = -2\frac{2}{9} - \frac{(-2\frac{2}{9})^3 - 3\times(-2\frac{2}{9}) + 4}{3\times(-2\frac{2}{9})^2 - 3}$

$\doteqdot -2.1962\ldots$ (3)

$x_3 = -2.1962\ldots - \frac{(-2.1962\ldots)^3 - 3\times(-2.1962\ldots) + 4}{3\times(-2.1962\ldots)^2 - 3}$

$= -2.1958$

The zero of this graph $= -2.20$ [to 2DP]

(ii) $x_0 = 0$ would be a bad initial approximation because there is a turning point between $x = 0$ & $x = -2$. Hence the tangent at $x = 0$ would intersect with the x axis further away from the root. (1)

(1)(b)(i) $T = 25 + Ae^{-kt}$

$\frac{dT}{dt} = -kAe^{-kt}$

$-k(T-25)$

$= -k(25 + Ae^{-kt} - 25)$ (1)

$= -kAe^{-kt}$

$=$ LHS.

(ii) $T = 80$ when $t = 0$

$80 = 25 + Ae^{0}$

$55 = A$.

$T = 25 + 55e^{-kt}$

$T = 70$ when $t = 10$

$70 = 25 + 55e^{-10k}$

$45 = 55e^{-10k}$

$\frac{45}{55} = e^{-10k}$

$\ln\left(\frac{45}{55}\right) = -10k$.

$\frac{-\ln\left(\frac{45}{55}\right)}{10} = k$. (5)

$\therefore k = -\frac{1}{10}\ln\left(\frac{45}{55}\right)$ $\qquad (\doteqdot -0.02006\ldots)$

(iii) Finding when $T = 50°C$:

$50 = 25 + 55e^{-kt}$, $\quad k = -\frac{1}{10}\ln\left(\frac{45}{55}\right)$ (3)

$25 = 55e^{-kt}$

$\frac{25}{55} = e^{-kt}$

$\ln\left(\frac{25}{55}\right) = -kt$

$\div -k$ $\qquad \div -k$, $\quad k = -\frac{1}{10}\ln\left(\frac{45}{55}\right)$

$39.29\ldots = t$

My coffee will be BELOW 50°C after 40 minutes.

Q.(2)(a)(i) $x = 3 + 6\sin\left(4t + \frac{\pi}{3}\right)$

when $t = 0$, $x = 3 + 6\sin\left(4\times 0 + \frac{\pi}{3}\right)$ (1)

$= 3 + 3\sqrt{2}$

(ii) Period of motion: $= \frac{2\pi}{4}$

$= \frac{\pi}{2}$.

Amplitude = 6. Centre of motion = 3. (3)
Hence particle is oscillating between
$x = -3$ & $x = 9$.

(iii) $v = 24\cos\left(4t + \frac{\pi}{3}\right)$ (3)

$a = -96\sin\left(4t + \frac{\pi}{3}\right)$

Note: $-16(x-3) = -16\left[3 + 6\sin\left(4t + \frac{\pi}{3}\right) - 3\right]$

$= -96\sin\left(4t + \frac{\pi}{3}\right)$

$= a$.

$\therefore \quad a = -16(x-3)$

(iv) Particle is at origin: $x = 0$

$3 + 6\sin\left(4t + \frac{\pi}{3}\right) = 0$ (5)

$6\sin\left(4t + \frac{\pi}{3}\right) = -3$.

$\sin\left(4t + \frac{\pi}{3}\right) = -\frac{1}{2}$ ~~Only first time wanted.~~

$4t + \frac{\pi}{3} = \frac{7\pi}{6}, \frac{11\pi}{6}, \ldots$

$4t = \frac{5\pi}{6}$.

$t = \frac{5\pi}{24}$.

PTO →

Q.(2)(a)(iv) [continued]:
Particle is first at origin after $\frac{5\pi}{24}$ seconds.

