



GIRRAWEEN HIGH SCHOOL

YEAR 12 - TASK 3

2006

MATHEMATICS

Extension 2

Time allowed – 90 minutes

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Board-approved calculators may be used.
- Start each question on a new sheet of paper.

Question 1 (28 marks)**Marks**

- (a) $\int \sin^4 x dx$ 4
- (b) $\int x^2 \cos x dx$ 4
- (c) $\int \tan^{-1} x dx$ 4
- (d) $\int \cos^2 x \cot x dx$ 4
- (e) $\int \frac{1}{2 + \cos x} dx$ 4
- (f) $\int \frac{5 + 3x}{1 - 9x^2} dx$ 4
- (g) $\int \frac{1}{\sqrt{5 + 4x - x^2}} dx$ 4

Question 2 (14 marks)

- (a) Show that
- (i) $\int x^n e^x dx = x^n e^x - n \int x^{n-1} e^x dx$ 2
- (ii) and hence solve $\int x^3 e^x dx$ 3
- (b) Calculate the area bounded by the curve $y = \frac{1}{x^2(x-3)}$,
and the x -axis and the ordinates $x = 4$ and $x = 6$. 5
- (c) Calculate the area of the region bounded by the curve $y = xe^{-x}$,
the x -axis and the line $x = 1$ 4

Question 3 (20 marks)

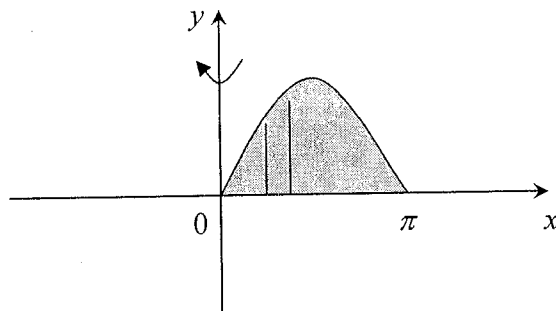
Marks

- (a) Find the volume generated when the area in the first quadrant bounded by $y = x$ and $y = x^3$ is revolved about the x -axis.

4

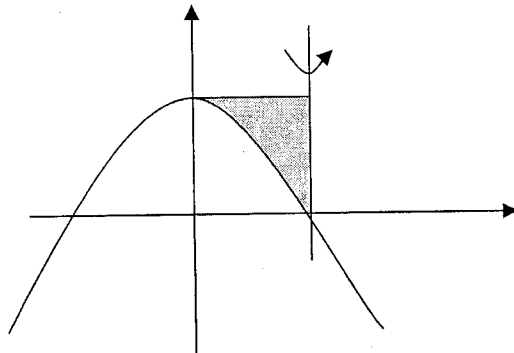
- (b) Find the volume generated when the area bounded by $y = \sin x$, $y = 0$, between $x = 0$ and $x = \pi$ is revolved about the y -axis, using cylindrical shells.

6



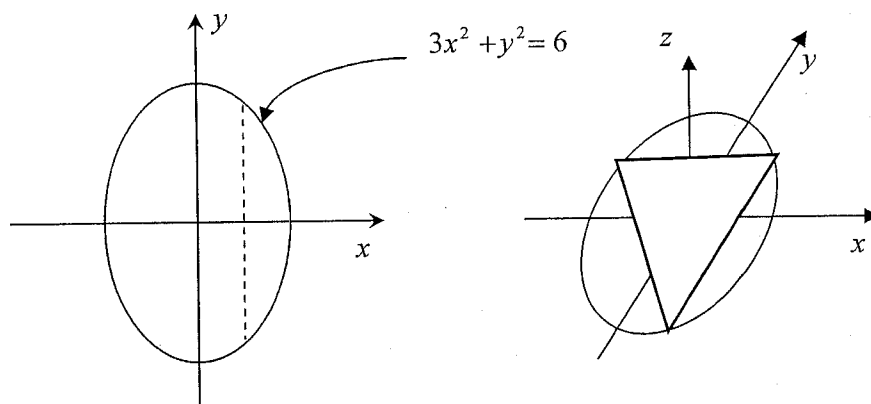
- (c) The region bounded by $y = 1 - x^2$ and the lines $x = 1, y = 1$ is rotated about $x = 1$. Find the volume generated.

5



- (d) The base of a solid is the region enclosed by the ellipse $3x^2 + y^2 = 6$. Find the volume of the solid if all the plane sections perpendicular to the x -axis are equilateral triangles.

