Some Different Looking Induction Problems

2006 Extension 1 HSC Q4d)

(i) Use the fact that $\tan(\alpha - \beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$ to show that

$$1 + \tan n\theta \tan(n+1)\theta = \cot\theta\left(\tan(n+1)\theta - \tan n\theta\right)$$

$$\tan\left((n+1)\theta - n\theta\right) = \frac{\tan(n+1)\theta - \tan n\theta}{1 + \tan(n+1)\theta\tan n\theta}$$

$$\tan\theta = \frac{\tan(n+1)\theta - \tan n\theta}{1 + \tan(n+1)\theta\tan n\theta}$$

$$\left(1 + \tan(n+1)\theta\tan n\theta\right)\tan\theta = \tan(n+1)\theta - \tan n\theta$$

$$1 + \tan(n+1)\theta\tan n\theta = \frac{\tan(n+1)\theta - \tan n\theta}{\tan\theta}$$

$$\underline{1 + \tan(n+1)\theta\tan n\theta = \cot\theta\left(\tan(n+1)\theta - \tan n\theta\right)}$$

2006 Extension 1 Examiners Comments;

(d) (i) This part was not done well, with many candidates presenting circular arguments.

However, a number of candidates successfully rearranged the given fact and easily saw the substitution required.

Many candidates spent considerable time on this part, which was worth only one mark.

(ii) Use mathematical induction to prove that for all integers $n \geq 1$

$$\tan\theta\tan 2\theta + \tan 2\theta\tan 3\theta + \ldots + \tan n\theta\tan(n+1)\theta$$
$$= -(n+1) + \cot\theta\tan(n+1)\theta$$

Step 1: Prove the result is true for $n = 1$

$LHS = \tan\theta\tan 2\theta$

$= 1 + \tan\theta\tan 2\theta - 1$

$= \cot\theta(\tan 2\theta - \tan\theta) - 1$

$= \cot\theta\tan 2\theta - 1 - 1$

$= \cot\theta\tan 2\theta - 2$

$RHS = -2 + \cot\theta\tan 2\theta$

Hence the result is true for $n = 1$

Step 2: Assume the result is true for $n = k$, where k is a positive integer

i.e. $\tan\theta\tan 2\theta + \tan 2\theta\tan 3\theta + \ldots + \tan k\theta\tan(k+1)\theta$
$$= -(k+1) + \cot\theta\tan(k+1)\theta$$

Step 3: Prove the result is true for $n = k + 1$

i.e. $\tan\theta\tan 2\theta + \tan 2\theta\tan 3\theta + \ldots + \tan(k+1)\theta\tan(k+2)\theta$
$$= -(k+2) + \cot\theta\tan(k+2)\theta$$

Proof:

$$\tan\theta\tan 2\theta+\tan 2\theta\tan 3\theta+\ldots+\tan(k+1)\theta\tan(k+2)\theta=$$

$$\tan\theta\tan 2\theta+\tan 2\theta\tan 3\theta+\ldots+\tan k\theta\tan(k+1)\theta+\tan(k+1)\theta\tan(k+2)\theta$$

$$=-(k+1)+\cot\theta\tan(k+1)\theta+\tan(k+1)\theta\tan(k+2)\theta$$

$$=-(k+1)+\cot\theta\tan(k+1)\theta+\cot\theta\left(\tan(k+2)\theta-\tan(k+1)\theta\right)-1$$

$$=-(k+2)+\cot\theta\tan(k+1)\theta+\cot\theta\tan(k+2)\theta-\cot\theta\tan(k+1)\theta$$

$$=-(k+2)+\cot\theta\tan(k+2)\theta$$

Hence the result is true for $n = k + 1$ if it is also true for $n = k$

Step 4: Since the result is true for n= 1, then the result is true for all positive integral values of n, by induction

2006 Extension 1 Examiners Comments;

(ii) Most candidates attempted this part. However, many only earned the mark for stating the inductive step.

They could not prove the expression true for $n=1$ as they were not able to use part (i), and consequently not able to prove the inductive step.

2004 Extension 1 HSC Q4a)

Use mathematical induction to prove that for all integers $n \geq 3$;

$$\left(1-\frac{2}{3}\right)\left(1-\frac{2}{4}\right)\left(1-\frac{2}{5}\right)\ldots\left(1-\frac{2}{n}\right)=\frac{2}{n(n-1)}$$

Step 1: Prove the result is true for $n = 3$

$$LHS = 1-\frac{2}{3} \qquad\qquad RHS = \frac{2}{3(2)}$$

$$= \frac{1}{3} \qquad\qquad = \frac{1}{3}$$

$$\therefore LHS = RHS$$

Hence the result is true for $n = 3$

Step 2: Assume the result is true for $n = k$, where k is a positive integer

$$i.e. \left(1-\frac{2}{3}\right)\left(1-\frac{2}{4}\right)\left(1-\frac{2}{5}\right)\ldots\left(1-\frac{2}{k}\right)=\frac{2}{k(k-1)}$$

Step 3: Prove the result is true for $n = k + 1$

$$i.e.\text{ Prove }\left(1-\frac{2}{3}\right)\left(1-\frac{2}{4}\right)\left(1-\frac{2}{5}\right)\cdots\left(1-\frac{2}{k+1}\right)=\frac{2}{(k+1)k}$$

Proof:

$$\left(1-\frac{2}{3}\right)\left(1-\frac{2}{4}\right)\left(1-\frac{2}{5}\right)\cdots\left(1-\frac{2}{k+1}\right)$$

$$=\left(1-\frac{2}{3}\right)\left(1-\frac{2}{4}\right)\left(1-\frac{2}{5}\right)\cdots\left(1-\frac{2}{k}\right)\left(1-\frac{2}{k+1}\right)$$

$$=\frac{2}{k(k-1)}\times\left(1-\frac{2}{k+1}\right)$$

$$=\frac{2}{k(k-1)}\times\frac{k+1-2}{(k+1)}$$

$$=\frac{2}{k(k-1)}\times\frac{(k-1)}{(k+1)}$$

$$=\frac{2}{k(k+1)}$$

Hence the result is true for $n = k + 1$ if it is also true for $n = k$

Step 4: Since the result is true for $n = 3$, then the result is true for all positive integral values of $n > 2$ by induction.

2004 Extension 1 Examiners Comments;

(a) Many candidates submitted excellent responses including all the necessary components of an induction proof.

Weaker responses were those that did not verify for $n = 3$, had an incorrect statement of the assumption for $n = k$ or had an incorrect use of the assumption in the attempt to prove the statement.

Poor algebraic skills made it difficult or even impossible for some candidates to complete the proof.

Rote learning of the formula $S(k) + T(k+1) = S(k+1)$ led many candidates to treat the expression as a sum rather than a product, thus making completion of the proof impossible.