

**GIRRAWEE HIGH SCHOOL
MATHEMATICS**

Year 12 Extension 2 Task 34?

Monday 21st June 2004

- Instructions: a) Write all your answers on your own paper.
b) Show all necessary working.
c) Marks may be deducted for careless or badly arranged work.

Time Allowed: 90 minutes

Question 1 (28 marks)

Marks

a) Find the following integrals:

- | | |
|---|---|
| (i) $\int \sin^3 2x dx$ | 4 |
| (ii) $\int x^2 e^{2x} dx$ | 4 |
| (iii) $\int \frac{x^2 dx}{\sqrt{(1-x^2)^3}}$ | 4 |
| (iv) $\int_0^{\frac{2\pi}{3}} \frac{1}{5+4\cos x} dx$ | 4 |
| (v) $\int \frac{x}{\sqrt{1-2x-x^2}} dx$ | 4 |
| (vi) $\int \sec^3 x dx$ | 4 |
| (vii) $\int \frac{2x+3}{(x-2)(x^2+3)} dx$ | 4 |

Question 2 (11 marks)

- a) (i) If $I_n = \int_1^e x(\log x)^n dx$ for $n \geq 0$ show that $I_n = \frac{e^2}{2} - \frac{n}{2} I_{n-1}$ for $n \geq 1$ 3
- (ii) Evaluate $\int_1^e x(\log x)^3 dx$ 3
- b) (i) Use the substitution $u = \frac{1}{x}$, show that $\int_{0.5}^1 \frac{\log x}{1+x^2} dx = \int_2^1 \frac{\log u}{1+u^2} du$ 3
- (ii) Hence, or otherwise, evaluate $\int_{0.5}^2 \frac{\log x}{1+x^2} dx$ 2

Question 3 (25 marks)

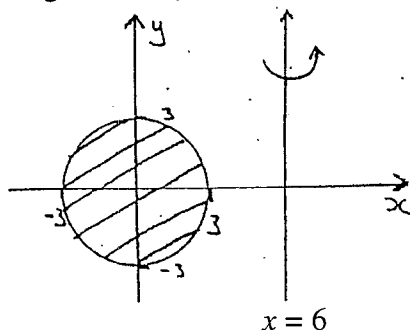
Marks

a) The area bounded by the curve $y = x^2 + 1$ and the line $y = 3 - x$ is rotated about the x -axis.

(i) Sketch the curve and line clearly labelling all the points of intersection. 3

(ii) By considering slices perpendicular to the x -axis, find the volume of the solid formed. 5

b) The diagram shows the region $x^2 + y^2 \leq 9$ and the line $x = 6$

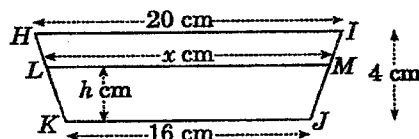


(i) Use the method of cylindrical shells to show that if the region $x^2 + y^2 \leq 9$ is rotated about the line $x = 6$, the volume V of the torus formed is given by; 3

$$V = 4\pi \int_{-3}^3 (6-x)\sqrt{9-x^2} dx$$

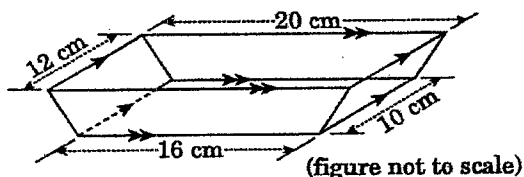
(ii) Hence find the volume of the torus. 3

c) A solid has a base in the form of an ellipse with major axis 10 units and minor axis 8 units. If every section perpendicular to the major axis is an isosceles triangle with altitude 6 units, find the volume of the solid. 5



d)

(i) A trapezium $H I J K$ has parallel sides $KJ = 16$ cm and $HI = 20$ cm. The distance between these sides is 4 cm. L lies in HK and M lies on IJ such that LM is parallel to KJ . The shortest distance from K to LM is h cm and LM has length x cm. Prove that $x = 16 + h$. 2



(ii) 4

The diagram above is of a cake tin with a rectangular base with sides 16 cm and 10 cm. Its top is also rectangular with dimensions 20 cm and 12 cm. The tin has depth 4 cm and each of its four side faces is a trapezium. Find its volume.

