GIRRAWEEN HIGH SCHOOL

MATHEMATICS

Year 12 Extension 1 Task 1 December 7^{th}, 2007

Time Allowed: 90 minutes

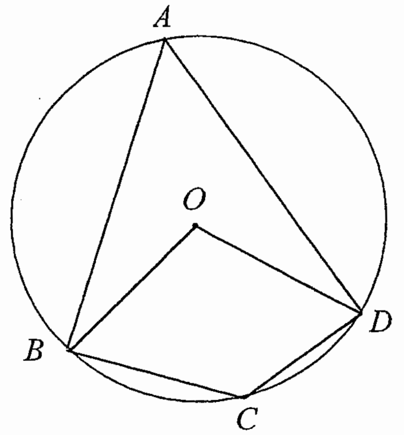
Instructions: Write all your answers on your own paper.
Start each question on a new piece of paper.
Show all necessary working.
Marks may be deducted for careless or badly arranged work.

Question 1 (14 marks)

a) Write the expansion of $(2+x)^4$. **3**

b) Expand and simplify: $(x-\frac{1}{x})^5$. **3**

c)

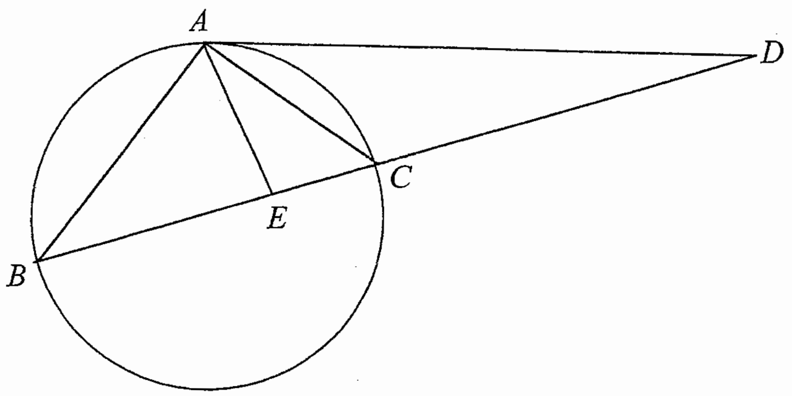


$ABCD$ is a quadrilateral inscribed in a circle with centre O. $\angle DAB = 36°$.
Find, giving reasons:

(i) the size of $\angle DOB$ **2**

(ii) the size of $\angle BCD$ **2**

d)



BC is a diameter of a circle. The tangent to the circle at A meets BC produced at D.
E is the point on BC such that AC bisects $\angle DAE$.
Show that $AE \perp BC$. **4**

Question 2 (15 marks)

a) Find the 4th term in the expansion of $(x^2 + \frac{1}{x})^8$ **3**

b) Find the middle term in the expansion of $(x^2 - \frac{1}{x})^{16}$ **3**

c) Find the term independent of x in the expansion of $(2x^3 + \frac{1}{x})^{12}$ **3**

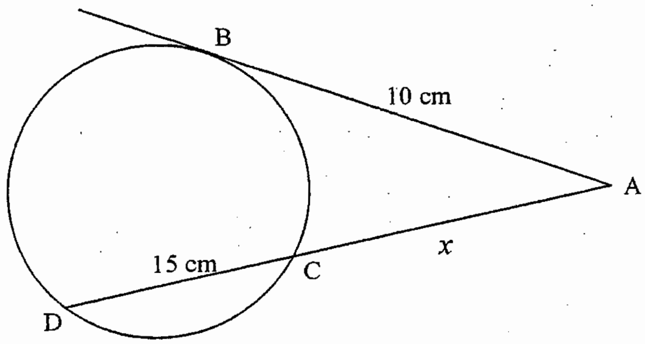
d) Prove by mathematical induction;

$$\frac{1}{1\times 4} + \frac{1}{4\times 7} + \frac{1}{7\times 10} + \ldots + \frac{1}{(3n-2)(3n+1)} = \frac{n}{3n+1}$$

for all positive integers $n \geq 1$ **6**

Question 3 (12 marks)

a)



$AB = 10\text{cm}$, $CD = 15\text{cm}$ and $AC = x$. Find the value of x. **3**

b) For the expansion of $(2+3x)^{10}$

(i) Show that the ratio of the coefficients of consecutive terms

$$T_{k+1} : T_k \quad \text{is} \quad \frac{3(11-k)}{2k}$$ **3**

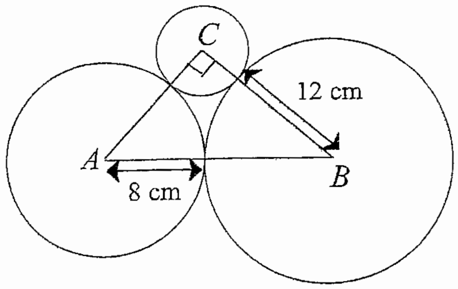
(ii) Hence, or otherwise find the largest coefficient. **2**
(You may leave the answer in the form ${}^{10}C_k 2^a 3^b$)

c) Prove by mathematical induction that $2 + 7^n$ is divisible by 3 for all positive integers $n \geq 1$ **4**

Please turn over for question 4........................

