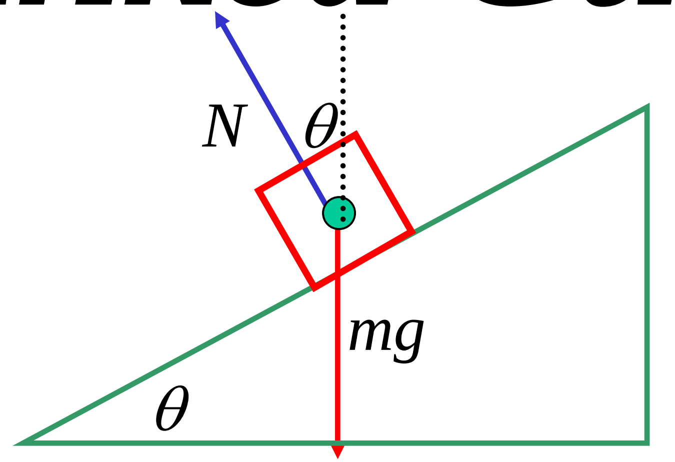
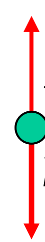
Motion Around A Banked Curve



$$\text{horizontal forces} = \frac{mv^2}{r}$$

$$N \sin\theta = \frac{mv^2}{r}$$

$$\text{vertical forces} = 0$$



$$N \cos\theta - mg = 0$$

$$N \cos\theta = mg$$

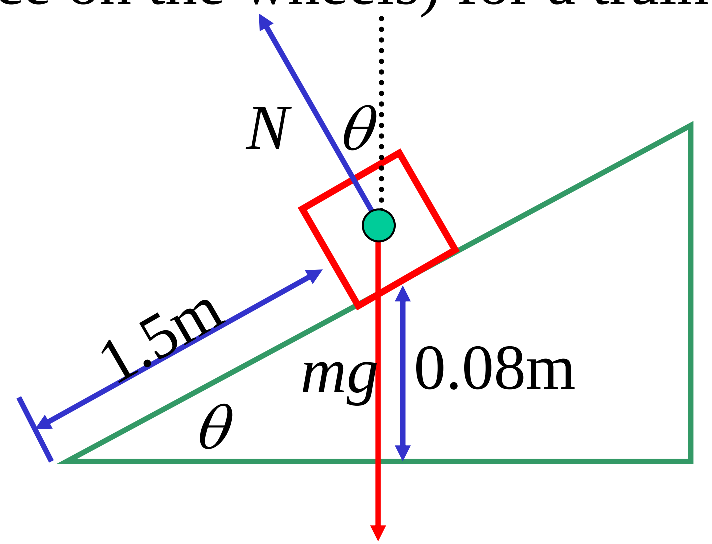
$$\tan\theta = \frac{mv^2}{r} \times \frac{1}{mg}$$

$$= \frac{v^2}{rg}$$

e.g. (i) A railway line has been constructed around a circular curve of radius 400m.

The distance between the rails is 1.5m and the outside rail is 0.08m above the inside rail.

Find the most favourable speed (the speed that eliminates a sideways force on the wheels) for a train on this curve.