Question 1 (28)

a) $\int \sin^3 2x \, dx$
 $= \int \sin 2x (1 - \cos^2 2x) \, dx$
 $u = \cos 2x \quad dv = -2 \sin 2x \, dx$
 $= -\frac{1}{2} \int (1 - u^2) \, du$
 $= -\frac{1}{2} (u - \frac{1}{3} u^3) + c$
 $= -\frac{1}{2} \cos 2x + \frac{1}{6} \cos^3 2x + c$

(ii) $\int x^2 e^{2x} \, dx$
 $u = x^2 \quad v = \frac{1}{2} e^{2x}$
 $du = 2x \, dx \quad dv = e^{2x} \, dx$
 $= \frac{1}{2} x^2 e^{2x} - \int x e^{2x} \, dx$
 $u = x \quad v = \frac{1}{2} e^{2x}$
 $du = dx \quad dv = e^{2x} \, dx$
 $= \frac{1}{2} x^2 e^{2x} - \frac{1}{2} x e^{2x} + \frac{1}{4} e^{2x} + c$

(iii) $\int \frac{x^2 \, dx}{\sqrt{(1-x^2)^3}}$
 $x = \sin \theta \quad dx = \cos \theta \, d\theta$
 $= \int \frac{\sin^2 \theta \cos \theta \, d\theta}{\cos^3 \theta}$
 $= \int \tan^2 \theta \, d\theta$
 $= \int (\sec^2 \theta - 1) \, d\theta$
 $= \tan \theta - \theta + c$
 $= \frac{x}{\sqrt{1-x^2}} - \sin^{-1} x + c$

(iv) $\int_0^{\frac{\pi}{2}} \frac{1}{5 + 4 \cos x} \, dx$
 $t = \tan \frac{x}{2}$
 $dx = \frac{2 \, dt}{1+t^2}$
 $= \int_0^{\sqrt{3}} \frac{2 \, dt}{5 + 4(\frac{1-t^2}{1+t^2})}$
 $= \int_0^{\sqrt{3}} \frac{2 \, dt}{5 + 5t^2 + 4 - 4t^2}$
 $= \int_0^{\sqrt{3}} \frac{2 \, dt}{9 + t^2}$
 $= \frac{2}{3} \left[\tan^{-1} \frac{t}{3} \right]_0^{\sqrt{3}}$
 $= \frac{2}{3} \left\{ \tan^{-1} \frac{\sqrt{3}}{3} - \tan^{-1} 0 \right\}$
 $= \frac{2}{3} \times \frac{\pi}{6}$
 $= \frac{\pi}{9}$

v) $\int \frac{x \, dx}{\sqrt{1-2x-x^2}}$
 $= \frac{1}{2} \int \frac{-2x-2}{\sqrt{1-2x-x^2}} \, dx - \int \frac{dx}{\sqrt{1-2x-x^2}}$
 $= -\sqrt{1-2x-x^2} - \sin^{-1} \left(\frac{x+1}{\sqrt{2}} \right) + c$

(vi) $\int \sec^3 x \, dx = \int \sec x \sec^2 x \, dx$
 $u = \sec x \quad v = \tan x$
 $du = \sec x \tan x \, dx \quad dv = \sec^2 x \, dx$
 $= \sec x \tan x - \int \sec^3 x \tan^2 x \, dx$
 $= \sec x \tan x - \int \sec^3 x \sec^2 x \, dx + \int \sec x \, dx$
 $= \sec x \tan x - \sec^3 x \sec x + \log(\sec x + \tan x) + \sec x \, dx$
 $= \frac{2}{3} \int \sec^3 x \, dx = \frac{2}{3} (\sec x \tan x + \log(\sec x + \tan x)) + c$