5



TASK 3 (MSC)

SOLUTIONS

Q1 a) $\int \sin^4 x dx$

$= \frac{1}{4} \int (1 - \cos 2x)^2 dx$

$= \frac{1}{4} \int (1 - 2\cos 2x + \cos^2 2x) dx$

$= \frac{1}{4} \int (1 - 2\cos 2x + \frac{1}{2}(1 + \cos 4x)) dx$

$= \frac{1}{4} \left[\frac{3x}{2} - \sin 2x + \frac{1}{8} \sin 4x \right] + C$ (4)

$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$\therefore \sin^4 x = \frac{1}{4}(1 - \cos 2x)^2$

$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

$\therefore \cos^2 2x = \frac{1}{2}(1 + \cos 4x)$

b) $\int x^2 \cos x dx$

$u = x^2 \quad v' = \cos x$
 $u' = 2x \quad v = \sin x$

$= x^2 \sin x - \int 2x \sin x dx$

$= x^2 \sin x - [2x \cdot (-\cos x) - \int 2 \cdot (-\cos x) dx]$

$= x^2 \sin x + 2x \cos x - 2 \sin x + C$ (4)

c) $\int \tan^{-1} x dx$

$u = \tan^{-1} x \quad v' = 1$
 $u' = \frac{1}{1+x^2} \quad v = x$

$= x \tan^{-1} x - \int \frac{x}{1+x^2} dx$

$= x \tan^{-1} x - \frac{1}{2} \log_e(1+x^2) + C$ (4)

Q2 $\int \cos^2 x \cot x dx$

$= \int \frac{\cos^3 x}{\sin x} dx$

$= \int (1 - \sin^2 x) \cdot \cos x dx$ Let $u = \sin x$
 $du = \cos x dx$

$= \int \frac{1-u^2}{u} du$

$= \int \frac{1}{u} - u du$

$= \log_e u - \frac{u^2}{2} + C$

$= \log_e(\sin x) - \frac{1}{2} \sin^2 x + C$ (4)

e) $\int \frac{1}{2+\cos x} dx$

$= \int \frac{1}{2+1-t^2} \cdot \frac{2dt}{1+t^2}$

$= \int \frac{2 dt}{2(1+t^2)+1-t^2}$

$= \int \frac{2 dt}{2+2t^2+1-t^2}$

$= 2 \int \frac{dt}{3+t^2}$

$= 2 \cdot \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{t}{\sqrt{3}} \right) + C$

$= \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{\tan \frac{x}{2}}{\sqrt{3}} \right) + C$

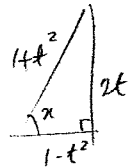
$= \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{1}{\sqrt{3}} \tan \frac{x}{2} \right) + C$ (4)

$t = \tan \frac{x}{2}$

$\frac{dt}{dx} = \frac{1}{2} \sec^2 \frac{x}{2}$

$dt = \frac{1}{2} (1 + \tan^2 \frac{x}{2}) dx$

$dx = \frac{2dt}{1+t^2}$



f) $\int \frac{5+3x}{1-9x^2} dx$

$= \int \frac{5+3x}{(1-3x)(1+3x)} dx$

$= \int \frac{3}{1-3x} + \frac{2}{1+3x} dx$

$= -\log_e(1-3x) + \frac{2}{3} \log_e(1+3x) + C$

$= \frac{2}{3} \log_e(1+3x) - \log_e(1-3x) + C$

$\frac{5+3x}{(1-3x)(1+3x)} = \frac{a}{(1-3x)} + \frac{b}{(1+3x)}$

$5+3x = a(1+3x) + b(1-3x)$

when $x = \frac{1}{3}$ $6 = 2a$

$a = 3$

when $x = -\frac{1}{3}$ $4 = 2b$

$b = 2$

(4)

g) $\int \frac{1}{\sqrt{5+4x-x^2}} dx$

$5+4x-x^2$

$= 9 - (x^2 - 4x + 4)$

$= 9 - (x-2)^2$

$= \int \frac{1}{\sqrt{9-(x-2)^2}} dx$

$= \int \frac{du}{\sqrt{3^2-u^2}}$

Let $u = (x-2)$

$du = dx$

$= \sin^{-1} \left(\frac{u}{3} \right) + C$

$= \sin^{-1} \left(\frac{x-2}{3} \right) + C$ (4)

(4)

