Girraween High School

Year 12 Mathematics Extension 2 **Task 4** June, 2003

Time 80 minutes

Instructions:

Complete the test on your own paper.

Show all necessary working.

Marks will be deducted for careless or badly arranged work.

Question 1. (43 marks)

a) Find:

(i) $\int \frac{\cos^2 x}{1-\sin x}dx$ (ii) $\int \frac{x^2+2x+3}{x-1}dx$ **8**

b) Find:

(i) $\int \frac{1}{1-\cos x}dx$ (ii) $\int \frac{1}{3+2x+x^2}dx$ **8**

c) Find

(i) $\int_e^{e^2} \ln x dx$ (ii) $\int_0^1 \sqrt{4-x^2}\,dx$ **8**

d) Given that $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ show that $\int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx = \frac{\pi^2}{4}$ **5**

e) For $f(x) = \frac{20}{(x+1)(x^2+4)}$:

(i) Decompose into partial fractions. **3**

(ii) Hence show that $\int_0^2 f(x)dx = \frac{\pi}{2} + \ln(\frac{81}{4})$ **4**

f) If $I_n = \int_0^t \frac{1}{(1+x^2)^n}dx$ n = 1, 2, 3, …

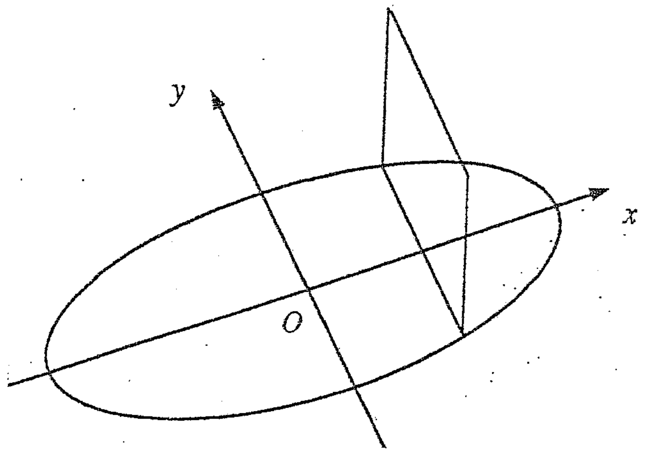
(i) Show $2nI_{n+1} = (2n-1)I_n + \frac{t}{(1+t^2)^n}$ n = 1, 2, 3, … **5**

(ii) Hence find the value of I_3, in terms of t. **2**

Question 2. (18 marks)

a) The region bounded by $y = x^2$, the y axis an the line $x = 1$ is rotated about the line y = -1. Find the volume of the solid of revolution. **5**

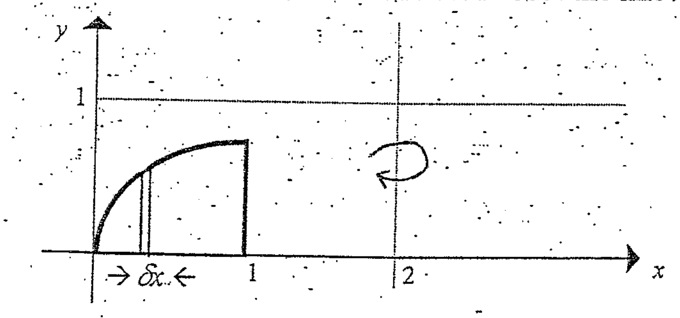
b) The base of a tent is in the shape of an ellipse with the equation $\frac{x^2}{4} + y^2 = 1$.
Vertical cross sections taken perpendicular to the major axis are the base of squares.



(i) Show the volume of the tent is given by $V = \int_{-2}^{2} (4 - x^2)dx$. **4**

(ii) Hence find the volume. **2**

c) The region shown bounded by the portion of the curve $y = \frac{x}{x+1}$, the x axis and the line $x = 1$ is rotated about the line $x = 2$.



