YEAR 12

GIRRAWEEN HIGH SCHOOL
MATHEMATICS

EXTENSION 2 TEST 3 JUNE, 2001

INSTRUCTIONS
a) Write all your answers on your own paper
b) Show all necessary working
c) Marks may be deducted for careless or badly arranged work
d) A table of standard integrals will be supplied

Time Allowed: 80 minutes

QUESTION 1 (28 Marks)

MARK

(a) (i) Use the table of standard integrals to find $\int \sec x \tan x\,dx$

(ii) Find $\int (\sec x + \tan x)^2 dx$ **4**

(b) Find the exact value of

(i) $\displaystyle\int_1^e \frac{(\ln x)^3}{x}dx$

(ii) $\displaystyle\int_0^{\frac{2\pi}{3}} \frac{1}{5+4\cos x}dx$

(iii) $\displaystyle\int_{\frac{3}{2}}^{3} \sqrt{9-u^2}\,du$

(iv) $\displaystyle\int_0^1 \frac{e^{-2x}}{e^{-x}+1}$ **18**

(c) Find

(i) $\int xe^{-x}dx$

(ii) $\displaystyle\int \frac{x+4}{x^2+2x+4}dx$

6

QUESTION 2(19 Marks) **MARK**

(a) (i) Write $\dfrac{4x^2-5x-7}{(x^2+x+2)(x-1)}$ in the form

$$\frac{Ax+B}{x^2+x+2}+\frac{C}{x-1}$$

(ii) hence evaluate $\displaystyle\int \frac{4x^2-5x-7}{(x^2+x+2)(x-1)}dx$ **5**

(b) If $I_n = \displaystyle\int_0^{\frac{\pi}{2}} \cos^n x \, dx$ prove that $I_n = \dfrac{n-1}{n} I_{n-2}$

Hence, evaluate I_5 **6**

(c) If $I = \displaystyle\int_0^{\frac{\pi}{2}} \sin^4 x \cos^2 x \, dx$

(i) Substitute $x = \dfrac{\pi}{2} - y$ to prove

$$I = \int_0^{\frac{\pi}{2}} \cos^4 y \sin^2 y \, dy$$

(ii) Hence prove

$$2I = \int_0^{\frac{\pi}{2}} \sin^2 t \cos^2 t \, dt$$

(iii) Hence deduce the value of I **8**

MARK

QUESTION 3 (19 Marks)

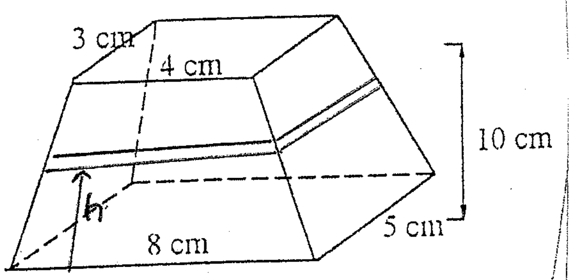
(a) Find the volume generated when the area bounded by the curve $y=6x-x^2$ and the x-axis is rotated about the line $x=3$. 5

(b) The diagram shows a cistern whose top and bottom rectangular faces measure 4cm by 3cm and 8cm by 5cm respectively. If these parallel faces are 10cm apart 7

(i) Show that the volume of the slice at height h is given by

$$\partial V = \left(\frac{2h^2}{25} - \frac{18h}{5} + 40\right)\partial h$$

(ii) Hence find the volume of the cistern.