Acceleration at this time

$= -96\sin\left(4\times\frac{5\pi}{24} + \frac{\pi}{3}\right)$ OR $= -16(x-3)$

$= -16(0-3)$

$= -96\sin\frac{7\pi}{6}$ $= 48 m/s^2$.

$= 48 m/s^2$.

(b)(i) LHS: RHS: (2)

$\frac{dP}{dt} = 500ke^{-kt}$ $\quad k(2000-P)$

$= k\times\left(2000 - \left(2000 - 500e^{-kt}\right)\right)$

$= 500ke^{-kt}$

$=$ LHS.

$\therefore \frac{dP}{dt} = k(2000-P)$

(ii) Initial population: P when $t = 0$ (2)

$= 2000 - 500e^{0}$

$= 1500$ goats

(iii) $\frac{dP}{dt} = k(2000-P)$ (2)

When $t = 0$, $\frac{dP}{dt} = 100$ & $P = 1500$

$100 = k(2000-1500)$

$100 = 500k$

$0.2 = k$.

(iv) Rate of change in population after 5 years: (2)

$\frac{dP}{dt} = 500ke^{-kt}$

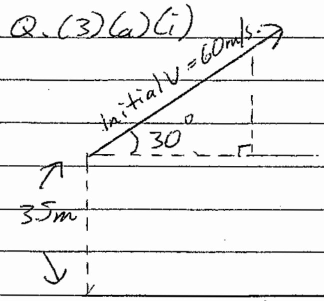
$= 500\times 0.2\times e^{-0.2\times 5}$

$\doteqdot 36.787$

$\doteqdot$ 37 goats per year

(v) As $t \to \infty$, $500e^{-kt} \to 0$. So population can never reach 2000. (1)

Q. (3)(a)(i)



$\ddot{x} = 0$

$\dot{x} = \int 0 . dt$

$= 0 + C$

As initial $\dot{x} = 60\cos 30°$

$C = 60\cos 30°$.

$\dot{x} = 60\cos 30°$

$x = \int 60\cos 30° . dt$

$= 60t\cos 30° + C$

As $x = 0$ when $t = 0$

$0 = 60(0)\cos 30° + C$

$0 = C$.

$\therefore x = 60t\cos 30°$

$\ddot{y} = -10$

$\dot{y} = \int -10 dt$

$\dot{y} = -10t + C$

As initial $\dot{y} = 60\sin 30°$,

$60\sin 30° = -10(0) + C$.

$60\sin 30° = C$

(4)

$\dot{y} = -10t + 60\sin 30°$

$y = \int -10t + 60\sin 30° . dt$

$= -5t^2 + 60t\sin 30° + C$

As $y = 35$ when $t = 0$

$35 = -5(0)^2 + 60(0)\sin 30° + C$

$35 = C$.

$y = -5t^2 + 60t\sin 30° + 35$

(ii) Greatest height achieved: $= y$ when $\dot{y} = 0$

$\dot{y} = -10t + 60\sin 30°$

If $\dot{y} = 0$

$-10t + 60\sin 30° = 0$

$60\sin 30° = 10t$

$6\sin 30° = t$

$3 = t$.

So max. height will be achieved after 3 seconds:

$y = -5t^2 + 60t\sin 30° + 35$ (3)

$= -5(3)^2 + 60(3)\sin 30° + 35$

$= 80m$.

Max. height achieved $= 80m$.

(iii) Projectile hits ground when $y = 0$

$-5t^2 + 60t\sin 30° + 35 = 0$

$-5t^2 + 30t + 35 = 0$

$t^2 - 6t - 7 = 0$

$(t-7)(t+1) = 0$.

↑ Can't be after −1 seconds.

