

**GIRRAWEE HIGH SCHOOL
MATHEMATICS**

Year 12 Extension 2 Task 3

Monday 23rd June 2002

- Instructions: a) Write all your answers on your own paper.
 b) Show all necessary working.
 c) Marks may be deducted for careless or badly arranged work.

Time Allowed: 80 minutes

Question 1 (27 marks)

Marks

a) Find the following integrals:

- | | |
|---|---|
| (i) $\int \tan^3 x dx$ | 4 |
| (ii) $\int \frac{dx}{x^3 \sqrt{x^2 - 4}}$ | 4 |
| (iii) $\int_0^{\frac{\pi}{2}} \frac{dx}{1 + \sin x + \cos x}$ | 4 |
| (iv) $\int_0^{\frac{\pi}{4}} x \sec^2 x dx$ | 4 |
| (v) $\int \frac{x+2}{\sqrt{x^2 - 6x + 7}} dx$ | 4 |
| (vi) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4 x \sin^5 x dx$ | 3 |
| (vii) $\int \frac{7x^2 - 5x + 4}{(x-1)(x^2+1)} dx$ | 4 |

Question 2 (9 marks)

a) The integral $I_n = \int_0^{\frac{\pi}{4}} x^n \cos 2x dx$;

(i) Prove that $I_n = \frac{1}{2} \left(\frac{\pi}{4} \right)^n - \frac{n(n-1)}{4} I_{n-2}$ 4

(ii) Using your answer to (i), evaluate $\int_0^{\frac{\pi}{4}} x^4 \cos 2x dx$ 3

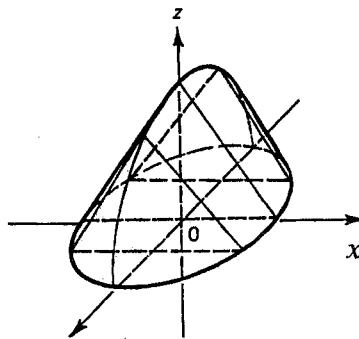
b) In your own words explain the error in the following solution: 2

$$\begin{aligned}
 \int \tan x dx &= \int \frac{\sin x dx}{\cos x} \\
 &= \int \sec x \sin x dx \\
 &= -\sec x \cos x + \int \sec x \tan x \cos x dx \\
 &= -\sec x \cos x + \int \tan x dx \\
 \int \tan x dx - \int \tan x dx &= -\sec x \cos x \\
 0 &= -1
 \end{aligned}$$

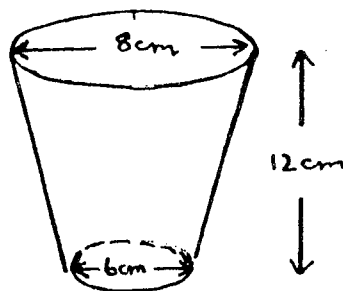
Question 3 (24 marks)

Marks

- a) Find the volume generated by rotating the area bounded by $y = \log x$, $y = 0$ and $x = 3$, about the x -axis. 5
- b) Consider the function $y = 2x - x^2$;
- (i) Show that when x is written as the subject, the resulting equation is $x = 1 \pm \sqrt{1 - y}$ 2
- (ii) By taking slices perpendicular to the y -axis, calculate the volume of the solid generated when the area bounded by $y = 2x - x^2$ and the x -axis is rotated about the y -axis. 5
- c) The solid shown has a circular base of radius 2 units. Vertical cross sections perpendicular to the y -axis are equilateral triangles. 5



- (i) Show that the area of one of these slices is given by $A(y) = \sqrt{3}(4 - y^2)$ 3
- (ii) Hence calculate the volume of the solid. 3
- d) A drinking glass is in the shape of a truncated cone in which the internal diameters of the top and the bottom are 8 cm and 6 cm respectively. The internal height of the glass is 12 cm.