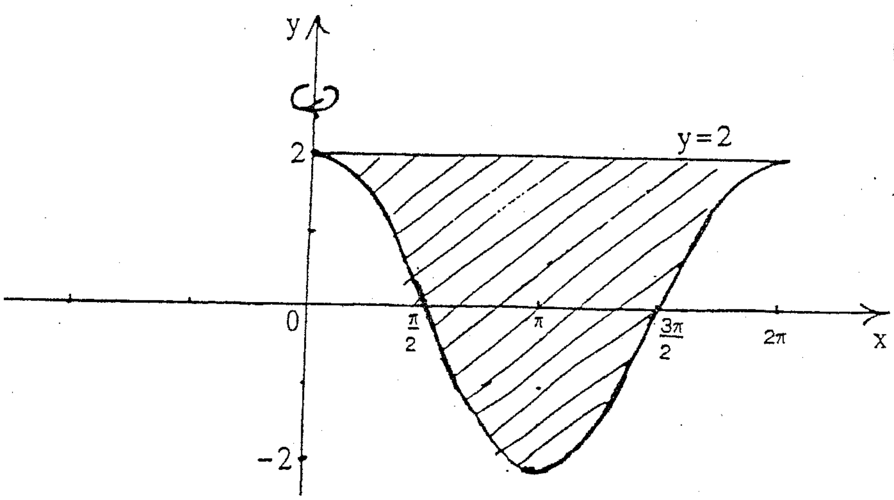
(c) A mathematically inclined microwave cooking enthusiast decided to design his own cake pan. The shape of the interior of the cake pan is obtained by rotating the region bounded by the curve $y=$ [illegible] 2 cos x 7

$0 \le x \le 2\pi$ and the line $y = 2$ through 360° about the y axis.

Use the method of cylindrical shells to show the volume of the cake is given by

$$4\pi \int_0^{2\pi} x(1-\cos x)\,dx$$

and hence calculate this volume.



Question 1

(a)(i) $\sec x$

(ii) $\int \sec^2 x + 2\sec x\tan x + \tan^2 x \, dx$

$= \int \sec^2 x + 2\sec x\tan x + \sec^2 x - 1 \, dx$

4

$= \int 2\sec^2 x + 2\sec x\tan x - 1 \, dx$

$= \underline{2\tan x + 2\sec x - x + c}$

(b)

(i) let $u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$

when $x = 1$ $u = 0$

$x = e$ $u = 1$

3

$\int_0^1 u^3 \, du$

$= \left[\frac{u^4}{4}\right]_0^1$

$= \underline{\frac{1}{4}}.$

(ii) let $t = \tan \frac{x}{2}$

$\cos x = \frac{1-t^2}{1+t^2}$

$dx = \frac{2}{1+t^2} dt$

when $x = 0$ $t = 0$

$x = \frac{2\pi}{3}$ $t = \sqrt{3}$

$\int_0^{\sqrt{3}} \frac{1}{5 + 4 \times \frac{1-t^2}{1+t^2}} \times \frac{2}{1+t^2} dt$

$= \int_0^{\sqrt{3}} \frac{2}{5+5t^2+4-4t^2} dt$

$= \int_0^{\sqrt{3}} \frac{2}{9+t^2} dt$

$= \left[\frac{2}{3}\tan^{-1}\frac{t}{3}\right]_0^{\sqrt{3}}$

$= \frac{2}{3}\tan^{-1}\frac{\sqrt{3}}{3} - 0$

5

$= \frac{2}{3} \times \frac{\pi}{6}$

$= \underline{\frac{\pi}{9}.}$

(iii) let $u = 3\sin\theta$

$\frac{du}{d\theta} = 3\cos\theta$

when $u = \frac{3}{2}$ $\theta = \frac{\pi}{6}$

$u = 3$ $\theta = \frac{\pi}{2}$

$\int_{\pi/6}^{\pi/2} \sqrt{9 - 9\sin^2\theta} \; 3\cos\theta \, d\theta$

$= \int_{\pi/6}^{\pi/2} 9\cos^2\theta \, d\theta$

5

$= \int_{\pi/6}^{\pi/2} \frac{9}{2}\left[1 + \cos 2\theta\right]$

$= \left[\frac{9}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right)\right]_{\pi/6}^{\frac{\pi}{2}}$

