HORNSBY GIRLS 4UNIT 2005

Total Marks – 120

Attempt Questions 1 – 8
All Questions are of equal value

Begin each question on a NEW SHEET of paper, writing your student number and question number at the top of the page. Extra paper is available.

Question 1 (15 marks) Use a SEPARATE sheet of paper.

Marks

(a) Find $\int \frac{dx}{x^2-4x+40}$ — 2

(b) Evaluate $\int_0^2 x^3 e^{x^2}\,dx$. — 3

(c) Find $\int \sin^3 x\,dx$ — 2

(d) Evaluate $\int_0^1 \frac{x}{\sqrt{4-x}}\,dx$ — 3

(e) (i) Find the real numbers a, b and c such that
$\frac{3x^2+2x+11}{(x^2+3)(1-x)} \equiv \frac{ax+b}{x^2+3}+\frac{c}{1-x}$. — 3

(ii) Hence find $\int \frac{3x^2+2x+11}{(x^2+3)(1-x)}\,dx$. — 2

Question 2 (15 marks) Use a SEPARATE sheet of paper.

Marks

(a) Given z is a complex number such that $z=1+i$

(i) Write z in mod-arg form — 2

(ii) Evaluate z^{12} — 2

(b) If $P(z)=z^4-30z^2+289$

(i) Show that $z=4+i$ is a zero of $P(z)$ — 2

(ii) Find all zeros of $P(z)$ over the complex field — 5

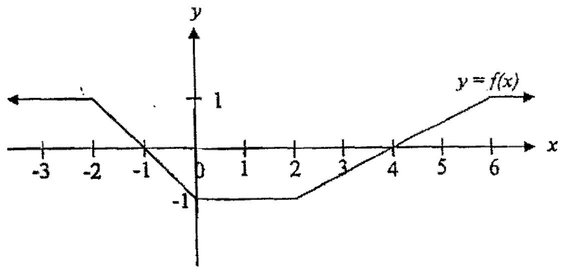
(c) $P(z)$ is a point on the argand diagram such that — 4

$$\arg\frac{z-i}{z+2}=\frac{\pi}{2}$$

Draw and describe the locus of $P(z)$.

Question 3 (15 marks) Use a SEPARATE sheet of paper. **Marks**

(a) The diagram below is a sketch of the function $y=f(x)$

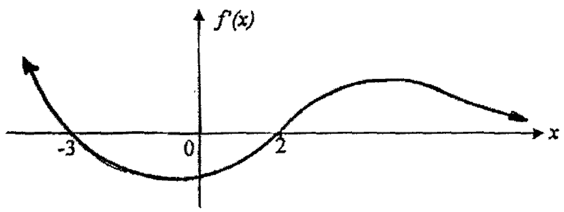


On separate diagrams sketch

(i) $y=|f(x)|$ **2**

(ii) $y=f(|x|)$ **1**

(b) The graph below represents the derivative $f'(x)$ of a certain function $f(x)$. Given that $f'(x)\to 0$ as $x\to\infty$, $f(0)=0$ and $f(1)<0$, sketch the graph of $f(x)$, noting the behaviour as $x\to\infty$. **3**



(c) (i) Sketch the curve $y=\dfrac{x^3+4}{x^2}$, showing any stationary points and asymptotic behaviour. **2**

(ii) Hence or otherwise, deduce the values of k, for which the equation $x^3-kx^2+4=0$ may have one real root. **1**

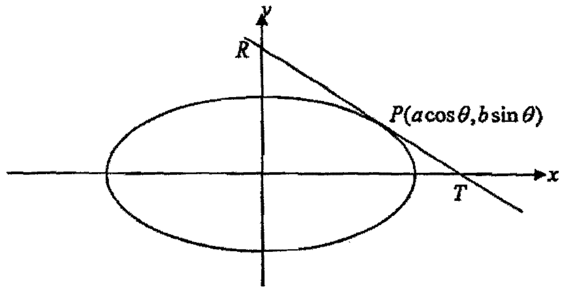
(d) (i) If $x=a$ is a multiple root of the polynomial equation $P(x)$ such that $P(x)=0$, prove that $P'(a)=0$. **3**

(ii) Find all roots of $P(x)=16x^3-12x^2+1$ given that two of the roots are equal. **3**

Question 4 (15 marks) Use a SEPARATE sheet of paper. **Marks**

(a) An ellipse has parametric equations $x=\sqrt{2}\cos\theta$ and $y=3\sin\theta$. **2**
Find the Cartesian equation and the eccentricity of the ellipse.

(b)



The ellipse with equation $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ shown in the diagram above has a tangent at the point $P(a\cos\theta, b\sin\theta)$. The tangent cuts the x-axis at T and the y-axis at R.