(i) Using the method of cylindrical shells, show that the volume δV of a typical shell at a distance x from the origin and with thickness δx is given by **3**
$\delta V = 2\pi(2 - x).\frac{x}{1+x}.\delta x$

(ii) Hence find the volume of the solid. **4**

Question 1

a) i) $\int \frac{\cos^2 x}{1-\sin x}\,dx$

$= \int \frac{(1-\cancel{\sin x})(1+\sin x)}{1-\cancel{\sin x}}\,dx$

$= \int (1+\sin x)\,dx$

$= x - \cos x + c$

ii) $\int \frac{x^2+2x+3}{x-1}$

$$\begin{array}{r} x+3 \\ x-1\,\overline{)\,x^2+2x+3} \\ \underline{x^2-x} \\ 3x+3 \\ \underline{3x-3} \\ 6 \end{array}$$

$\therefore = \int \left(x+3+\frac{6}{x-1}\right)dx$

$= \frac{x^2}{2} + 3x + 6\ln(x-1) + c$

b i) $\int \frac{1}{1-\cos x}\,dx$

If $t = \tan\frac{x}{2}$

$\frac{dt}{dx} = \sec^2\frac{x}{2}\cdot\frac{1}{2}$

$= \frac{1+\tan^2 x/2}{2}$

$= \frac{1+t^2}{2}$

$\therefore\ dx = \frac{2\,dt}{1+t^2}$

$\therefore \int \frac{1}{1-\cos x}\,dx = \int \frac{1}{1-\frac{(1-t^2)}{1+t^2}} \times \frac{2\,dt}{1+t^2}$

$= \int \frac{1+\cancel{t^2}}{1+t^2-(1-t^2)}\cdot\frac{2\,dt}{1+\cancel{t^2}}$

$= \int \frac{2\,dt}{2t^2}$

$= \int \frac{dt}{t^2}$

$= -\frac{1}{t} + c$

$= -\cot\left(\frac{t}{2}\right) + c$

ii) $\int \frac{1}{3+2x+x^2}\,dx$

$= \int \frac{1}{2+(1+x)^2}\,dx$

$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{1+x}{\sqrt{2}}\right) + c$

c i) $\int_e^{e^2} \ln x \cdot \frac{dy}{dx}\,dy$

$= \left[x\ln x\right]_e^{e^2} - \int_e^{e^2} x\cdot\frac{1}{x}\,dx$

$= \left[x\ln x\right]_e^{e^2} - \left[x\right]_e^{e^2}$

$= \left[x\ln x - x\right]_e^{e^2}$

$= (e^2\ln e^2 - e^2) - (e\ln e - e)$

$= e^2\cdot 2\ln e - e^2 - e\ln e + e$

Now $\ln e = 1$

$\therefore = 2e^2 - e^2 - e + e$

$= e^2$

ii) $\int_0^1 \sqrt{4-x^2}\,dx$

let $x = 2\sin u$

$\frac{dx}{du} = 2\cos u$

$dx = 2\cos u\,du$

$x = 1 \quad u = \pi/6$

$x = 0 \quad u = 0$

$\therefore \int_0^{\pi/6} \sqrt{4-4\sin^2 u}\cdot 2\cos u\,du$

$= \int_0^{\pi/6} 2\sqrt{1-\sin^2 u}\cdot 2\cos u\,du$

$= 2\int_0^{\pi/6} 2\cos^2 u\,du$

Now $2\cos^2 u = \cos 2u + 1$

$\therefore 2\int_0^{\pi/6} (\cos 2u + 1)\,du$

$= 2\left[\frac{\sin 2u}{2} + u\right]_0^{\pi/6}$

$= 2\left[\frac{\sin \pi/3}{2} + \frac{\pi}{6} - 0\right]$

$= 2\left[\frac{\sqrt{3}}{4} + \frac{\pi}{6}\right]$

$= \frac{\sqrt{3}}{2} + \frac{\pi}{3}$

d) $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx = \int \frac{(\pi-x)\sin(\pi-x)\,dx}{1+\cos^2(\pi-x)}$

Now $\sin(\pi-x) = \sin x$

$\cos(\pi-x) = -\cos x$

$\therefore = \int_0^{\pi} \frac{(\pi-x)\sin x}{1+(-\cos x)^2}\,dx$

$= \int_0^{\pi} \frac{\pi\sin x - x\sin x}{1+\cos^2 x}\,dx$

$= \int_0^{\pi} \frac{\pi\sin x}{1+\cos^2 x}\,dx - \int_0^{\pi} \frac{x\sin x\,dx}{1+\cos^2 x}$

$\therefore 2\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx = \pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx$

