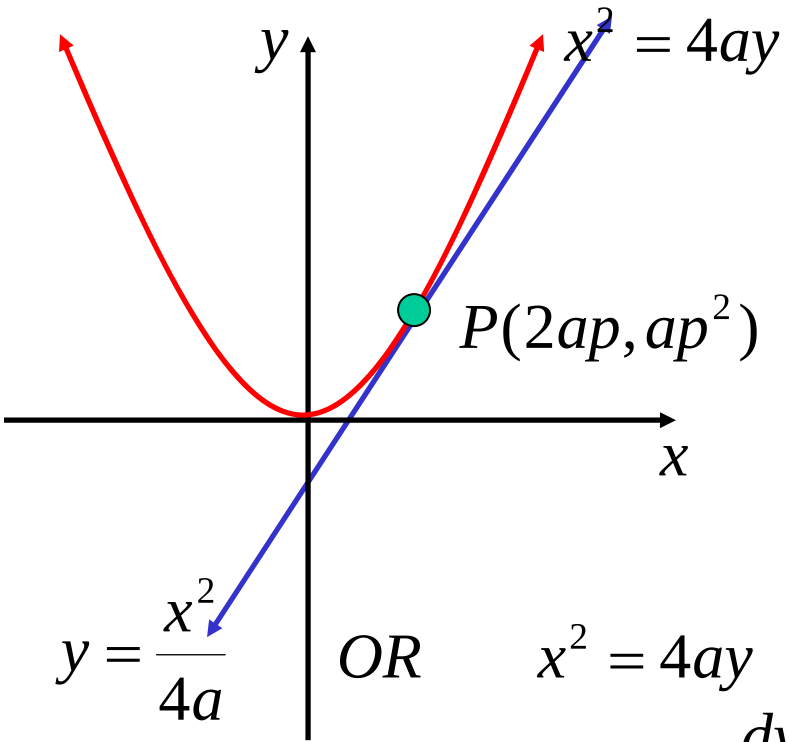
Tangents & Normals

(i) Using Parametrics

(1) Tangent



$$y = \frac{x^2}{4a}$$

$$\frac{dy}{dx} = \frac{x}{2a}$$

OR

$$x^2 = 4ay$$

$$2x = 4a\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{x}{2a}$$

when $x = 2ap, \frac{dy}{dx} = p$

OR

$$x = 2at \qquad y = at^2$$

$$\frac{dx}{dt} = 2a \qquad \frac{dy}{dt} = 2at$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$\frac{dy}{dx} = 2at \times \frac{1}{2a}$$