(i) Show that the equation of the tangent at the point P is **2**
$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1.$$

(ii) If T is the point of intersection between the tangent at point P and one of the directrices of the ellipse, show that $\cos\theta=e$. **3**

(iii) Hence find the angle that the focal chord through P makes with the x-axis. **1**

(iv) Using similar triangles or otherwise, show that $RP=e^2RT$. **3**

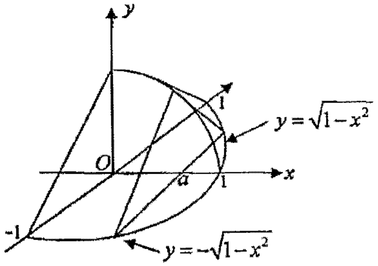
(c) The area between the curve $y=\ln(x+1)$ and the x-axis, between $x=0$ and $x=1$ is rotated about the y-axis. **4**
Find the volume of the solid of revolution formed using the method of cylindrical shells.

Question 5 (15 marks) Use a SEPARATE sheet of paper. **Marks**

(a) (i) Write down the value of $\int_0^a \sqrt{a^2-x^2}\,dx$. 1

(ii) Explain why $\int_{-a}^{a} x\sqrt{a^2-x^2}\,dx$ is equal to zero. 1

(b)



The base of a solid is the semi-circular region in the x - y plane with the straight edge running from the point $(0,-1)$ to the point $(0,1)$ and the point $(1,0)$ on the curved edge of the semicircle.

Each cross-section perpendicular to the x-axis is an isosceles triangle with each of the two equal side lengths three quarters the length of the third side.

(i) Show that the area of the triangular cross-section at $x=a$ is $\frac{\sqrt{5}}{2}(1-a^2)$. 2

(ii) Hence find the volume of the solid. 2

(c) The point $T\left(ct,\frac{c}{t}\right)$ lies on the hyperbola $xy=c^2$. The tangent at T meets the x-axis at P and the y-axis at Q. The normal at T meets the line $y=x$ at R.

(i) Prove that the tangent at T has equation $x+t^2y=2ct$. 2

(ii) Find the coordinates of P and Q. 2

(iii) Write down the equation of the normal at T. 1

(iv) Show that the x coordinate of R is $x=\frac{c}{t}\left(t^2+1\right)$. 2

(v) Prove that ΔPQR is isosceles. 2

Question 6 (15 marks) Use a SEPARATE sheet of paper. **Marks**

(a) The equation $x^3+2x-1=0$ has roots α,β,γ.

Find a polynomial equation in x whose roots are:

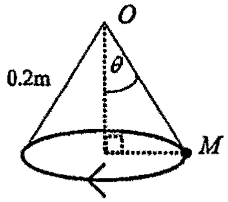
(i) $-\alpha,-\beta,-\gamma$ 1

(ii) $\alpha^2,\beta^2,\gamma^2$ 2

(iii) $\pm\alpha,\pm\beta,\pm\gamma$ 2

(b) Find a and b if $(1+i)$ is a root of $x^2+(a+2i)x+5-ib=0$ 3

(c) A body M, of mass 650g, is fixed to point O by a light wire 0.2m long. The body rotates in a horizontal plane at 72 revolutions per minute. Taking $g=10m/s^2$,



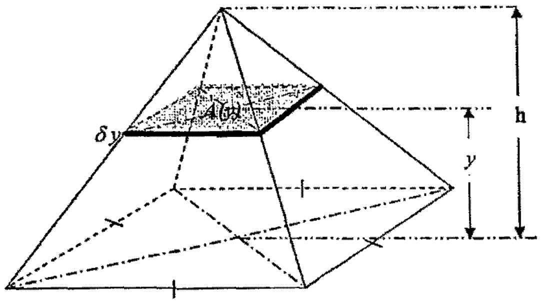
(i) Prove that $\tan\theta=\frac{72\pi^2\sin\theta}{625}$. 3

(ii) Find θ to the nearest minute. 2

(iii) The mass of the body is to be doubled but the speed of rotation is to remain the same. What will happen to the value of θ? 2

Question 7 (15 marks) Use a SEPARATE sheet of paper. **Marks**

(a) The great pyramid of Cheops at Giza in Egypt is approximately 150 m high and its base is a square with an area of approximately 5 hectares.



(i) Show that the area of the cross section $A(y)$, at y is given by **1**

$$A(y) = (5\times 10^4)\times\left(\frac{h-y}{h}\right)^2$$

(ii) Find the volume of the pyramid by using the slicing technique. **4**

(b) A particle of mass 1 kg is projected vertically upwards with an initial velocity of 100 m/s in a medium in which the resistance force is equal to 0.01 times the square of the body's velocity, i.e. $0.01v^2$. Use $g = 10m/s^2$.