Q2

$$a) i) \int x^n e^x dx$$

$$= x^n e^x - \int n x^{n-1} e^x dx$$

$$u = x^n \quad v' = e^x$$

$$u' = n x^{n-1} \quad v = e^x$$

$$I_n = x^n e^x - n \int x^{n-1} e^x dx$$

$$I_n = x^n e^x - n I_{n-1} \quad (2)$$

$$ii) I_3 = \int x^3 e^x dx$$

$$I_3 = x^3 e^x - 3 I_2 \quad ; \quad I_2 = x^2 e^x - 2 I_1$$

$$= x^3 e^x - 3 [x^2 e^x - 2 I_1] \quad ; \quad I_1 = x e^x - I_0$$

$$= x^3 e^x - 3 x^2 e^x + 6 [x e^x - I_0] \quad I_0 = \int x^0 e^x dx$$

$$= x^3 e^x - 3 x^2 e^x + 6 x e^x - 6 e^x + C \quad ; \quad I_0 = \int e^x dx$$

$$(3) \quad I_0 = e^x$$

$$c) y = x e^{-x}$$

when $y=0, x=0$

$$A = \int_0^1 x e^{-x} dx$$

$$u = x \quad v' = e^{-x}$$

$$u' = 1 \quad v = -e^{-x}$$

$$= -x e^{-x} - \int -e^{-x} dx$$

$$= [-x e^{-x} - e^{-x}]_0^1$$

$$= (-e^{-1} - e^{-1}) - (0 - 1)$$

$$= -\frac{1}{e} - \frac{1}{e} + 1$$

$$= 1 - \frac{2}{e}$$

(4)

$$b) y = \frac{1}{x^2(x-3)} = \frac{ax+b}{x^2} + \frac{c}{x-3}$$

$$1 = (ax+b)(x-3) + cx^2$$

$$A = \int_4^6 \frac{1}{x^2(x-3)} dx$$

when $x=3$

$$1 = 9c$$

$$c = \frac{1}{9}$$

$$= \int_4^6 \left[-\frac{1}{9} \frac{x-3}{x^2} + \frac{1}{9} \frac{1}{x-3} \right] dx \quad x=0$$

$x=0$

$$1 = -3b$$

$$b = -\frac{1}{3}$$

$$= \frac{1}{9} \int_4^6 \left[-\frac{x-3}{x^2} + \frac{1}{x-3} \right] dx \quad x=1$$

$x=1$

$$1 = (a+b)(-2) + c$$

$$= \frac{1}{9} \int_4^6 \left[-\frac{1}{x} - 3x^{-2} + \frac{1}{x-3} \right] dx \quad a = \frac{1}{9}$$

$a = \frac{1}{9}$

$$1 = -2a - 2b + c$$

$b = -\frac{1}{3}$

$$1 = -2a + \frac{2}{3} + \frac{1}{9}$$

$$= \frac{1}{9} \left[-\log_e(x) + \frac{3}{x} + \log_e(x-3) \right]_4^6 \quad 9 = -18a + b + 1$$

$$9 = -18a + b + 1$$

$$= \frac{1}{9} \left[\log_e \left(\frac{x-3}{x} \right) + \frac{3}{x} \right]_4^6 \quad -18a = 2$$

$$-18a = 2$$

$$a = -\frac{1}{9}$$

$$= \frac{1}{9} \left[\left(\log_e \frac{1}{2} + \frac{1}{2} \right) - \left(\log_e \frac{1}{4} + \frac{3}{4} \right) \right]$$

$$= \frac{1}{9} \left[\log_e 2 - \frac{1}{4} \right]$$

(5)

$$(Q3) V = \lim_{\Delta x \rightarrow 0} \sum_{i=1}^n \pi (R^2 - r^2) \Delta x$$

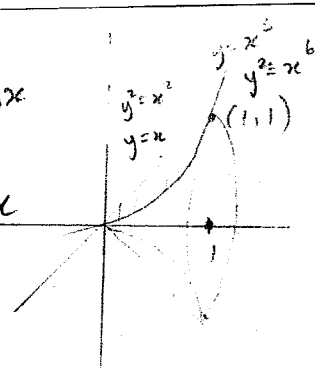
$$V = \lim_{\Delta x \rightarrow 0} \sum_{i=1}^n \pi (x^2 - x^6) \Delta x$$

$$= \pi \int_0^1 x^2 - x^6 dx$$

$$= \pi \left[\frac{x^3}{3} - \frac{x^7}{7} \right]_0^1$$

$$= \pi \left[\frac{1}{3} - \frac{1}{7} - (0 - 0) \right]$$

$$= \frac{4\pi}{21}$$



(4)

$$V = \lim_{\Delta x \rightarrow 0} \sum_{i=1}^n 2\pi x_i \cdot y_i \cdot \Delta x$$