$$= t$$

when $t = p, \frac{dy}{dx} = p$

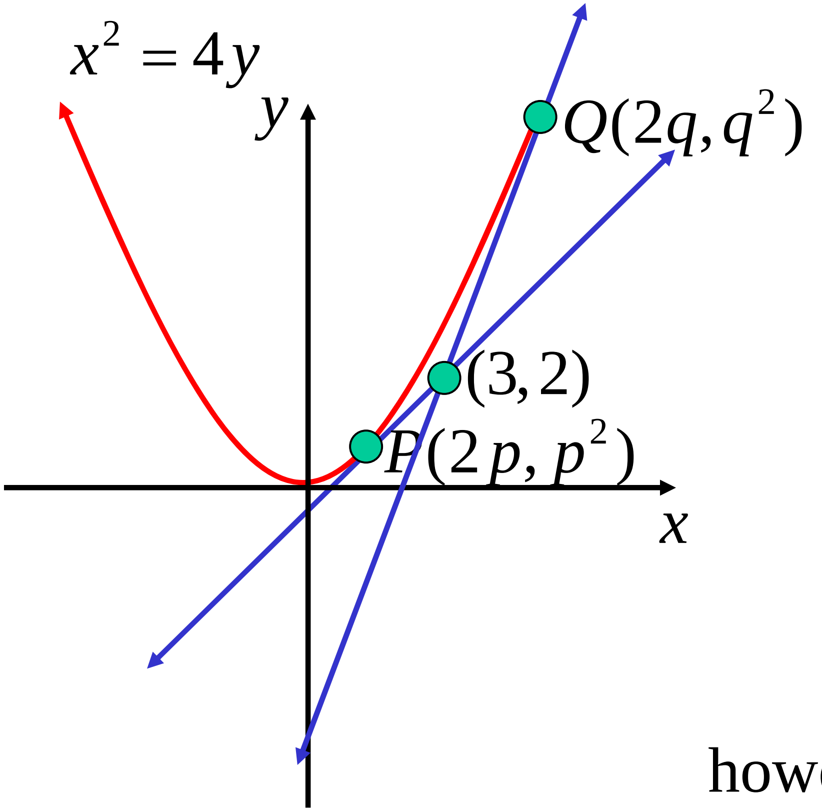
$\therefore$ slope of tangent is p

$$y - ap^2 = p(x - 2ap)$$

$$y - ap^2 = px - 2ap^2$$

$$\boxed{y = px - ap^2}$$

(2) Tangents from an external point



$$y = \frac{x^2}{4}$$

$$\frac{dy}{dx} = \frac{x}{2}$$

$$\text{when } x = 2p, \frac{dy}{dx} = p$$

$$y - p^2 = p(x - 2p)$$

$$y - p^2 = px - 2p^2$$

$$y = px - p^2$$

however tangent passes through (3,2)

$$2 = 3p - p^2$$

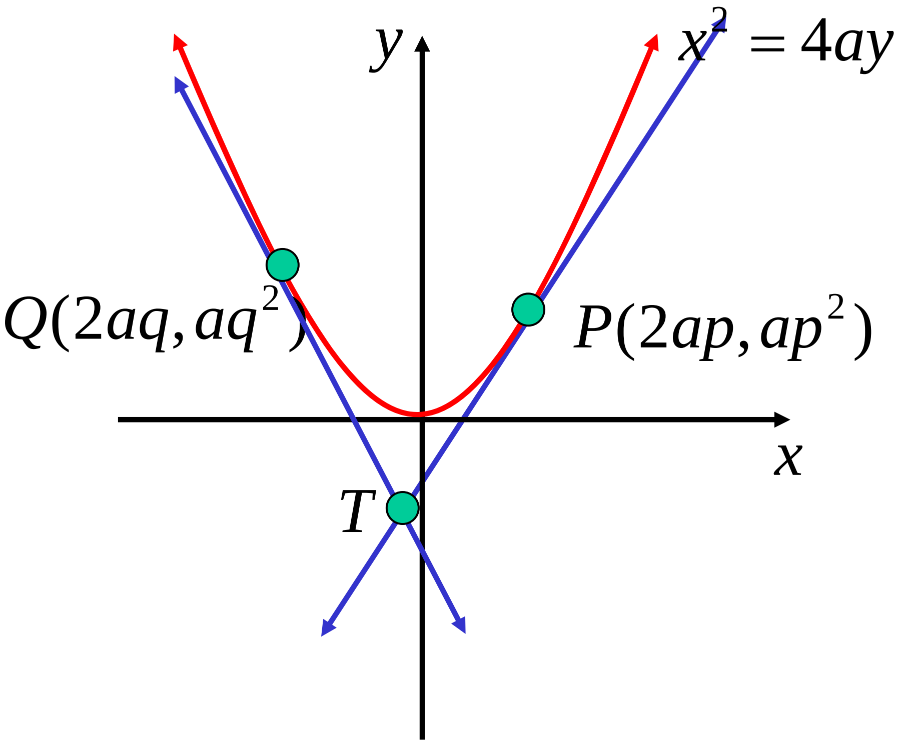
$$p^2 - 3p + 2 = 0$$

$$(p-2)(p-1) = 0$$

$$p = 1 \quad \text{or} \quad p = 2$$

$\therefore$ tangents are $y = x - 1$ and $y = 2x - 4$

(3) Intersection of tangents



1 Show tangent at P is $y = px - ap^2$

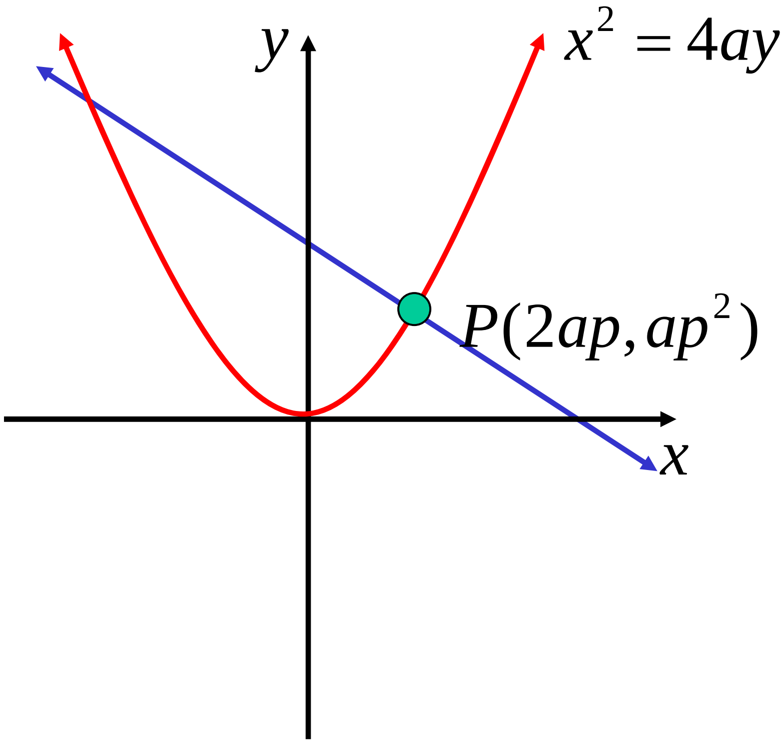
2 $\therefore$ tangent at Q is $y = qx - aq^2$