(i) Show that the maximum height reached by the particle is $50\log_e 11$ metres. **4**

(ii) Will the downward velocity of the particle on its return to the point of projection be greater than, less than, or equal to 100m/s? Justify your answer. **2**

(iii) Calculate the actual downward velocity of the particle on its return to the point of projection. **4**

Question 8 (15 marks) Use a SEPARATE sheet of paper. **Marks**

(a) Use the following identity to answer the following questions.

$$\tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{1-6\tan^2\theta+\tan^4\theta}.$$

(i) Solve $x^4 + 4\sqrt{3}x^3 - 6x^2 - 4\sqrt{3}x + 1 = 0$. **3**

(ii) Hence show that

(1) $\tan\frac{\pi}{24} + \tan\frac{7\pi}{24} + 4\sqrt{3} = \tan\frac{5\pi}{24} + \tan\frac{11\pi}{24}$ **1**

(2) $\tan\frac{\pi}{24}\tan\frac{5\pi}{24} = \cot\frac{7\pi}{24}\cot\frac{11\pi}{24}$ **1**

(iii) Find the polynomial of least degree that has zeros **3**

$$\left(\cot\frac{\pi}{24}\right)^2, \left(\cot\frac{7\pi}{24}\right)^2, \left(\cot\frac{13\pi}{24}\right)^2, \left(\cot\frac{19\pi}{24}\right)^2.$$

(b) Let $I_n = \int_0^1 x(x^2-1)^n\,dx$ for $n = 0,1,2,\ldots$

(i) Use integration by parts to show that **3**

$$I_n = \frac{-n}{n+1}I_{n-1} \text{ for } n \geq 1.$$

(ii) Hence or otherwise show that **2**

$$I_n = \frac{(-1)^n}{2(n+1)} \text{ for } n \geq 0.$$

(iii) Explain why $I_{2n} > I_{2n+1}$ for $n \geq 0$ **1**

(iv) Explain whether or not $I_n > I_{n+2}$ for all $n \geq 0$. **1**

End of Examination

2008 Extension 2 Trial Solutions

Q1 a) $\int \frac{dx}{x^2-4x+40} = \int \frac{dx}{(x-2)^2+36}$

$= \frac{1}{6}\tan^{-1}\left(\frac{x-2}{6}\right)$

b) $\int_0^2 x^3 e^{x^2}\,dx = \left[\frac{x^2 e^{x^2}}{2}\right]_0^2 - \int_0^2 \frac{2x e^{x^2}}{2}\,dx$ $\quad u = x^2 \quad v = \frac{1}{2}e^{x^2}$

$= \left[\frac{4e^4}{2} - 0\right] - \frac{1}{2}\left[e^{x^2}\right]_0^2$ $\quad du = 2x \quad dv = xe^{x^2}$

$= 2e^4 - \frac{1}{2}(e^4 - 1)$

$= \frac{1}{2}(3e^4 + 1)$

c) $\int \sin^3 x\,dx = \int \sin^2 x.\sin x\,dx$

$= \int (1-\cos^2 x)\sin x\,dx$

$= \int \sin x - \cos^2 x.\sin x\,dx$

$= -\cos x + \frac{\cos^3 x}{3}$

d) $\int_0^1 \frac{x}{\sqrt{4-x}}\,dx$ $\quad$ let $u = 4-x \Rightarrow x = 4-u$

$du = -1$

$= \int_4^3 \frac{4-u}{\sqrt{u}}.-1\,du$ $\quad x=0,\ u=4$

$= \int_3^4 (4u^{-1/2} - u^{1/2})\,du$ $\quad x=1,\ u=3$

$= \left[8u^{1/2} - \frac{2}{3}u^{3/2}\right]_3^4$

$= \left(8\sqrt{4} - \frac{2}{3}.4\sqrt{4} - \left(8\sqrt{3} - \frac{2}{3}.3\sqrt{3}\right)\right)$

$= 16 - \frac{16}{3} - 6\sqrt{3}$

$= \frac{1}{3}(32 - 18\sqrt{3})$

e) i) $\frac{3x^2+2x+11}{(x^2+3)(1-x)} = \frac{(ax+b)(1-x)+c(x^2+3)}{(x^2+3)(1-x)}$

$ax - ax^2 + b - bx + cx^2 + 3c = 3x^2 + 2x + 11$

$c - a = 3 \Rightarrow a = c-3$ ①

$a - b = 2 \rightarrow c - 3 - b = 2 \therefore c - b = 5$ — ②

$3c + b = 11$ — ③

②+③: $4c = 16,\ c = 4$ $\quad$ ie $a = 1,\ b = -1,\ c = 4$

in ② $4 - b = 5,\ b = -1$

in ① $a = 4 - 3 = 1$

ii) $\int \frac{3x^2+2x+11}{(x^2+3)(1-x)}\,dx = \int \frac{x-1}{x^2+3} + \frac{4}{1-x}\,dx$