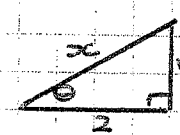


- (i) Show that the cross section taken x cm above the base has radius $\left(3 + \frac{x}{12}\right)$ cm. 2
- (ii) Hence find the volume of fluid the glass can hold. 4

Question 1 (27)

(i) $\int \tan^3 x \, dx$
 $= \int \tan x (\sec^2 x - 1) \, dx$
 $= \int \tan x \sec^2 x \, dx - \int \tan x \, dx$
 $= \int u \, du + \log |\cos x| + c \quad u = \tan x$
 $= \frac{1}{2} u^2 + \log |\cos x| + c \quad du = \sec^2 x \, dx$
 $= \frac{1}{2} \tan^2 x + \log |\cos x| + c \quad (4)$

(ii) $\int \frac{dx}{x^3 \sqrt{x^2 - 4}}$ $x = 2 \sec \theta$
 $dx = 2 \sec \theta \tan \theta \, d\theta$
 $= \int \frac{2 \sec \theta \tan \theta \, d\theta}{8 \sec^3 \theta \sqrt{4 \sec^2 \theta - 4}}$
 $= \frac{1}{8} \int \frac{\sec \theta \tan \theta \, d\theta}{\sec^3 \theta \tan \theta}$
 $= \frac{1}{8} \int \cos^2 \theta \, d\theta$
 $= \frac{1}{16} \int (1 + \cos 2\theta) \, d\theta$
 $= \frac{1}{16} \theta + \frac{1}{32} \sin 2\theta + c$
 $= \frac{1}{16} \theta + \frac{1}{16} \sin \theta \cos \theta + c$
 $= \frac{1}{16} \sec^{-1} \frac{x}{2} + \frac{1}{16} \cdot \frac{\sqrt{x^2 - 4}}{x} \cdot \frac{2}{x} + c$
 $= \frac{1}{16} \sec^{-1} \frac{x}{2} + \frac{\sqrt{x^2 - 4}}{8x^2} + c \quad (4)$



(iii) $\int_0^{\frac{\pi}{2}} \frac{dx}{1 + \sin x + \cos x}$ $t = \tan \frac{x}{2}$
 $dx = \frac{2dt}{1+t^2}$
 $= \int_0^1 \frac{2dt}{1+t^2}$
 $= \int_0^1 \frac{2dt}{1 + \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}}$
 $= \int_0^1 \frac{2dt}{1+t^2+2t+1-t^2}$
 $= \int_0^1 \frac{2dt}{2+2t}$
 $= \int_0^1 \frac{dt}{1+t}$
 $= [\log(1+t)]_0^1$
 $= \log 2 - \log 1$
 $= \log 2 \quad (4)$

(iv) $\int_0^{\frac{\pi}{4}} x \sec^2 x \, dx$ $u = x$ $v = \tan x$
 $du = dx$ $dv = \sec^2 x \, dx$
 $= [x \tan x]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan x \, dx$
 $= [x \tan x + \log |\cos x|]_0^{\frac{\pi}{4}}$
 $= \frac{\pi}{4} \tan \frac{\pi}{4} + \log \cos \frac{\pi}{4} - 0 + \log \cos 0$
 $= \frac{\pi}{4} + \log \frac{1}{\sqrt{2}} + \log 1$
 $= \frac{\pi}{4} + \log \left(\frac{1}{\sqrt{2}} \right) \quad (4)$

(vi) $\int \frac{x+2}{\sqrt{x^2-6x+7}} \, dx$ $u = x^2 - 6x + 7$
 $du = (2x-6) \, dx$
 $= \frac{1}{2} \int \frac{2x-6}{\sqrt{x^2-6x+7}} \, dx + 5 \int \frac{dx}{\sqrt{(x-3)^2 - 2}}$
 $= \frac{1}{2} \int u^{-\frac{1}{2}} \, du + 5 \log(x-3 + \sqrt{x^2-6x+7}) + c$
 $= \sqrt{x^2-6x+7} + 5 \log(x-3 + \sqrt{x^2-6x+7}) + c$

(vii) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4 x \sin^5 x \, dx$
 $\cos^4 x = \text{even function}$
 $\sin^5 x = \text{odd function}$
 $\therefore \cos^4 x \sin^5 x = \text{odd function}$
 $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4 x \sin^5 x \, dx = 0 \quad (3)$