3 Solve simultaneously

$$px - y = ap^2$$
$$qx - y = aq^2$$
$$(p-q)x = a(p^2 - q^2)$$
$$(p-q)x = a(p+q)(p-q)$$
$$x = a(p+q)$$
$$y = ap(p+q) - ap^2$$
$$y = ap^2 + apq - ap^2$$
$$= apq$$

$$\therefore T = \{a(p+q), apq\}$$

(4) Normal



(1) Show the slope of tangent at P is p

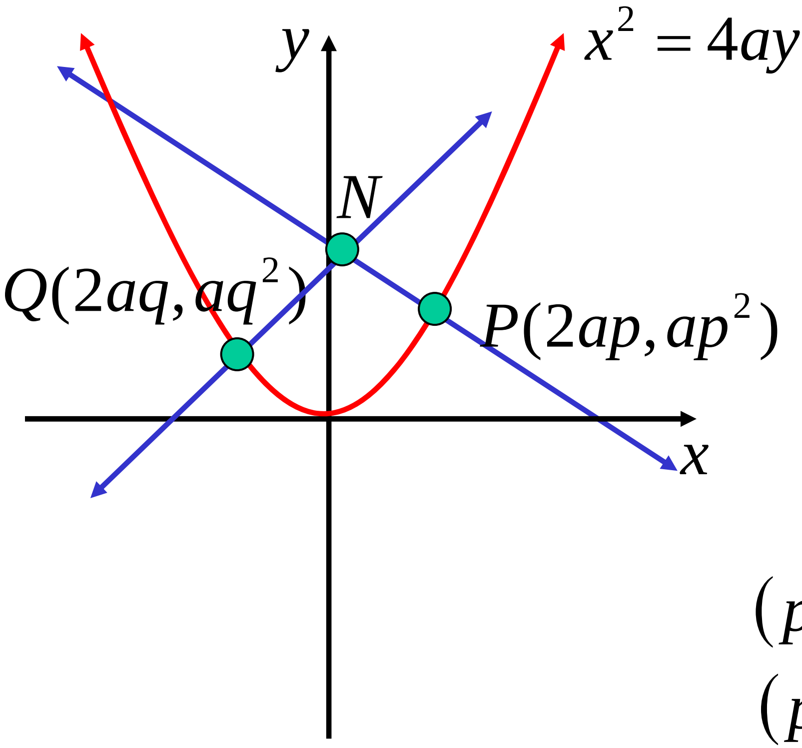
(2) $\therefore$ slope of normal is $-\dfrac{1}{p}$

$$y - ap^2 = \frac{-1}{p}(x - 2ap)$$

$$py - ap^3 = -x + 2ap$$

$$\boxed{x + py = ap^3 + 2ap}$$

(5) Intersection of normals



1 Show normal at P is $x + py = ap^3 + 2ap$

2 $\therefore$ normal at Q is $x + qy = aq^3 - 2aq$

3 Solve simultaneously

$$x + py = ap^3 + 2ap$$

$$\underline{x + qy = aq^3 + 2aq}$$

$$(p-q)y = a(p^3 - q^3) + 2a(p-q)$$

$$(p-q)y = a(p-q)(p^2 + pq + q^2) + 2a(p-q)$$

$$y = a(p^2 + pq + q^2 + 2)$$

$$x + ap(p^2 + pq + q^2 + 2) = ap^3 + 2ap$$

$$x = ap^3 + 2ap - ap^3 - ap^2q - apq^2 - 2ap$$

$$= -apq(p+q)$$

$$\therefore N = \{-apq(p+q), a(p^2 + pq + q^2 + 2)\}$$

Exercise 9F; 1ac, 2ac, 3, 6, 8, 9ac, 11, 12, 13