$= \int \frac{1}{2}.\frac{2x}{x^2+3} - \frac{1}{x^2+3} + \frac{4}{1-x}$

$= \frac{1}{2}\ln(x^2+3) - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) - 4\ln(1-x)$, $x < 1$

SOLUTIONS QUESTION 2

(a) (i) Given $z = 1+i$

$|z| = \sqrt{1^2+1^2}$
$= \sqrt{2}$

$\arg z = \tan^{-1}\frac{1}{1}$
$= \frac{\pi}{4}$

$\therefore z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$ (2)

(ii) $z^{12} = \left[\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^{12}$
$= 2^6(\cos 3\pi + i\sin 3\pi)$ De Moivre's Thm
$= 64(-1+0)$
$= -64$ (2)

(b) $P(z) = z^4 - 30z^2 + 289$

(i) If $P(4+i)=0$ then $4+i$ is a zero of $P(z)$

$P(4+i) = (4+i)^4 - 30(4+i)^2 + 289$
$= ((4+i)^2)^2 - 30(15+8i) + 289$ substitution & partial simp.
$= (15+8i)^2 - 450 - 240i + 289$ — (1)
$= 225 + 240i + 64i^2 - 450 - 240i + 289$
$= 225 + 240i - 64 - 450 - 240i + 289$ — (1)
$= 0$

$\therefore (4+i)$ is a zero of $P(z)$

(ii) Since $4+i$ is a zero, so also is $4-i$ (complex conj.) (1)
ie $[z-(4+i)][z-(4-i)]$ is a factor
$= [(z-4)-i][(z-4)+i]$ is a fact
$= (z-4)^2 - i^2$ " " "
$= z^2 - 8z + 17$ " " " (1)

(ii) (continued)

$$\begin{array}{r}
z^2 + 8z + 17 \\
z^2-8z+17 \overline{)\, z^4 \qquad - 30z^2 \qquad + 289} \\
z^4 - 8z^3 + 17z^2 \qquad\qquad \\
\hline
8z^3 - 47z^2 \qquad + 289 \\
8z^3 - 64z^2 + 136z \qquad \\
\hline
17z^2 - 136z + 289 \\
17z^2 - 136z + 289 \\
\hline
0
\end{array}$$
(1)

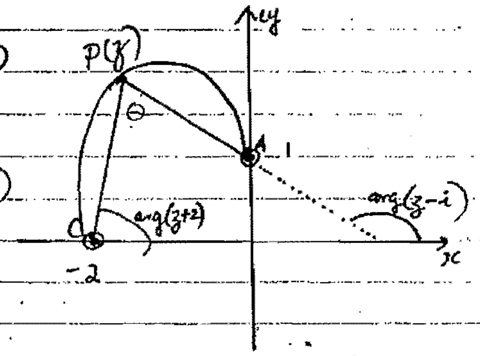
$\therefore z^2 + 8z + 17$ is a factor of $P(z)$

$\therefore z^2 + 8z + 17 = (z^2 + 8z + 16) + 1$ (Alt: use quad. form)
$= (z+4)^2 - i^2$
$= (z+4-i)(z+4+i)$
$= (z-(-4+i))(z-(-4-i))$ (1)

$\therefore$ remaining zero's are $-4+i$, $-4-i$

$\therefore$ zeros of $P(z)$ are $4+i$, $4-i$, $-4+i$, $-4-i$ (1)

(c)



(2)

Geometrically
$\arg\left(\frac{z-i}{z+2}\right) = \frac{\pi}{2}$

$\arg(z-i) - \arg(z+2) = \frac{\pi}{2}$

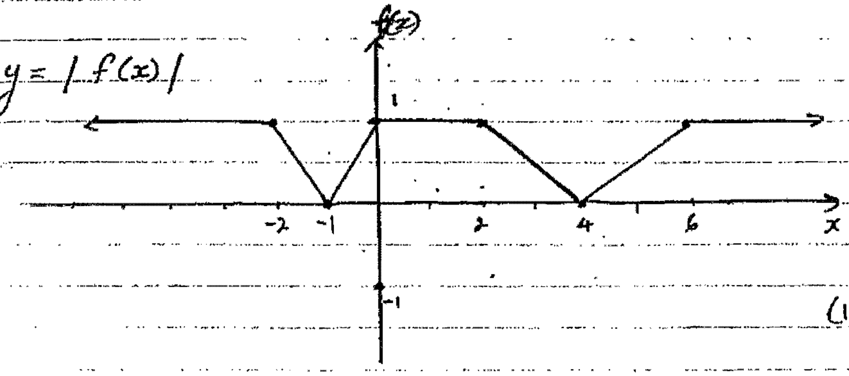
NB $\arg(z-i) > \arg(z+2)$

From diagram
Now $\theta + \arg(z+2) = \arg(z-i)$
(ext. angle of Δ)
$\therefore \theta = \arg(z-i) - \arg(z+2)$
$= \frac{\pi}{2}$