$$\text{horizontal forces} = \frac{mv^2}{r}$$

$$N \sin\theta = \frac{mv^2}{r}$$

$$\tan\theta = \frac{mv^2}{r} \times \frac{1}{mg}$$

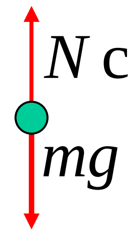
$$= \frac{v^2}{rg}$$

$$\text{But } \sin\theta = \frac{0.08}{1.5}$$

$$= \frac{4}{75}$$

$$\therefore \tan\theta = \frac{4}{\sqrt{5609}}$$

$$\text{vertical forces} = 0$$



$$N \cos\theta - mg = 0$$

$$N \cos\theta = mg$$

$$\therefore \frac{v^2}{(400)(9.8)} = \frac{4}{\sqrt{5609}}$$

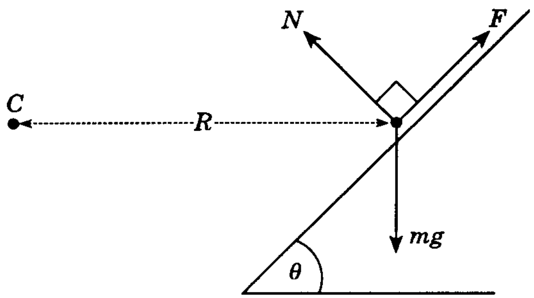
$$v^2 = \frac{4(400)(9.8)}{\sqrt{5609}}$$

$$v = 14.47\,\text{m/s}$$

$$v = 52\,\text{km/h}$$

(*ii*) (1995)

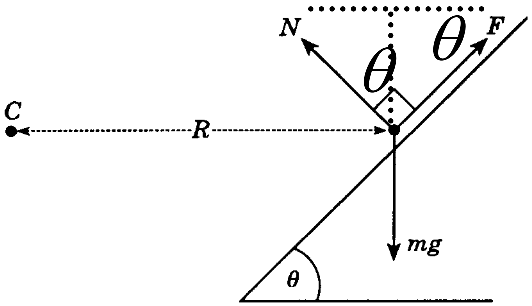
A particle of mass m travels at a constant speed v round a circular track of radius R, centre C. The track is banked inwards at an angle θ, and the particle does not move up or down the bank.



The reaction exerted by the track on the particle has a normal component N, and a component F due to friction, directed up or down the bank. The force F lies in the range $-\mu N$ to μN, where μ is a positive constant and N is the normal component; the sign of F is positive when F is directed up the bank.

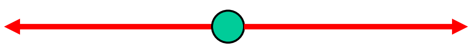
The acceleration due to gravity is g. The acceleration related to the circular motion is of magnitude $\frac{v^2}{R}$ and is directed towards the centre of the track.

a) By resolving forces horizontally and vertically, show that;



$$\frac{v^2}{Rg} = \frac{N\sin\theta - F\cos\theta}{N\cos\theta + F\sin\theta}$$

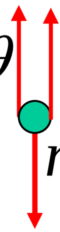
$$\text{horizontal forces} = \frac{mv^2}{R}$$



$N\sin\theta$ $F\cos\theta$

$$N\sin\theta - F\cos\theta = \frac{mv^2}{R}$$

$$\text{vertical forces} = 0$$



$N\cos\theta$ $F\sin\theta$ mg

$$N\cos\theta + F\sin\theta - mg = 0$$

$$N\cos\theta + F\sin\theta = mg$$

$$\frac{v^2}{Rg} = \frac{mv^2}{R} \times \frac{1}{mg}$$

$$\frac{v^2}{Rg} = \frac{N\sin\theta - F\cos\theta}{N\cos\theta + F\sin\theta}$$

b) Show that the maximum speed v_{max} at which the particle can travel without slipping up the track is given by; $\dfrac{v_{max}^2}{Rg} = \dfrac{\tan\theta + \mu}{1 - \mu\tan\theta}$

[You may suppose that $\mu\tan\theta < 1$]

As it is the friction that resists the particle moving up or down the slope, then if the particle is not slipping up, then friction must be at a maximum in the opposite direction, i.e. $F = -\mu N$

$$\frac{v_{max}^2}{Rg} = \frac{N\sin\theta + \mu N\cos\theta}{N\cos\theta - \mu N\sin\theta}$$

$$= \frac{\dfrac{\sin\theta}{\cos\theta} + \mu\dfrac{\cos\theta}{\cos\theta}}{\dfrac{\cos\theta}{\cos\theta} - \mu\dfrac{\sin\theta}{\cos\theta}}$$

$$= \frac{\tan\theta + \mu}{1 - \mu\tan\theta}$$

c) Show that if $\mu \geq \tan\theta$, then the particle will not slide down the track, regardless of its speed.