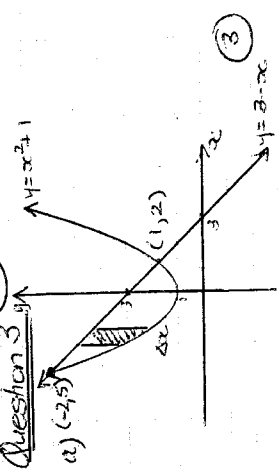
(vii) $\frac{A}{x-2} + \frac{Bx+C}{(x-2)(x^2+3)}$
 $A(x^2+3) + (Bx+C)(x-2) = 2x^2+3$
 $x=2 \quad x=0 \quad x=1$
 $7A=7 \quad 3A-2C=3 \quad 4A-B-C=5$
 $A=1 \quad 3-2C=3 \quad 4-B-C=5$
 $C=0 \quad B=-1$

$\int \frac{2x+3}{(x-2)(x^2+3)} \, dx = \int \left[\frac{1}{x-2} - \frac{x}{x^2+3} \right] \, dx$
 $= \log|x-2| - \frac{1}{2} \log|x^2+3| + c$

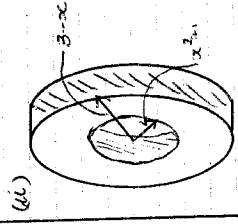
Question 2 (11)

i) $I_n = \int x^n \log(x) \, dx$
 $u = (\log x)^n \quad v = \frac{1}{2} x^2$
 $du = n(\log x)^{n-1} \frac{1}{x} \, dx \quad dv = x \, dx$
 $I_n = \left[\frac{1}{2} x^2 (\log x)^n \right] - \frac{n}{2} \int x (\log x)^{n-1} \, dx$
 $= \frac{x^2}{2} - \frac{n}{2} I_{n-1}$

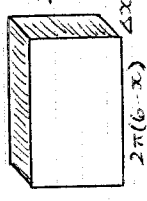
Question 3 (25)



a) $x^2 + 1 = 3 - x$
 $x^2 + x - 2 = 0$
 $(x+2)(x-1) = 0$
 $x = -2 \text{ or } x = 1$



(ii) $A(x) = \pi [(3-x)^2 - (x^2+1)^2]$
 $\Delta V = \pi (9 - 6x + x^2 - x^4 - 2x^2 - 1)$
 $= \pi (8 - 6x - x^2 - x^4) \Delta x$
 $V = \lim_{\Delta x \rightarrow 0} \sum_{x=2}^1 \pi (8 - 6x - x^2 - x^4) \Delta x$
 $= \pi \int_2^1 (8 - 6x - x^2 - x^4) \, dx$
 $= \pi \left[8x - 3x^2 - \frac{1}{3} x^3 - \frac{1}{5} x^5 \right]_2^1$
 $= \pi \left\{ \left(8 - 3 - \frac{1}{3} - \frac{1}{5} \right) - \left(16 - 12 + \frac{8}{3} + \frac{32}{5} \right) \right\}$
 $= \frac{117\pi}{5} \text{ units}^3$



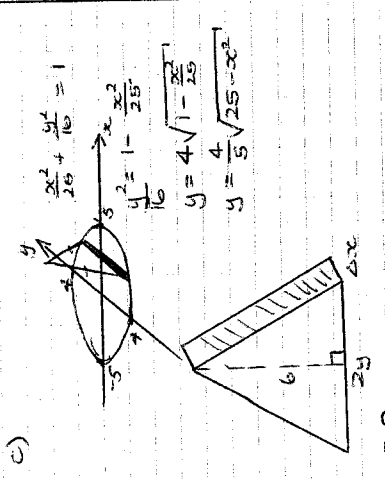
b) $A(x) = 4\pi(6-x)\sqrt{9-x^2}$
 $\Delta V = 4\pi(6-x)\sqrt{9-x^2} \Delta x$
 $V = \lim_{\Delta x \rightarrow 0} \sum_{x=3}^0 4\pi(6-x)\sqrt{9-x^2} \Delta x$
 $= 4\pi \int_3^0 (6-x)\sqrt{9-x^2} \, dx$