$\therefore P(z)$ is a semicircle with AC as diameter and P above AC. Points A, C are excluded (2)
from the locus ... $(-1, -\frac{1}{2})$ radius $\sqrt{5}/2$

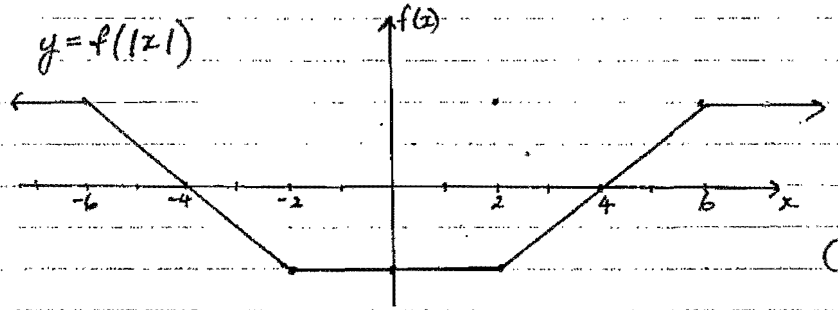
Solutions Question 3

(a) (i) $y = |f(x)|$



(1)

(ii) $y = f(|x|)$



(1)

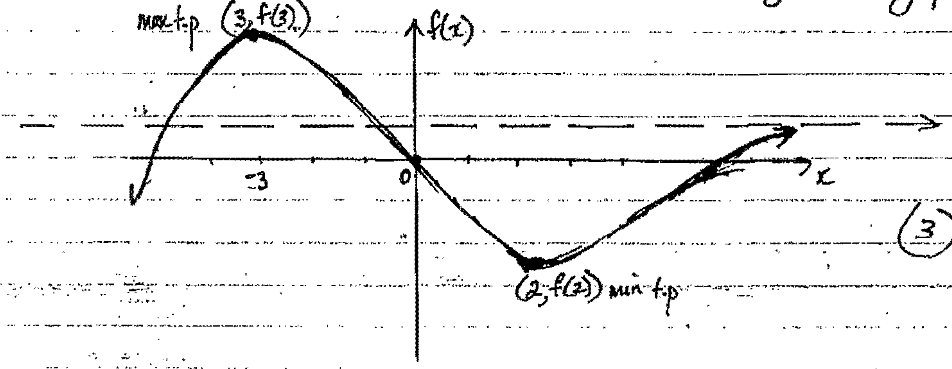
(b) $f'(2) = 0 \quad f'(2^-) < 0 \quad f'(2^+) > 0 \quad -\ 0\ + \quad \therefore x = 2$ is min

$f'(-3) = 0 \quad f'(-3^-) > 0 \quad f'(-3^+) < 0 \quad +\ 0\ - \quad \therefore x = -3$ is max

Curve passes through $(0,0)$

Curve below x axis when $x = 1$

as $x \to \infty$, $f'(x) \to 0$ ie flattens out to horizontal asymptote.



(3)

(c) (i) $y = \dfrac{x^3+4}{x^2}$

$= x + 4x^{-2}$

zero's $x^3 = -4$

$x = \sqrt[3]{-4}$

T.P's $\left(\frac{dy}{dx} = 0\right)$

$\dfrac{dy}{dx} = 1 - \dfrac{8}{x^3}$

$\dfrac{dy}{dx} = 0$ when $x = 2$

Inflexions $\left(\frac{d^2y}{dx^2} = 0\right)$

$\dfrac{d^2y}{dx^2} = \dfrac{24}{x^4} > 0 \therefore$ no inflexions

test t.p.