Question 4 (15 marks)

a)



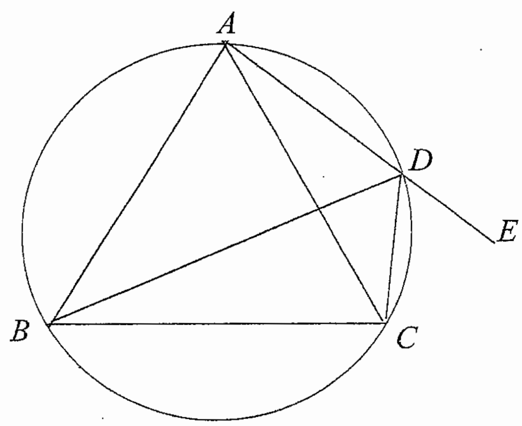
Three circles with centres A, B and C touch externally in pairs with $\angle BCA = 90°$.
The circles with centres A and B have radii 8cm and 12cm respectively.

(i) If the circle with centre C has radius x cm, show that

$$x^2 + 20x - 96 = 0$$ **2**

(ii) Hence, find the radius of the circle with centre C. **2**

b)



ABC is a triangle in which $BC = AC$. D is a point on the minor arc AC of the circle passing through A, B and C. AD is produced to E.

(i) Copy the diagram.
(ii) Give a reason why $\angle CDE = \angle ABC$. **1**
(iii) Show that DC bisects $\angle BDE$. **3**

c) Find the coefficient of x^2 in the expansion of $(1+x)^2(1+x)^8$ **3**

d) Prove by mathematical induction:

$5^n \geq 3n + 7$ for all positive integers $n > 1$ **4**

YEAR 12 EXT I TASK I 2007 – SOLUTIONS

Q1. a) $(2+x)^4$

$= 2^4 + 4\cdot 2^3 x + 6\cdot 2^2 x^2 + 4\cdot 2x^3 + x^4$

$= 2^4 + 32x + 24x^2 + 8x^3 + x^4$ (3)

b) $(x-\frac{1}{x})^5$

$= x^5 + 5x^4(\frac{-1}{x}) + 10x^3(\frac{-1}{x})^2 + 10x^2(\frac{-1}{x})^3 + 5x(\frac{-1}{x})^4 + (\frac{-1}{x})^5$

$= x^5 - 5x^3 + 10x - \frac{10}{x} + \frac{5}{x^3} - \frac{1}{x^5}$ (3)

c) i) $\angle DOB = 72^\circ$ ($\angle$ at the centre is twice $\angle$ at the circumference on arc BD) (2)

ii) $\angle BCD = 144^\circ$ (opposite $\angle$s of cyclic quadrilateral supplementary) (2)

d) $\angle DAC = \angle ABC$ ($\angle$ in alternate segment)

$\angle DAC = \angle EAC$ (AC bisects $\angle DAE$)

$\therefore \angle ABC = \angle EAC$ (both $= \angle DAC$)

$\angle BAC = 90^\circ$ ($\angle$ on semi-circle)

$\angle BAE + \angle EAC = 90^\circ$ (adjacent complementary $\angle$s)

$\therefore \angle BAE + \angle ABC = 90^\circ$ ($\angle ABC = \angle EAC$)

$\therefore \angle AEB = 90^\circ$ ($\angle$ sum of $\triangle ABE$)

$\therefore AB \perp BC$ (4)

Q2) a) $(x^2+\frac{1}{x})^8$

$T_4 = {}^8C_3 (x^2)^5 (\frac{1}{x})^3$

$= {}^8C_3\, x^7$

$= 56x^7$ (3)

b) $(x^2-\frac{1}{x})^{16}$

17 terms, middle: T_9

$T_9 = {}^{16}C_8 (x^2)^8 (\frac{-1}{x})^8$

$= {}^{16}C_8\, x^8$

Middle Term $= 12\,870\, x^8$ (3)

c) $(2x^3 + \frac{1}{x})^{12}$

$T_{k+1} = {}^{12}C_k (2x^3)^{12-k} x^{-k}$

$= {}^{12}C_k\, 2^{12-k} x^{36-3k} x^{-k}$

$= {}^{12}C_k\, 2^{12-k} x^{36-4k}$

For term independent of x:

$36 - 4k = 0$

$k = 9 \quad \therefore T_{10}$

$T_{10} = {}^{12}C_9\, 2^3$

$= 1760$ (3)

d) $\frac{1}{1\cdot 4} + \frac{1}{4\cdot 7} + \frac{1}{7\cdot 10} + \dots + \frac{1}{(3n-2)(3n+1)} = \frac{n}{3n+1}$, $n \geq$

Step 1: Show true for $n=1$

LHS $= \frac{1}{(3-2)(3+1)} = \frac{1}{4}$

RHS $= \frac{1}{3+1} = \frac{1}{4}$

LHS = RHS

$\therefore$ true for $n=1$

Step 2: Assume true for $n=k$

ie. $\frac{1}{1\cdot 4} + \frac{1}{4\cdot 7} + \frac{1}{7\cdot 10} + \dots + \frac{1}{(3k-2)(3k+1)} = \frac{k}{3k+1}$

Step 3: Prove true for $n=k+1$

ie. $\frac{1}{1\cdot 4} + \frac{1}{4\cdot 7} + \frac{1}{7\cdot 10} + \dots + \frac{1}{(3k-2)(3k+1)} + \frac{1}{(3k+1)(3k+4)} = \frac{k+1}{3k+4}$

LHS $= \frac{k}{3k+1} + \frac{1}{(3k+1)(3k+4)}$

$= \frac{k(3k+4)+1}{(3k+1)(3k+4)}$

$= \frac{3k^2+4k+1}{(3k+1)(3k+4)}$

$= \frac{(3k+1)(k+1)}{(3k+1)(3k+4)}$

$= \frac{k+1}{3k+4}$ = RHS.

Hence, if true for $n=k$, also true for $n=k+1$.

Step 4:

$\therefore$ By the principle of mathemat[ical] induction, true for all $n \geq 1$. (6)

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YEAR 12 EXT I TASK 1 2007 – SOLUTIONS

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