$v_{\min}$ is the minimum speed the particle can travel without sliding down the track. In this case friction must be a maximum up the slope i.e. $F = \mu N$

$$\frac{v_{\min}^2}{Rg} = \frac{\tan\theta - \mu}{1 + \mu\tan\theta}$$

If $\mu \geq \tan\theta$;

$$\therefore \frac{v_{\min}^2}{Rg} \leq 0$$

$$v_{\min}^2 = 0$$

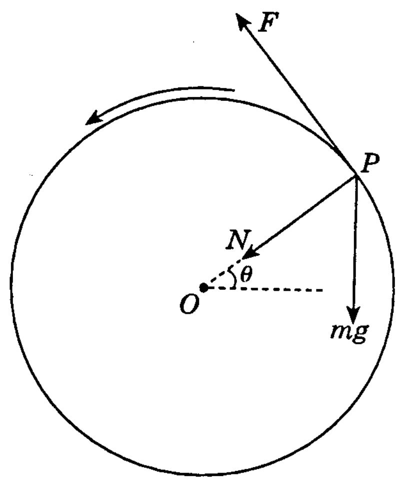
$$v_{\min} = 0$$

Thus if the minimum velocity the particle can travel without sliding down the track is 0, the particle will not slide down the track, regardless of its speed.

(*iii*) (1996)

A circular drum is rotating with uniform angular velocity round a horizontal axis. A particle P is rotating in a vertical circle, without slipping, on the inside of the drum.

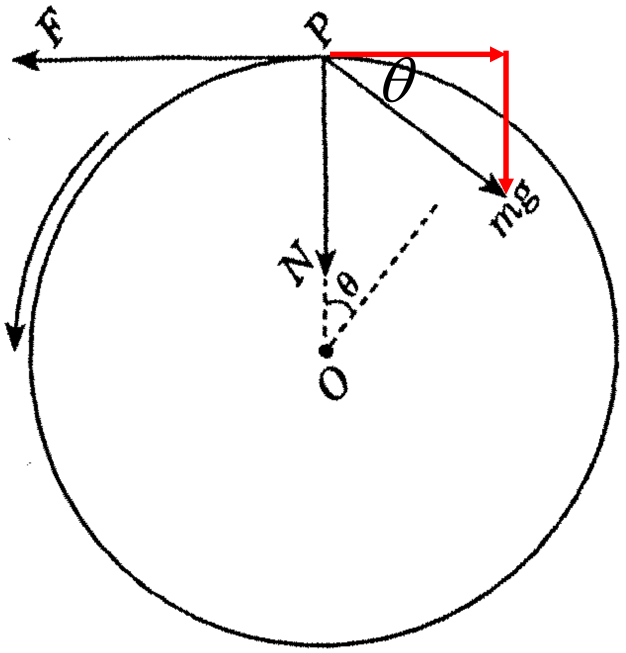
The radius of the drum is r metres and its angular velocity is ω radians/second. Acceleration due to gravity is g metres/second^2, and the mass of P is m kilograms.



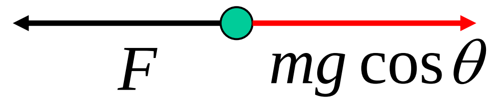
The centre of the drum is O, and OP makes an angle of θ to the horizontal.

The drum exerts a normal force N on P, as well as frictional force F, acting tangentially to the drum, as shown in the diagram.

By resolving forces perpendicular to and parallel to OP, find an expression for $\frac{F}{N}$ in terms of the data.