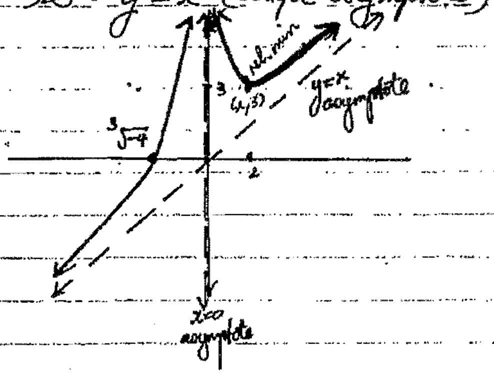
$\dfrac{d^2y}{dx^2} > 0$ when $x = 2$ $\therefore$ min t.p. at $(2,3)$

Asymptotic Behaviour

vertical $x = 0$ $\quad x \to 0^+ \ y \to \infty$

$x \to 0^- \ y \to \infty$

other as $x \to \infty \therefore y = x$ (oblique asymptote)



(3)

(ii) From the graph $x^3 - kx^2 + 4 = 0 \Rightarrow k = \dfrac{x^3+4}{x^2}$ what horizontal line $y = k$ will only have one solution? any $k < 3$. (1)

(d) (i) If 'a' is a mult. root of $P(x)$ then

$$P(x) = (x-a)^r \cdot Q(x)$$
$$P'(x) = (x-a)^r \cdot Q'(x) + Q(x) \cdot r(x-a)^{r-1}$$
$$= (x-a)^{r-1}\left[(x-a)Q'(x) + rQ(x)\right]$$
$$= (x-a)^{r-1} S(a) \quad \text{where } S(a) = (x-a)Q'(x) + rQ(x)$$

$$\therefore P'(a) = (a-a)^{r-1} S(a)$$
$$= 0 \times S(a)$$
$$= 0 \quad \text{as required.} \qquad (3)$$

(ii) $P(x) = 16x^3 - 12x^2 + 1$

$P'(x) = 48x^2 - 24x$

when $P'(x) = 0$

$48x^2 - 24x = 0$

$24x(2x-1) = 0$

$\therefore x = 0, \frac{1}{2}$

Since $P(0) = 1$

and $P(\frac{1}{2}) = 0$ $x = \frac{1}{2}$ is double root

Now $\alpha + \beta + \gamma = \frac{12}{16}$ but $\alpha = \beta = \frac{1}{2}$

$\frac{1}{2} + \frac{1}{2} + \gamma = \frac{12}{16}$

$\gamma = -\frac{1}{4}$

$\therefore$ 3 roots are $\frac{1}{2}, \frac{1}{2}, -\frac{1}{4}$ (3)

Q4. a) $x = \sqrt{2}\cos\theta$ $\quad y = 3\sin\theta$

$\cos\theta = \frac{x}{\sqrt{2}}$ $\quad \sin\theta = \frac{y}{3}$

$$\left(\frac{x}{\sqrt{2}}\right)^2 + \left(\frac{y}{3}\right)^2 = 1 \qquad (\cos^2\theta + \sin^2\theta = 1)$$
$$\therefore \frac{x^2}{2} + \frac{y^2}{9} = 1$$
$$a^2 = b^2(1-e^2), \quad a < b$$
$$2 = 9(1-e^2)$$
$$1 - e^2 = \frac{2}{9}$$
$$e^2 = \frac{7}{9}$$
$$\therefore e = \frac{\sqrt{7}}{3}$$

b) i) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\therefore \frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = -\frac{2x}{a^2} \times \frac{b^2}{2y}$$
$$= \frac{-b^2x}{a^2y}$$

at $P(a\cos\theta, b\sin\theta)$, $\frac{dy}{dx} = \frac{-b^2 a\cos\theta}{a^2 b\sin\theta}$

$$= \frac{-b\cos\theta}{a\sin\theta}$$

$\therefore$ equation of tangent is $y - b\sin\theta = \frac{-b\cos\theta}{a\sin\theta}(x - a\cos\theta)$

$$a\sin\theta y - ab\sin^2\theta = -b\cos\theta x + ab\cos^2\theta$$
$$b\cos\theta x + a\sin\theta y = ab(\sin^2\theta + \cos^2\theta)$$
$$\therefore \frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1 \quad \text{— (1)}$$

ii) at T, $y = 0$ & T lies on tangt $\therefore$ $\frac{x\cos\theta}{a} = 1$

ie $x = \frac{a}{\cos\theta}$

$\therefore T\left(\frac{a}{\cos\theta}, 0\right)$

equation of directrix is $x = \frac{a}{e}$

$$\therefore \frac{a}{\cos\theta} = \frac{a}{e}$$

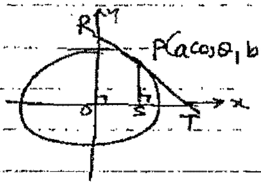
ie $\cos\theta = e$

14b) iii) $\because$ since $\cos\theta = e$, the x-coordinate of P is ae

The focus has coordinates $S(ae, 0)$

$\therefore$ The focal chord makes an angle of $90°$ with x-axis

iv) $\triangle TOR \;|||\; \triangle TSP$ (equiangular)



$\therefore \dfrac{RT}{OT} = \dfrac{RP}{OS}$ (ratio of intercepts)