$= \frac{9\pi}{4} - \frac{9\pi}{12} + \frac{\sqrt{3}}{4}$

$= \frac{3\pi}{2} + \frac{\sqrt{3}}{4}.$

(iv) Let $u = e^{-x}$

$$\frac{du}{dx} = -e^{-x}$$

$$\int^{e^{-1}} \frac{-u}{u+1}\,du.$$

when $x=0$ $u=1$

$x=1$ $u=e^{-1}$

$$\int_1^{e^{-1}} \frac{-(u+1)}{u+1} + \frac{1}{u+1}\,du.$$

$$= \int_1^{e^{-1}} -1 + \frac{1}{u+1}\,du$$ (5)

$$= \Big[-u + \ln(u+1)\Big]_1^{e^{-1}}$$

$$= -e^{-1} + \ln 2 + 1 + \ln(e^{-1}+1)$$

$$= 1 - \frac{1}{e} + \ln\left(\frac{2e+1}{e}\right)$$

(c)(i) $\int x \frac{d}{dx}(-e^{-x})\,dx$

$$= -xe^{-x} + \int e^{-x}\,dx$$ (3)

$$= -xe^{-x} - e^{-x} + c$$

(ii) $\int \frac{1/2(2x+2)}{x^2+2x+4} + \frac{3}{x^2+2x+4}\,dx$

(3)

$$= \int \frac{1/2(2x+2)}{x^2+2x+4} + \frac{3}{(x+1)^2+3}\,dx$$

$$= \frac{1}{2}\ln(x^2+2x+4) + \frac{3}{\sqrt3}\tan^{-1}\frac{x+1}{\sqrt3}$$

Question 2

(a)

$$\frac{4x^2-5x-7}{(x^2+x+2)(x-1)} = \frac{Ax+B}{x^2+x+2} + \frac{C}{x-1}$$

$$4x^2-5x-7 = (Ax+B)(x-1) + C(x^2+x+2)$$

$$Ax^2 - Ax + Bx - B + Cx^2 + Cx + 2C = 4x^2 - 5x$$

$\therefore$ $A + C = 4$

$B - A + C = -5$

$-B + 2C = -7$

OR

when $x=1$ $\quad -8 = 4C$

$C = -2$

(3)

$x=0$ $\quad -7 = -B + 2C$

$\therefore -7 = -B - 4$

$\therefore B = 3$

$x=-1$ $\quad 2 = 2A + 6 - 4$

$\therefore A = 6$

$$\int \frac{6x+3}{x^2+x+2} - \frac{2}{x-1}\,dx$$

$$\int \frac{3(2x+1)}{x^2+x+2} - \frac{2}{x-1}$$

$$= 3\ln(x^2+x+2) - 2\ln(x-1)$$

(2)

(b) $\int \cos^n x\,dx = \int_0^{\pi/2} \cos^{n-1} x \frac{d}{dx}(\sin x)$

$$\therefore I_n = \Big[\cos^{n-1}x\sin x\Big]_0^{\pi/2} + \int_0^{\pi/2} (n-1)\cos^{n-2}x\sin^2 x\,dx$$

$$= (n-1)\int_0^{\pi/2} \cos^{n-2}x(1-\cos^2 x)\,dx$$

$$= (n-1)\int_0^{\frac{\pi}{2}} \cos^{(n-2)}x - \cos^n x$$

$$= (n-1)\, I_{n-2} - (n-1) I_n$$

(b) CONT

$$I_n + (n-1)I_n = (n-1)I_{n-2}$$

$$nI_n = (n-1)I_{n-2}$$

$$I_n = \frac{n-1}{n} I_{n-2}$$ (3)

$$I_5 = \frac{4}{5} I_3$$

$$I_3 = \frac{2}{3} I_1$$

$$I_1 = \int_0^{\pi/2} \cos x \, dx$$

$$= \Big[\sin x \Big]_0^{\pi/2}$$

$$= 1$$ (3)

