GIRRAWEEN HIGH SCHOOL
MATHEMATICS

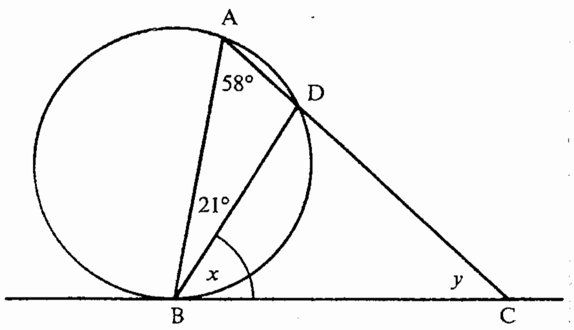
Year 12 Extension 1 Task 1 November 27th 2006

Time Allowed: 90 minutes

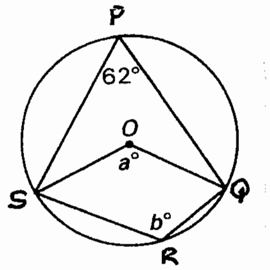
Instructions: Write all your answers on your own paper.
Start each question on a new page.
Show all necessary working.
Marks may be deducted for careless or badly arranged work.

Question1 (12 marks)

a) Find the values of x and y, giving reasons. **4**



b) Find the values of a and b, giving reasons. **4**



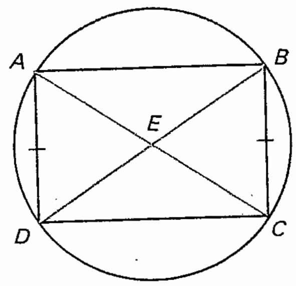
c) Prove by mathematical induction that

$$4+10+18+\ldots+n(n+3)=\frac{1}{3}n(n+1)(n+5)$$ **4**

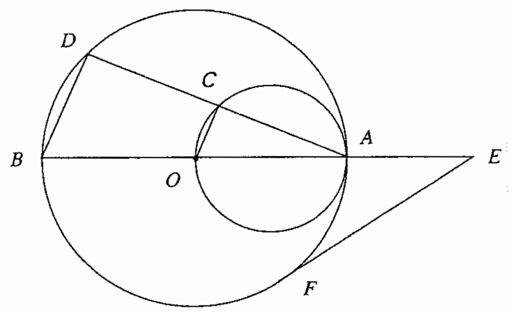
for all positive integers n

Question 2 (16 marks)

a) ABCD is a cyclic quadrilateral. AD = BC. The diagonals meet at E.
Prove that AB is parallel to DC. **6**



b) Given that AE = OA, O is the centre of the larger circle and EF is a tangent to the circle,



(i) prove that OC is parallel to BD **3**

(ii) show that EF = $\sqrt{3}$ AE **3**

(iii) prove that ΔOCA is similar to ΔBDA **3**

(iv) if OB = 8cm and BD = 7cm, find the length of OC. **1**

Question 3 (14 marks)

a) Write the expansion of $(1+2x)^5$. **2**

b) Find the 4th term in the expansion of $(x^3 - \frac{2}{x})^8$ **3**

c) Find the term independent of x in the expansion of $(3x + \frac{2}{x^2})^9$ **3**

d) If $(1+\sqrt{3})^4 = a + b\sqrt{3}$, find the values of a and b. **3**

e) Find the coefficient of x^6 in the expansion of $(2x-3)^{20}$. **3**
Leave your answer in unexpanded form.

Question 4 (13 marks)

a) For the expansion of $(2+5x)^{12}$,

(i) derive the ratio for $\frac{T_{k+1}}{T_k}$ **3**

(ii) hence, find the largest coefficient. **2**

(You may leave your answer in the form ${}^{12}C_k 2^a 5^b$)

b) Prove by mathematical induction that $11^n - 1$ is divisible by 10 for all $n \geq 1$. **4**

c) Prove by mathematical induction that **4**

$$3^n \geq 2n+1 \quad \text{for} \quad n \geq 1$$

SOLUTIONS

Ext 1 HSC Task 1 2006

Question 1 (12 marks)

a) $x = 58°$ (∠ in alternate segment) ②

$y = 180° - (58+21+58)$

$= 43°$ (∠ sum of Δ) ②

b) $a = 124°$ (∠ at the centre is twice ∠ at circum. on same arc) ②

$b = 118°$ (opp. ∠s of cycl. quad suppl.) ②

c) $4 + 10 + 18 + \dots + n(n+3) = \frac{1}{3}n(n+1)(n+5)$

Show result is true for $n=1$

LHS $= 1(1+3) = 4$

RHS $= \frac{1}{3}(1)(2)(6) = \frac{12}{3} = 4$

∴ true for $n=1$

Assume true for $n=k$

ie, $4+10+18+\dots+k(k+3) = \frac{1}{3}k(k+1)(k+5)$

Prove true for $n=k+1$

ie, $4+10+18+\dots k(k+3)+(k+1)(k+4) = \frac{1}{3}(k+1)(k+2)(k+6)$

LHS $= 4+10+18+\dots+k(k+3)+(k+1)(k+4)$

$= \frac{1}{3}k(k+1)(k+5) + (k+1)(k+4)$

$= \frac{1}{3}(k+1)[k(k+5) + 3(k+4)]$

$= \frac{1}{3}(k+1)(k^2+8k+12)$

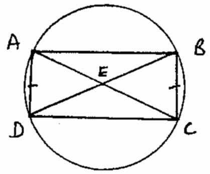
$= \frac{1}{3}(k+1)(k+2)(k+6) =$ RHS

Hence, if the result is true for $n=k$, then it is true for $n=k+1$

Since the result is true for $n=1$, then it is true for $n=1+1=2$, and if true for $n=2$, also true for $n=2+1=3$ and so on for all positive integral values of n ④