$T\left(\frac{a}{e}, 0\right)$ and $S(ae, 0)$

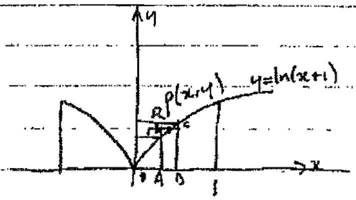
$\therefore OT = \frac{a}{e}$ and $OS = ae$

$\therefore \dfrac{RT}{\frac{a}{e}} = \dfrac{RP}{ae}$

$RT = \dfrac{RP}{\cancel{a}e} \cdot \dfrac{\cancel{a}}{e}$

$\therefore RP = e^2 RT$

c)



let ABCD be rectangle of height y and width δx where $P(x,y)$ is the midpt of the rectangle

let R be outer radius & r be inner radius

$\delta V = \pi(R^2 - r^2) \times h$

$= \pi\,(R+r)(R-r) \times y$

$= 2\pi\left(\frac{R+r}{2}\right)(R-r) \times y$

$= 2\pi \cdot x \cdot \delta x \cdot y$

$\therefore V = \lim_{\delta x \to 0} \sum_{x=0}^{1} 2\pi x y\, \delta x$, $y = \ln(x+1)$

$= \int_0^1 2\pi x \cdot \ln(x+1)\, dx$ $\quad u = \ln(x+1)$ $\quad v = \frac{x^2}{2}$ $\quad du = \frac{1}{x+1}$ $\quad dv = x$

$= 2\pi\left(\left[\frac{x^2}{2}\ln(x+1)\right]_0^1 - \int \frac{x^2}{2(x+1)}\,dx\right)$

$= 2\pi\left(\frac{1}{2}\ln 2 - \frac{1}{2}\int_0^1 \left(x - 1 + \frac{1}{x+1}\right) dx\right)$

$= 2\pi\left(\frac{1}{2}\ln 2 - \frac{1}{2}\left[\frac{x^2}{2} - x + \ln(x+1)\right]_0^1\right)$

$= 2\pi\left(\frac{1}{2}\ln 2 - \frac{1}{2}\left[\frac{1}{2} - 1 + \ln 2 - (0 - 0 + 0)\right]\right)$

$\therefore$ Volume $= \frac{\pi}{2}$ units^3

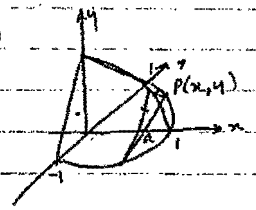
Q5 a) i) $\int_{-a}^{a} \sqrt{a^2 - x^2}\, dx = \frac{1}{2} \cdot \pi \cdot a^2$, area of semicircle centre $(0,0)$, radius a

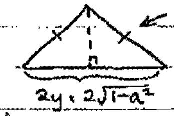
$= \frac{\pi a^2}{2}$

ii) $f(x) = x\sqrt{a^2 - x^2}$ is an odd f^n $\therefore \int_{-a}^{0} f(x)\,dx = -\int_0^a f(x)\,dx$

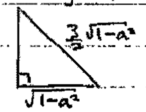
$\therefore \int_{-a}^{a} f(x)\,dx = 0$

b) i)





$\leftarrow \frac{3}{4} \times 2\sqrt{1-a^2} = \frac{3\sqrt{1-a^2}}{2}$



height $= \sqrt{\frac{9}{4}(1-a^2) - (1-a^2)}$

$= \sqrt{\frac{5}{4}(1-a^2)}$

$= \frac{\sqrt{5}}{2}\sqrt{1-a^2}$

$\therefore$ Area of $\triangle = \frac{1}{2} \times 2\sqrt{1-a^2} \times \frac{\sqrt{5}}{2}\sqrt{1-a^2}$

$= \frac{\sqrt{5}}{2}(1-a^2)$

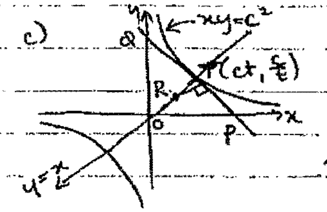
ii) $V = \int_0^1 \frac{\sqrt{5}}{2}(1-x^2)\,dx$

$= \frac{\sqrt{5}}{2}\left[x - \frac{x^3}{3}\right]_0^1$

$= \frac{\sqrt{5}}{2}\left(1 - \frac{1}{3}\right)$

$\therefore$ Vol $= \frac{\sqrt{5}}{3}$ units^3

c)



i) $xy = c^2 \Rightarrow y = c^2 x^{-1}$

$\frac{dy}{dx} = -c^2 x^{-2}$

at $T(ct, \frac{c}{t})$ $\quad M_T = \frac{-c^2}{(ct)^2} = -\frac{1}{t^2}$

$\therefore$ eqn of tangent is $y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$