$$\therefore I_3 = \frac{2}{3}$$

$$I_5 = \frac{4}{5} \times \frac{2}{3}$$

$$= \frac{8}{15}.$$

(c) let $x = \pi/2 - y$

$$\frac{dx}{dy} = -1$$

when $x = 0$ $y = \pi/2$

$x = \pi/2$ $y = 0$

$$-\int_{\pi/2}^{0} \sin^4(\pi/2 - y)\cos^2(\pi/2 - y)\, dy$$

$$= \int_0^{\pi/2} \cos^4 y \sin^2 y \, dy$$

Since $\sin(\pi/2 - y) = \cos y$

and $\cos(\pi/2 - y) = \sin y$.

(ii)

I

and $-\int_{\pi/2}^{0} f(y)\,dy = \int_0^{\pi/2} f(y)\,dy$.

Now

(ii)

$$2I = \int_0^{\pi/2} \sin^4 t \cos^2 t \, dt + \int_0^{\pi/2} \cos^4 t \sin^2 t \, dt$$

from part(i) and given.

$$2I = \int_0^{\pi/2} \sin^2 t \cos^2 t (\sin^2 t + \cos^2 t)\, dt$$

$$= \int_0^{\pi/2} \sin^2 t \cos^2 t \, dt$$

$$\therefore 2I = \int_0^{\pi/2} \frac{\sin^2 2t}{4} \, dt$$

$$= \frac{1}{8} \int_0^{\pi/2} 1 - \cos 2t \, dt$$

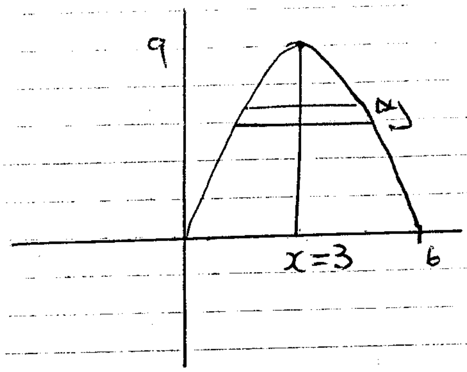
$$= \frac{1}{8} \left[t - \frac{\sin 2t}{2} \right]_0^{\pi/2}$$

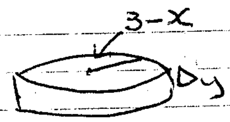
$$= \pi/16$$

$$\therefore I = \pi/32.$$

Question 3

(a) $y = x(6-x)$





$A(y) = \pi(3-x)^2$

Now

$6x - x^2 = y$

$\therefore 9 - 6x + x^2 = -y + 9$

$\therefore V = \lim_{\Delta y \to 0} \sum_{y=0}^{9} \pi(9-y)\,\Delta y$

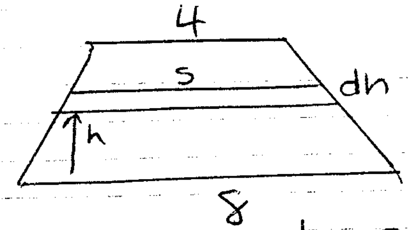
$\therefore V = \int_0^9 9\pi - y\pi \, dy$

$= \left[9\pi y - \frac{y^2\pi}{2}\right]_0^9$

$= 81\pi - \frac{81\pi}{2}$

$= \frac{81\pi}{2}$.

(b)



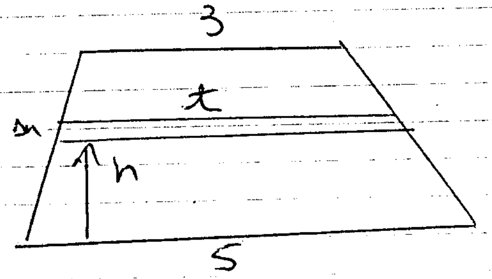
of form $s = mh + B$

when $h=0$ $s=8$ $8=B$

$h=10$ $s=4$ $4=10m+8$

$m = -2/5$.