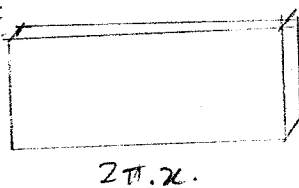
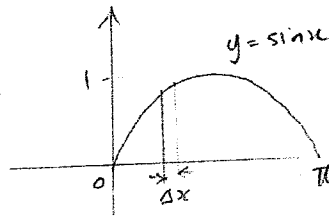
$$V = 2\pi \int_0^{\pi} x \cdot \sin x dx$$

$$= 2\pi \left[-x \cos x - \int -\cos x dx \right]_0^{\pi}$$

$$= 2\pi \left[-x \cos x + \sin x \right]_0^{\pi}$$

$$= 2\pi \left[(-\pi + 0) - (0 + 0) \right]$$

$$= -2\pi^2$$



$$u = x \quad v = \sin x$$

$$u' = 1 \quad v' = -\cos x$$

(6)

$$V = \lim_{\Delta y \rightarrow 0} \sum_{i=1}^n (c - x_i)^2 \Delta y$$

$$c) V = \lim_{\Delta x \rightarrow 0} \sum_{i=1}^n (1 - \sqrt{1-y})^2 \Delta y$$

$$= \pi \int_0^1 (1 - \sqrt{1-y})^2 dy$$

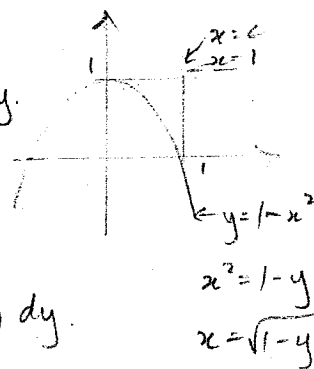
$$= \pi \int_0^1 1 - 2\sqrt{1-y} + (1-y) dy$$

$$= \pi \int_0^1 2 - y - 2\sqrt{1-y} dy$$

$$= \pi \left[2y - \frac{y^2}{2} - \frac{2(1-y)^{3/2}}{3/2(-1)} \right]_0^1$$

$$= \pi \left[\left(2 - \frac{1}{2} + 0 \right) - \left(0 - 0 + \frac{4}{3} \right) \right]$$

$$= \frac{\pi}{6}$$



$$d) 3x^2 + y^2 = 6$$

$$\frac{x^2}{2} + \frac{y^2}{6} = 1$$

$$\therefore a = \sqrt{2} \quad b = \sqrt{6}$$

$$V = \lim_{\Delta x \rightarrow 0} \sum_{i=1}^n A(x_i) \Delta x$$

$$V = \lim_{\Delta x \rightarrow 0} \sum_{i=1}^n \sqrt{3} (6 - 3x^2) \Delta x$$

$$V = \int_{-\sqrt{2}}^{\sqrt{2}} \sqrt{3} (6 - 3x^2) dx$$

$$= \sqrt{3} \int_{-\sqrt{2}}^{\sqrt{2}} 6 - 3x^2 dx$$

$$= 3\sqrt{3} \int_{-\sqrt{2}}^{\sqrt{2}} 2 - x^2 dx$$

even fn.

$$= 6\sqrt{3} \int_0^{\sqrt{2}} 2 - x^2 dx$$

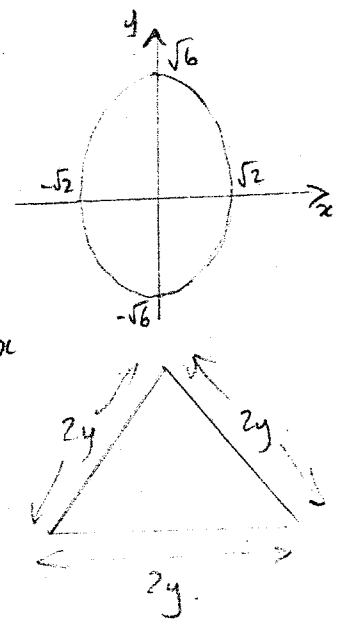
$$= 6\sqrt{3} \left[2x - \frac{x^3}{3} \right]_0^{\sqrt{2}}$$

$$= 6\sqrt{3} \left[2\sqrt{2} - \frac{2\sqrt{2}}{3} - (0) \right]$$

$$= 6\sqrt{3} \left(\frac{6\sqrt{2} - 2\sqrt{2}}{3} \right)$$

$$= 2\sqrt{3} (4\sqrt{2})$$

$$= 8\sqrt{6}$$



$$\text{Area} = \frac{1}{2} (2y)(2y) \sin 60^\circ$$

$$= 2y^2 \cdot \left(\frac{\sqrt{3}}{2} \right)$$

$$= \sqrt{3} y^2$$

$$\text{Since } y^2 = 6 - 3x^2$$

$$\therefore \text{Area} = \sqrt{3} (6 - 3x^2)$$

(5)