(viii) $\int \frac{7x^2 - 5x + 4}{(x-1)(x^2+1)} \, dx$
 $A(x^2+1) + (Bx+C)(x-1) = 7x^2 - 5x + 4$
 $x=1 \quad x=i$
 $2A = 6 \quad (B+C)(i-1) = -7-5i+4$
 $A = 3 \quad (-B-C) + i(C-B) = -3-5i$
 $B+C = 3$
 $B-C = 5$
 $2B = 8$
 $B = 4 \quad \therefore C = -1$

$\int \frac{7x^2 - 5x + 4}{(x-1)(x^2+1)} \, dx$
 $= \int \left[\frac{3}{x-1} + \frac{4x}{x^2+1} - \frac{1}{x^2+1} \right] \, dx$
 $= 3 \log|x-1| + 2 \log|x^2+1| - \tan^{-1} x + c \quad (4)$

Question 2 (9)

a) $I_n = \int x^n \cos 2x \, dx$
 $u = x^n$
 $dv = \cos 2x \, dx$
 $du = n x^{n-1} dx$
 $v = \frac{1}{2} \sin 2x$
 $\int x^n \cos 2x \, dx = \frac{1}{2} x^n \sin 2x - \int \frac{1}{2} n x^{n-1} \sin 2x \, dx$

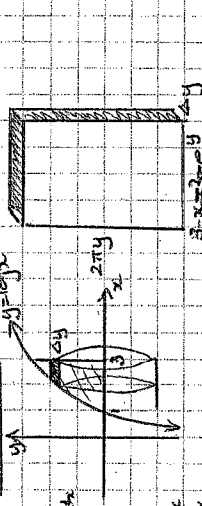
$u = x^{n-1}$
 $dv = \sin 2x \, dx$
 $du = (n-1)x^{n-2} dx$
 $v = -\frac{1}{2} \cos 2x$
 $\int x^{n-1} \sin 2x \, dx = -\frac{1}{2} x^{n-1} \cos 2x + \int \frac{1}{2} (n-1) x^{n-2} \cos 2x \, dx$
 $= \frac{1}{2} \left(\frac{\pi}{4} \right)^n \sin \pi + \frac{1}{4} n \left(\frac{\pi}{4} \right)^{n-1} \cos \pi - 0 = -\frac{n(n-1)}{4} I_{n-2}$
 $= \frac{1}{2} \left(\frac{\pi}{4} \right)^n - \frac{n(n-1)}{4} I_{n-2}$ (4)

(ii) $\int x^4 \cos 2x \, dx$
 $= \frac{1}{2} \left(\frac{\pi}{4} \right)^4 - \frac{4(3)}{4} I_2$
 $= \frac{\pi^4}{512} - \frac{3}{2} \left(\frac{\pi}{4} \right)^2 + \frac{2(1)}{4} I_0$
 $= \frac{\pi^4}{512} - \frac{3\pi^2}{32} + \frac{1}{2} \int \cos 2x \, dx$
 $= \frac{\pi^4}{512} - \frac{3\pi^2}{32} + \frac{1}{4} \left[\sin 2x \right]_0^{\frac{\pi}{2}}$
 $= \frac{\pi^4}{512} - \frac{3\pi^2}{32} + \frac{1}{4} (\sin \pi - \sin 0)$
 $= \frac{\pi^4}{512} - \frac{3\pi^2}{32} + \frac{1}{4}$ (3)

b) $\int \tan x = -\sec x \cos x + c + \int \tan x \, dx$
 $\int \tan x = -\sec x \cos x + c - \sec x \cos x + c$
 $0 = -1 + c$

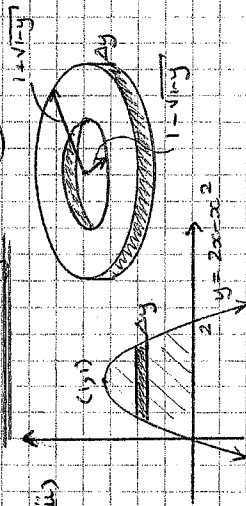
is the error is neglecting the constant of integration. (2)

Question 3 (24)