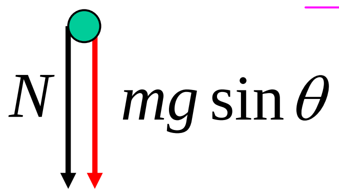
$$\text{forces} \perp OP = 0$$



$$mg\cos\theta - F = 0$$

$$F = mg\cos\theta$$

$$\text{forces} \parallel OP = mr\omega^2$$



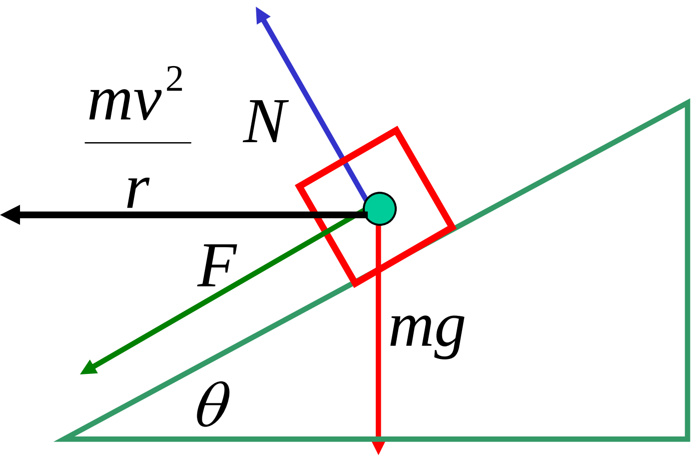
$$N + mg\sin\theta = mr\omega^2$$

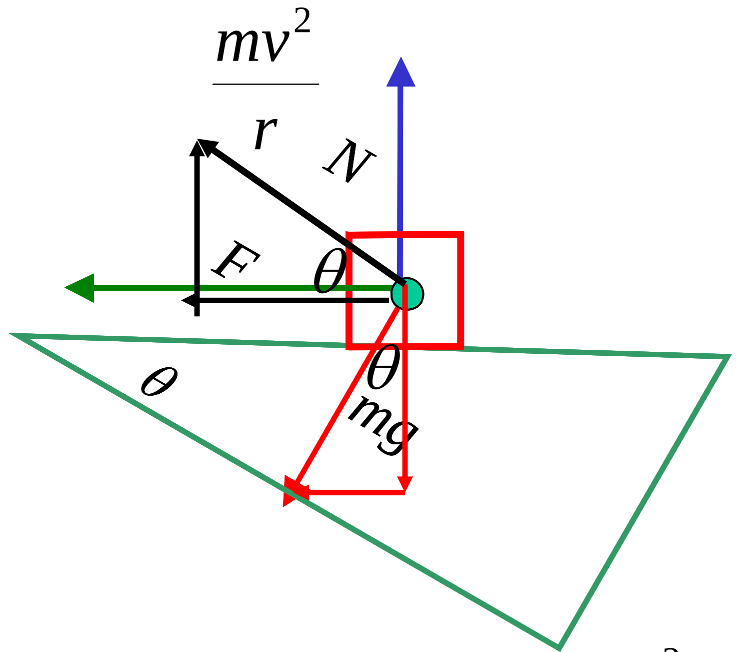
$$N = mr\omega^2 - mg\sin\theta$$

$$\frac{F}{N} = \frac{mg\cos\theta}{mr\omega^2 - mg\sin\theta}$$

$$\frac{F}{N} = \frac{g\cos\theta}{r\omega^2 - g\sin\theta}$$

Resolving Forces Along The Bank



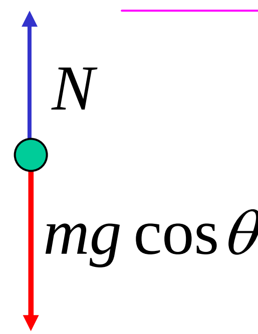


$$\text{forces along the bank} = \frac{mv^2}{r}\cos\theta$$

$$\text{forces} \perp \text{the bank} = \frac{mv^2}{r}\sin\theta$$

$$F + mg\sin\theta = \frac{mv^2}{r}\cos\theta$$

$$F = \frac{mv^2}{r}\cos\theta - mg\sin\theta$$



$$N - mg\cos\theta = \frac{mv^2}{r}\sin\theta$$

$$N = \frac{mv^2}{r}\sin\theta + mg\cos\theta$$

Exercise 9D; all