(iii) $\int x(\log x)^3 \, dx = I_3$
 $= \frac{x^2}{2} - \frac{3}{4} I_2$
 $= \frac{x^2}{2} - \frac{3x^2}{4} + \frac{3}{4} I_1$
 $= \frac{-x^2}{4} + \frac{3x^2}{4} - \frac{3}{4} I_0$
 $= \frac{x^2}{2} - \frac{3}{4} \int x \, dx$
 $= \frac{x^2}{2} - \frac{3}{4} \left[\frac{1}{2} x^2 \right]$
 $= \frac{x^2}{2} - \frac{3}{8} x^2 + \frac{3}{8} = \frac{x^2+3}{8}$

b) $\int_0.5^1 \frac{\log x}{1+x^2} \, dx$
 $u = \frac{1}{x} \Rightarrow x = \frac{1}{u}$
 $dx = -\frac{du}{u^2}$
 $= \int_2^1 \frac{-\log(\frac{1}{u})}{1 + \frac{1}{u^2}} \frac{du}{u^2}$
 $= \int_2^1 \frac{\log u}{u^2 + 1} \, du$

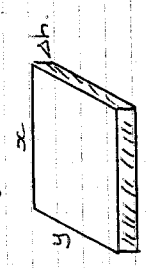
(ii) $\int_0.5^2 \frac{\log x}{1+x^2} \, dx$
 $= \int_0.5^2 \frac{\log x}{1+x^2} \, dx + \int_2^1 \frac{\log x}{1+x^2} \, dx$
 $= \int_0.5^2 \frac{\log x}{1+x^2} \, dx - \int_1^2 \frac{\log x}{1+x^2} \, dx$
 $= -\int_1^2 \frac{\log x}{1+x^2} \, dx + \int_1^2 \frac{\log x}{1+x^2} \, dx$
 $= 0$

(ii) $V = 4\pi \int_0^3 (6-x)\sqrt{9-x^2} dx$
 $= 24\pi \int_0^3 \sqrt{9-x^2} dx - 4\pi \int_0^3 x\sqrt{9-x^2} dx$
 Semicircle odd/even = odd/even function
 $= 24\pi \times \frac{1}{2} \pi (3)^2 - 0$
 $= 108\pi^2 \text{ units}^3$ (3)



$A(x) = \frac{1}{2} \times \frac{8}{5} \sqrt{25-x^2} \times 6$
 $= \frac{24}{5} \sqrt{25-x^2}$
 $\Delta V = \frac{24}{5} \sqrt{25-x^2} \Delta x$
 $V = \lim_{\Delta x \rightarrow 0} \sum_{x=0}^5 \frac{24}{5} \sqrt{25-x^2} \Delta x$
 $= \frac{24}{5} \int_0^5 \sqrt{25-x^2} dx$
 $= \frac{24}{5} \times \frac{1}{2} \pi (5)^2$
 $= 60\pi \text{ units}^3$ (5)

d) $m = \frac{4}{30-16}$
 $= 1$
 $h-0 = 1(x-16)$
 $= x-16$
 $x = h+16$ (2)
 (ii) y relationship
 $m = \frac{4}{12-10}$
 $= 2$
 $h-0 = 2(y-10)$
 $h = 2y-20$
 $y = \frac{1}{2}h+10$



$A(h) = (h+16)(\frac{1}{2}h+10)$
 $= \frac{1}{2}h^2 + 18h + 160$
 $\Delta V = (\frac{1}{2}h^2 + 18h + 160) \Delta h$
 $V = \lim_{\Delta h \rightarrow 0} \sum_{h=0}^4 (\frac{1}{2}h^2 + 18h + 160) \Delta h$
 $= \int_0^4 (\frac{1}{2}h^2 + 18h + 160) dh$
 $= [\frac{1}{6}h^3 + 9h^2 + 160h]_0^4$
 $= \frac{32}{3} + 144 + 640$
 $= 794 \frac{2}{3} \text{ units}^3$ (4)