Now if $u = \cos x$

$\frac{du}{dx} = -\sin x$

$\therefore -du = \sin x\,dx$

$x = 0 \quad u = 1$

$x = \pi \quad u = -1$

$$\therefore = \pi \int_{1}^{-1} \frac{-du}{1+u^2}$$

$$= \pi \int_{-1}^{1} \frac{du}{1+u^2}$$

$$= \pi \left[\tan^{-1} u\right]_{-1}^{1}$$

$$= \pi \cdot \left[\tan^{-1}(1) - (\tan^{-1}(-1)\right]$$

$$= \pi \left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right]$$

$$= \pi \cdot \frac{\pi}{2}$$

$$= \frac{\pi^2}{2}$$

$$\therefore \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\,dx = \frac{\pi^2}{4}$$

e) $f(x) = \dfrac{20}{(x+1)(x^2+4)}$

i)

$$\frac{20}{(x+1)(x^2+4)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+4}$$

$$\therefore 20 = A(x^2+4) + (Bx+C)(x+1)$$

If $x = -1$

$$20 = 5A + 0$$

$$A = 4$$

If $x = 0$

$$20 = 4A + C$$

$$\therefore C = 4 \quad \text{as } A = 4$$

If $x = 1$

$$20 = 5A + (B+4)\cdot 2$$

$$20 = 20 + (B+4)\cdot 2$$

$$B = -4$$

$$\therefore f(x) = \frac{4}{x+1} + \frac{(-4x+4)}{x^2+4}$$

ii). $\displaystyle\int_0^2 f(x)\,dx = \int_0^2 \frac{4}{x+1} - \frac{(4x-4)}{x^2+4}\,dx$

$$= \Big[4\ln(x+1)\Big]_0^2 - \int_0^2 \frac{4x}{x^2+4}\,dx + \int_0^2 \frac{4}{x^2+4}\,dx$$

$$= \big[4\ln 3 - 0\big] - 2\Big[\ln(x^2+4)\Big]_0^2 + 2\left[\tan^{-1}\frac{x}{2}\right]_0^2$$

$$= 4\ln 3 - 2\big[\ln 8 - \ln 4\big] + 2\big[\tan^{-1} 1 - \tan^{-1} 0\big]$$

$$= 4\ln 3 - 2\cdot\ln 2 + 2\left(\frac{\pi}{4} - 0\right)$$

$$= 4\ln 3 - 2\ln 2 + \frac{\pi}{2}$$

$$= \frac{\pi}{2} + \ln 3^4 - \ln 2^2$$

$$= \frac{\pi}{2} + \ln\left(\frac{81}{4}\right)$$

f) i) $I_n = \displaystyle\int_0^t \frac{1}{(1+x^2)^n}\,dx$

$$= \int_0^t \frac{1}{(1+x^2)^n}\cdot\frac{d(x)}{dx}\,dx$$

$$= \left[x\cdot\frac{1}{(1+x^2)^n}\right]_0^t - \int_0^t x\cdot -n(1+x^2)^{-n-1}\cdot 2x\,dx$$

$$= \frac{t}{(1+t^2)^n} + 2n\int_0^t \frac{x^2}{(1+x^2)^{n+1}}\,dx$$

$$I_n = \frac{t}{(1+t^2)^n} + 2n\int_0^t \frac{1+x^2-1}{(1+x^2)^{n+1}}\,dx$$

$$= \frac{t}{(1+t^2)^n} + 2n\int_0^t \frac{1}{(1+x^2)^n}\,dx - 2n\int_0^t \frac{1}{(1+x^2)^{n+1}}\,dx$$

$$\therefore I_n = \frac{t}{(1+t^2)^n} + 2n I_n - 2n I_{n+1}$$

$$2n I_{n+1} = 2n I_n - I_n + \frac{t}{(1+t^2)^n}$$

$$2n I_{n+1} = (2n-1) I_n + \frac{t}{(1+t^2)^n}$$

ii) Now $n = 2$ gives

$$4 I_3 = 3 I_2 + \frac{t}{(1+t^2)^2}$$

Now $n = 1$ gives

$$2 I_2 = I_1 + \frac{t}{1+t^2}$$

$\therefore$

but $I_1 = \displaystyle\int_0^t \frac{1}{1+x^2}\,dx$

$$= \left[\tan^{-1} x\right]_0^t$$

$$= \tan^{-1} t$$

$$\therefore 2 I_2 = \tan^{-1} t + \frac{t}{1+t^2}$$

$$I_2 = \frac{1}{2}\left(\tan^{-1} t + \frac{t}{1+t^2}\right)$$

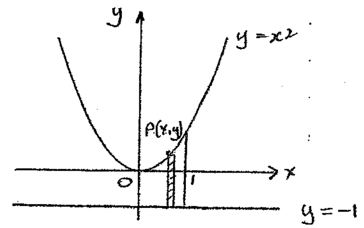
Now

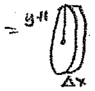
$$4 I_3 = 3\cdot\frac{1}{2}\left[\tan^{-1} t + \frac{t}{1+t^2}\right] + \frac{t}{(1+t^2)^2}$$

$$\therefore I_3 = \frac{3}{8}\tan^{-1} t + \frac{3t}{8(1+t^2)} + \frac{t}{4(1+t^2)^2}$$