$A(y) = 2\pi y(3 - e^y)$
 $\Delta V = \lim_{\Delta y \rightarrow 0} \sum_{y=0}^{\Delta y} (6\pi y - 2\pi y e^y) \Delta y$
 $V = \lim_{\Delta y \rightarrow 0} \sum_{y=0}^{\Delta y} (6\pi y - 2\pi y e^y) \Delta y$
 $= 2\pi \int_0^{\log 3} (6y - 2y e^y) dy$
 $= 2\pi \left[3y^2 - y e^y + \int \frac{d}{dy} (y e^y) dy \right]_0^{\log 3}$
 $= 2\pi \left[\frac{3}{2} y^2 - y e^y + y e^y \right]_0^{\log 3}$
 $= 2\pi \left[\frac{3}{2} (\log 3)^2 - (\log 3) 3 + 3 - 1 \right]$
 $= \pi (3(\log 3)^2 - 6 \log 3 + 4) \text{ units}^3$

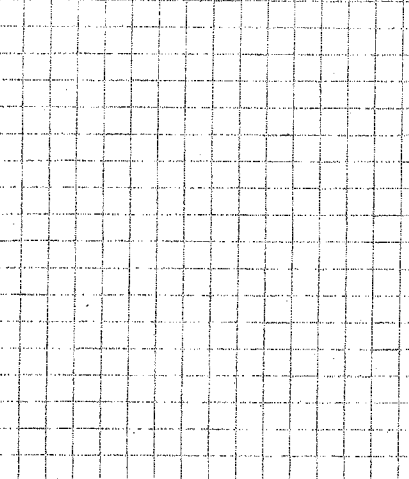
b) $y = 2x - x^2$
 $x^2 - 2x + y = 0$
 $x = \frac{2 \pm \sqrt{4 - 4y}}{2}$
 $x = \frac{2 \pm 2\sqrt{1-y}}{2}$
 $x = 1 \pm \sqrt{1-y}$ (2)



$A(y) = \pi (1 + \sqrt{1-y})^2 - \pi (1 - \sqrt{1-y})^2$
 $= \pi (1 + 2\sqrt{1-y} + (1-y)^2 - (1 - 2\sqrt{1-y} + (1-y)^2))$
 $= 4\pi \sqrt{1-y}$
 $\Delta V = 4\pi \sqrt{1-y} \Delta y$
 $V = \lim_{\Delta y \rightarrow 0} \sum_{y=0}^{\Delta y} 4\pi \sqrt{1-y} \Delta y$
 $= 4\pi \int_0^1 (1-y)^{\frac{1}{2}} dy$
 $= 4\pi \left[\frac{2}{3} (1-y)^{\frac{3}{2}} \right]_0^1$
 $= \frac{8\pi}{3} (0 - 1)$
 $= -\frac{8\pi}{3} \text{ units}^3$ (5)

$V = \lim_{\Delta x \rightarrow 0} \sum_{x=0}^{\Delta x} \pi \left(9 + \frac{1}{2} x + \sqrt{4x^2} \right) \Delta x$
 $= \pi \int_0^{12} \left(9 + \frac{1}{2} x + \sqrt{4x^2} \right) dx$
 $= \pi \left[9x + \frac{1}{4} x^2 + \sqrt{4x^3} \right]_0^{12}$
 $= \pi (108 + 36 + 4 - 0)$
 $= 148 \pi \text{ cm}^3$ (4)

(ii) $\Delta V = \sqrt{3} (4 - y^2) \Delta y$
 $V = \lim_{\Delta y \rightarrow 0} \sum_{y=0}^{\Delta y} \sqrt{3} (4 - y^2) \Delta y$
 $= 2\sqrt{3} \int_0^2 (4 - y^2) dy$
 $= 2\sqrt{3} \left[4y - \frac{1}{3} y^3 \right]_0^2$
 $= 2\sqrt{3} \left(8 - \frac{8}{3} - 0 \right)$
 $= \frac{32\sqrt{3}}{3} \text{ units}^3$ (3)



$r - 6 = \frac{x}{12}$
 $r - 6 = \frac{x}{12}$
 $r = 6 + \frac{x}{12}$ (2)

(iii) $A(x) = \pi \left(3 + \frac{x}{12} \right)^2$
 $= \pi \left(9 + \frac{x}{6} + \frac{x^2}{144} \right)$
 $\Delta V = \pi \left(9 + \frac{x}{6} + \frac{x^2}{144} \right) \Delta x$