$\therefore$ → Projectile hits ground after 7 seconds. (3)

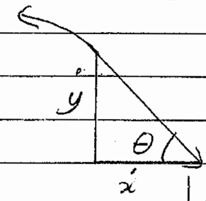
Where particle hits ground: $x = 60t\cos 30°$

$= 60 \times 7 \times \cos 30°$

$\doteq 363.7...$

→ Projectile hits ground approximately 364m horizontally from where launched.

(iv)



Projectile hits ground at angle θ.

$\tan\theta = \left|\frac{\dot{y}}{\dot{x}}\right|$

After 7 seconds, $\dot{y} = -10t + 60\sin 30°$

$\doteq -40$.

$\dot{x} = 60\cos 30°$

$(\doteq 51.96..)$

$\tan\theta = \left|\frac{\dot{y}}{\dot{x}}\right|$

$\doteq \left|\frac{40}{51.96..}\right|$

$= 0.769..$

$\theta \doteq 37°35'$.

(4)

Speed when hits ground:

$= \sqrt{(\dot{x})^2 + (\dot{y})^2}$

$= \sqrt{(51.96..)^2 + (-40)^2}$

$= \sqrt{4300}$

$\doteq 65.57..m/s$.

The projectile hits the ground at approximately 65.6 m/s at an angle of $37°35'$.

Q. (3)(b) (i) Period $= \frac{2\pi}{n} = \frac{2\pi}{3}$

Hence $n = 3$.

As $\ddot{x} = -n^2 x$ for SHM

$\ddot{x} = -3^2 x$

$\ddot{x} = -9x.$ (2)

(ii) $\ddot{x} = a = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = -9x.$

$\therefore \frac{1}{2}v^2 = \int -9x \, dx$

$\frac{1}{2}v^2 = -\frac{9}{2}x^2 + C_1 \quad C = 2C_1.$

$v^2 = -9x^2 + C.$

As $v = 12$ when $x = 3$,

$12^2 = -9(3)^2 + C$ (3)

$144 = -81 + C$

$225 = C.$

$v^2 = 225 - 9x^2.$

or $v^2 = 9(25 - x^2)$

(iii) Amplitude of motion: Will reach limits where $v = 0$

$v^2 = 9(25 - x^2)$

$0 = 9(25 - x^2)$ (4)

$0 = 25 - x^2$

$x = \pm 5.$

Amplitude of motion $= 5.$

Particle will reach greatest speed where $a = 0$.

As $a = -9x = 0$, $x = 0$.

Greatest speed: $v^2 = 9(25 - x^2)$

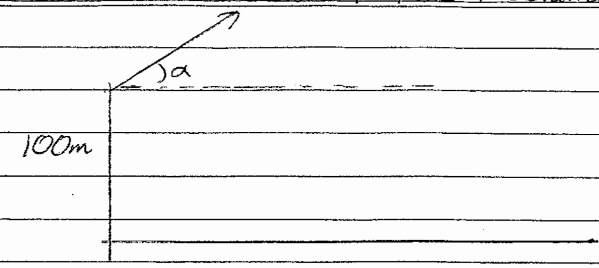
$= 9(25 - 0^2)$

$v^2 = 225.$

$v = \pm 15.$

Greatest speed of particle $= 15$ m/s.

Q. (4)



(a) Range $= 3$ km.

So $x = 3000$ when $y = 0$.

Time when $x = 3000$,

$200t\cos\alpha = 3000$

$t = \frac{3000}{200\cos\alpha}$

$= \frac{15}{\cos\alpha}.$ (4)

As $y = 0$ when $t = \frac{15}{\cos\alpha}$,

$-5\left(\frac{15}{\cos\alpha}\right)^2 + 200 \times \frac{15}{\cos\alpha} \times \sin\alpha + 100 = 0$

$-5 \times \frac{225}{\cos^2\alpha} + 3000\frac{\sin\alpha}{\cos\alpha} + 100 = 0$

$-1125(1 + \tan^2\alpha) + 3000\tan\alpha + 100 = 0$ as $\frac{1}{\cos^2\alpha} = 1 + \tan^2\alpha$, $\frac{\sin\alpha}{\cos\alpha} = \tan\alpha$