Question 2.

a)



$\Delta V =$  $= \pi(y+1)^2\cdot\Delta x$

$$\text{Volume} = \lim_{\Delta x \to 0} \sum_0^1 \pi(y+1)^2\Delta x$$

$$= \pi\int_0^1 (y+1)^2\,dx$$

$$y+1 = x^2+1$$

$$\therefore = \pi\int_0^1 (x^2+1)^2\,dx$$

$$= \pi\int_0^1 (x^4+2x^2+1)\,dx$$

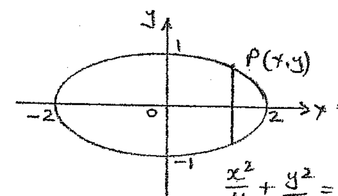
$$= \pi\left[\frac{x^5}{5} + \frac{2x^3}{3} + x\right]_0^1$$

$$= \pi\left[\frac{1}{5} + \frac{2}{3} + 1 - 0\right]$$

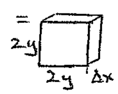
$$= \pi\left[\frac{3+10+15}{15}\right]$$

$$= \frac{28\pi}{15}\ \text{unit}^3$$

b)



$$\frac{x^2}{4} + \frac{y^2}{1} = 1$$

$\Delta V =$  $= 4y^2\Delta x$

$$V = \lim_{\Delta x \to 0} \sum_{-2}^{2} 4y^2 \Delta x$$

$$= \int_{-2}^{2} 4y^2\,dx$$

Now $y^2 = 1 - \frac{x^2}{4}$

$$\therefore\ V = \int_{-2}^{2} 4\left(1 - \frac{x^2}{4}\right) dy$$

$$V = \int_{-2}^{2} (4 - x^2)\,dx$$

ii)

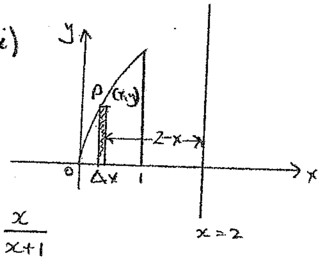
$$V = \left[4x - \frac{x^3}{3}\right]_{-2}^{2}$$

$$= \left[\left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right)\right]$$

$$= \frac{16}{3} + \frac{16}{3}$$

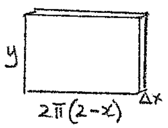
$$= \frac{32}{3} \text{ or } 10\frac{2}{3} \text{ unit}^3$$

c) i)



$y = \frac{x}{x+1}$

$\Delta V =$



$$= 2\pi(2-x)\cdot y \cdot \Delta x$$

$$= 2\pi(2-x)\cdot \frac{x}{1+x} \cdot \Delta x$$

ii) $V = \lim_{\Delta x \to 0} \sum_{0}^{1} 2\pi(2-x)\cdot\frac{x}{1+x}\,\Delta x$

$$= 2\pi \int_0^1 \frac{(2-x)x}{1+x}\,dx$$

$$= 2\pi \int_0^1 \frac{-x^2+2x}{x+1}\,dx$$

$$\begin{array}{r} -x + 3 \\ x+1 \overline{)\, -x^2 + 2x + 0} \\ -x^2 - x \\ \hline 3x + 0 \\ 3x + 3 \\ \hline -3 \end{array}$$

$$\therefore\ = 2\pi \int_0^1 -x + 3 - \frac{3}{x+1}\,dx$$

$$= 2\pi\left[-\frac{x^2}{2} + 3x - 3\ln(x+1)\right]_0^1$$

$$= 2\pi\left[\left(-\frac{1}{2} + 3 - 3\ln 2\right) - (0 + 0 - 3\ln 1)\right]$$

$$= 2\pi\left[-\frac{1}{2} + 3 - 3\ln 2\right]$$

$$= \pi\left[-1 + 6 - 6\ln 2\right)$$

$$= \pi\left[5 - 6\ln 2\right]$$

or $5\pi - 6\pi\ln 2$