$\therefore s = -\frac{2}{5}h + 8$



Let $t = Lh + c$

when $h=0$ $t=5$ $\therefore c=5$

$h=10$ $t=3$ $3 = 10L + 5$

$L = -1/5$

$t = -\frac{1}{5}h + 5$

(4)

$\therefore \delta V = \left(-\frac{2}{5}h+8\right)\left(-\frac{1}{5}h+5\right)\delta h$

$= \left(\frac{2h^2}{25} - 2h - \frac{8h}{5} + 40\right)\delta h$

$= \left(\frac{2h^2}{25} - \frac{18h}{5} + 40\right)\delta h$

$\therefore V = \lim_{\Delta h \to 0} \sum_{h=0}^{10} \frac{2h^2}{25} - \frac{18h}{5} + 40\,dh$

$= \int_0^{10} \frac{2h^2}{25} - \frac{18h}{5} + 40\,dh$

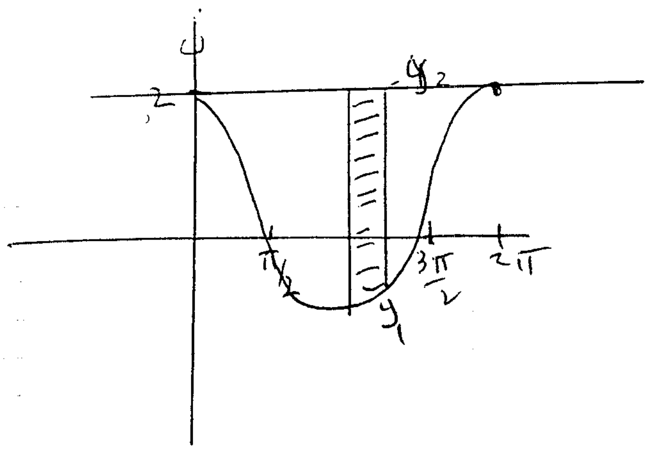
$= \left[\frac{2h^3}{75} - \frac{9h^2}{5} + 40h\right]_0^{10}$

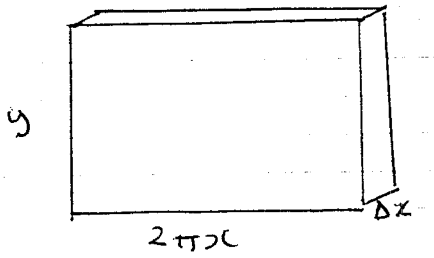
$= 26\,^2/_3 - 180 + 400$

(3)

$= 246\,^2/_3\ \text{cm}^3$

(c)





$A(x) = 2\pi x y$

but $y = y_2 - y_1$

$= 2 - 2\cos x$

$\therefore V = \lim\limits_{\Delta x \to 0} \sum\limits_{x=0}^{2\pi} 2\pi x(2-2\cos x)\Delta x$

$V = 2\pi \int_0^{2\pi} x(2-2\cos x)\,dx$

$= 4\pi \int_0^{2\pi} x(1-\cos x)\,dx$

$= 4\pi \int_0^{2\pi} x - x\cos x\,dx$

3

Now $\int_0^{2\pi} x\cos x\,dx = \int_0^{2\pi} x\frac{d}{dx}(\sin x)\,dx$

$= \Big[x\sin x\Big]_0^{2\pi} - \int_0^{2\pi} \sin x\,dx$

$= \Big[\cos x\Big]_0^{2\pi}$

$= \cos 2\pi - \cos 0$

$= 1.$

$\therefore v = 4\pi \int_0^{2\pi} x\,dx$

$= 4\pi \left[\frac{x^2}{2}\right]_0^{2\pi}$

$= \underline{8\pi^3 \text{ units}^3}.$

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