Question 2 (16 marks)

a)



In Δs ADE and BCE

∠ADE = ∠BCE (∠s in same segment equ

∠AED = ∠BEC (vert. opp ∠s)

AD = BC (given)

∴ ΔADE ≡ ΔBCE (AAS)

∴ AE = CE (matching sides of cong. Δs)

DE = BE (matching sides of cong. Δs)

∴ ABCD is a parallelogram (diagonals bisect each other)

∴ AB // DC (opp. sides of parallelogram) ⑥

b) OA is the diameter of smaller circle (BE passes through point of contact)

i) ∠OCA = 90° (∠ in semi-circle)

∠ODC = 90° (∠ in semi-circle)

∴ OC // BD (corresp. ∠s equal) ③

ii) AE = BO + OA + AE

$EF^2 = BE \cdot EA$ (square of tange equal to product of intercepts)

$= 3AE \cdot EA$

$= 3AE^2$

∴ $EF = \sqrt{3}\,AE$ ③

iii) OB = 8cm,)7cm

In Δs OCA & BDA

∠A is common

∠BDC = ∠OCA (Corresp. ∠s, OC//BD)

∴ ΔOCA ||| ΔBDA (equiangular) ③

iv) ∴ $\frac{BD}{OD} = \frac{BA}{OA}$ (ratio of matching sides of similar Δs)

$\frac{7}{OC} = \frac{16}{8}$

∴ OC = 3.5 cm ①

Question 3 (14 marks)

a) $(1+2x)^5$

$= 1+5(2x)+10(2x)^2+10(2x)^3+5(2x)^4+(2x)^5$

$= 1+10x+40x^2+80x^3+80x^4+32x^5$ ②

b) $T_4 = {}^8C_3\,(x^3)^5\left(\frac{-2}{x}\right)^3$

$= 56\,x^{15}\left(\frac{-8}{x^3}\right)$

$= -448\,x^{12}$ ③

c) $T_{k+1} = {}^9C_k\,(3x)^{9-k}\left(\frac{2}{x^2}\right)^k$

$= {}^9C_k\,(3^{9-k})(x^{9-k})\frac{2^k}{x^{2k}}$

$= {}^9C_k\,3^{9-k}\,2^k\,x^{9-3k}$

For term independent of x

$9-3k=0$

$k=3$

$T_4 = {}^9C_3\,3^6 \cdot 2^3$

$= 489\,888$ ③

d) $(1+\sqrt{3})^4 = 1+4\sqrt{3}+6(\sqrt{3})^2+4(\sqrt{3})^3+(\sqrt{3})^4$

$= 1+4\sqrt{3}+18+12\sqrt{3}+9$

$= 28+16\sqrt{3}$

∴ $a = 28$, $b = 16$ ③

e) $T_{15} = {}^{20}C_{14}\,(2x)^6(-3)^{14}$

∴ coefficient of x^6

$= {}^{20}C_{14}\,2^6 \cdot 3^{14}$ ③

Question 4

a) i) $(2+5x)^{12}$

$T_{k+1} = {}^{12}C_k\,2^{12-k}(5x)^k$

$T_k = {}^{12}C_{k-1}\,2^{13-k}(5x)^{k-1}$

$\frac{T_{k+1}}{T_k} = \frac{{}^{12}C_k\,2^{12-k}(5x)^k}{{}^{12}C_{k-1}\,2^{13-k}(5x)^{k-1}}$

$= \frac{12!}{(12-k)!\,k!} \times \frac{(13-k)!\,(k-1)!}{12!} \times \frac{5x}{2}$

$= \frac{13-k}{k} \times \frac{5x}{2}$ ③

ii) $\frac{T_{k+1}}{T_k} = \frac{5(13-k)}{2k}$

For greatest coefficient

$\frac{5(13-k)}{2k} > 1$

$65-5k > 2k$

$k < 9.29$

∴ $k=9$

Greatest coeff $= {}^{12}C_9\,2^3 5^9$ ②

b) $11^n - 1$ is divisible by 10 for $n \geq 1$

Show true for $n=1$

$11^1 - 1 = 10$ divisible by 10

$\therefore$ true for $n=1$

Assume true for $n=k$

ie. $11^k - 1 = 10p$ where p is an integer

Prove true for $n=k+1$

$11^{k+1} - 1 = 10q$ where q is an integer

$LHS = 11^{k+1} - 1$

$= 11 \cdot 11^k - 1$

$= 11(10p+1) - 1$

$= 110p + 11 - 1$

$= 110p + 10$

$= 10(11p+1) = 10q$ where $q = 11p+1$

$= RHS$

$\therefore$ divisible by 10

Hence, if true for $n=k$, then true for $n=k+1$.

Since true for $n=1$, then it is true for $n=2$ and if true for $n=2$, also true for $n=2+1=3$ and so on for all $n \geq 1$

(4)

c) $3^n \geq 2n+1$ for ≥ 1

Show true for $n=1$

$LHS = 3^1 = 3$

$RHS = 2(1)+1 = 3$

$\therefore$ true for $n=1$

Assume true for $n=k$

ie. $3^k \geq 2k+1$

Prove true for $n=k+1$

ie $3^{k+1} \geq 2(k+1)+1$

$\geq 2k+3$

$LHS = 3^{k+1} = 3 \cdot 3^k$

$\geq 3(2k+1)$

$\geq 6k+3$

$\geq 2k+3$

$\therefore 3^{k+1} \geq 2(k+1)+1$

Hence, if true for $n=k$, then true for $n=k+1$

Since true for $n=1$, then it is true for $n=2$ and if true for $n=2$, also true for $n=2+1=3$ and so on for all $n \geq 1$

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