$45 + 45\tan^2\alpha - 120\tan\alpha - 4 = 0$

$45\tan^2\alpha - 120\tan\alpha + 41 = 0.$

(b) Possible angles: $\tan\alpha = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ (2)

$= \frac{120 \pm \sqrt{(-120)^2 - 4 \times 45 \times 41}}{2 \times 45}$

$= \frac{120 \pm \sqrt{7020}}{90}.$

$\tan\alpha = 0.407\ldots$ $\quad \tan\alpha = 2.264\ldots$

$\alpha = 21°55'.$ $\quad \alpha = 66°10'.$

The projectile would hit its target if fired at angles of $21°55'$ or $66°10'$

Q. (5)(a) $(1+x)^3(1+x)^5 = (1+x)^8$

$$\left({}^3C_0 + {}^3C_1 x + {}^3C_2 x^2 + {}^3C_3 x^3\right)\left({}^5C_0 + {}^5C_1 x + {}^5C_2 x^2 + {}^5C_3 x^3 + {}^5C_4 x^4 + {}^5C_5 x^5\right)$$

$$= \left({}^8C_0 + {}^8C_1 x + {}^8C_2 x^2 + {}^8C_3 x^3 + {}^8C_4 x^4 + {}^8C_5 x^5 + {}^8C_6 x^6 + {}^8C_7 x^7 + {}^8C_8 x^8\right)$$

Taking co-efficients of x^6 in expansion: (3)

$${}^3C_1 x \times {}^5C_5 x^5 + {}^3C_2 x^2 \times {}^5C_4 x^4 + {}^3C_3 x^3 \times {}^5C_3 x^3 = {}^8C_6 x^6$$

$$\therefore {}^3C_1 \times {}^5C_5 + {}^3C_2 \times {}^5C_4 + {}^3C_3 \times {}^5C_3 = {}^8C_6.$$

(b)(i) $(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \dots + {}^nC_n x^n$

(3)

Substituting in $x = 1$,

$$(1+1)^n = {}^nC_0 + {}^nC_1 \times 1 + {}^nC_2 \times 1^2 + \dots + {}^nC_n \times 1^n$$

$$2^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_n.$$

(ii) $\int (1+x)^n \, dx = \int {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \dots + {}^nC_n x^n \, dx$

Putting one C in instead of C_1 & C_2.

$$\frac{(1+x)^{n+1}}{n+1} + C = {}^nC_0 x + \frac{{}^nC_1 x^2}{2} + \frac{{}^nC_2 x^3}{3} + \dots + \frac{{}^nC_n x^{n+1}}{n+1}$$

Letting $x = 0$ in order to find C: (4)

$$\frac{(1+0)^{n+1}}{n+1} + C = {}^nC_0 \times 0 + \frac{{}^nC_1 \times 0^2}{2} + \dots$$

$$\frac{1}{n+1} + C = 0$$

$$\therefore C = -\frac{1}{n+1}.$$

Hence $\dfrac{(1+x)^{n+1}}{n+1} - \dfrac{1}{n+1} = {}^nC_0 x + \dfrac{{}^nC_1}{2} x^2 + \dfrac{{}^nC_2}{3} x^3 + \dots + \dfrac{{}^nC_n}{n+1} x^{n+1}$.

Sub. in $x = 1$,

$$\frac{(1+1)^{n+1} - 1}{n+1} = {}^nC_0 (1) + \frac{{}^nC_1}{2}(1)^2 + \frac{{}^nC_2}{3}(1)^3 + \dots + \frac{{}^nC_n}{n+1}(1)^{n+1}$$

$$\frac{2^{n+1} - 1}{n+1} = {}^nC_0 + \frac{{}^nC_1}{2} + \frac{{}^nC_2}{3} + \dots + \frac{{}^nC_n}{n+1}.$$