$t^2 y - ct = -x + ct$

$x + t^2 y = 2ct$

ii) at P $y = 0$ $\therefore x = 2ct$ $\therefore P(2ct, 0)$

at Q, $x = 0$ $\therefore t^2 y = 2ct$ $\therefore Q\left(0, \frac{2c}{t}\right)$

iii) $M_N = t^2$ $\therefore$ eqn is $y - \frac{c}{t} = t^2(x - ct)$

ie $ty - c = t^3 x - ct^4$

$t^3 x - ty = ct^4 - c$

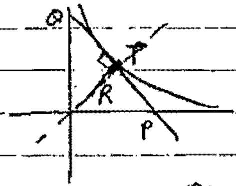
iv) R lies on $y=x$ and the normal at T, $t^3x - ty = ct^4 - c$

$\therefore t^3x - tx = c(t^4-1)$

$tx(t^2-1) = c(t^2-1)(t^2+1)$

$x = \frac{c}{t}(t^2+1)$

v) R lies on $y=x$ $\therefore$ coordinates of R $\left(\frac{c}{t}(t^2+1), \frac{c}{t}(t^2+1)\right)$



$P(2ct, 0)$ $\quad Q(0, \frac{2c}{t})$

Midpt of PQ is $\left(\frac{2ct+0}{2}, \frac{0+\frac{2c}{t}}{2}\right)$

$= \left(ct, \frac{c}{t}\right)$, which is the point T

$\therefore$ Any point on the normal at T will lie on the perpendicular bisector of PQ.

$\therefore \triangle PRQ$ is isosceles

Question 6 Solutions

(a)(i) let $x = -X$

$(-X)^3 + 2(-X) - 1 = 0$

$-X^3 - 2X - 1 = 0$

$\therefore x^3 + 2x + 1 = 0$ $\quad$ (1)

(ii) let $x = \sqrt{X}$

$(\sqrt{X})^3 + 2\sqrt{X} - 1 = 0$

$X\sqrt{X} + 2\sqrt{X} = 1$ $\quad$ (sq. b.s.

$X^3 + 4X^2 + 4X = 1$

$\therefore x^3 + 4x^2 + 4x - 1 = 0$ $\quad$ (2

(iii) let $x = \pm X$ (Note $x = -X$) in part (i) and $x = X$ is original equation

$\therefore (X^3+2X-1)(X^3+2X+1) = 0$ must have $x = \pm X$

$\therefore (x^3+2x-1)(x^3+2x+1) = 0$

$x^6 + 4x^4 + 4x^2 - 1 = 0$ $\quad$ (2)

(b) Note $1-i$ is NOT a root of $x^2 + (a+2i)x + 5 - ib = 0$ since all coefficients are NOT real

$1+i$ is a root of $z^2 + (a+2i)z + 5 - ib = 0$ (given)

$\therefore (1+i)^2 + (a+2i)(1+i) + 5 - ib = 0$

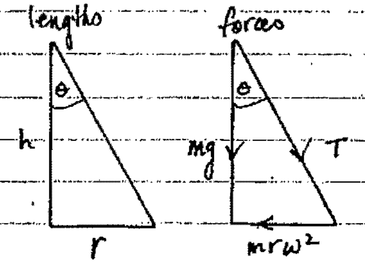
$\therefore 1 + 2i - 1 + a + 2i + ai - 2 + 5 - ib = 0$

$(a+3) + (4+a-b)i = 0$

$\therefore a + 3 = 0$ as $Re = 0$ $\Rightarrow a = -3$

$4 + a - b = 0$ as $Im = 0$ $\Rightarrow b = 1$ $\quad$ (3

(c)



(i) $T\sin\theta = mr\omega^2$ (hor.)

$T\cos\theta = mg$ (vert)

$\therefore \tan\theta = \frac{r\omega^2}{g}$

$r = 0.2\sin\theta$

$= \frac{\sin\theta}{5}$

$\omega = \frac{72 \times 2\pi}{60} = \frac{12\pi}{5}$ rad/s

$\therefore \tan\theta = \frac{\sin\theta}{5} \times \frac{144\pi^2}{25} \times \frac{1}{10}$

$= \frac{72\pi^2 \sin\theta}{625}$ as req.

(3)

(ii) $\tan\theta = \dfrac{72\pi^2 \sin\theta}{625}$ from (i)

$\dfrac{\sin\theta}{\cos\theta} = \dfrac{72\pi^2 \sin\theta}{625}$

$\therefore \cos\theta = \dfrac{625}{72\pi^2}$

$\theta = 28°25'$ (2)

(iii) from (i) $\tan\theta = \dfrac{r\omega^2}{g}$

$\dfrac{r}{h} = \dfrac{r\omega^2}{g}$

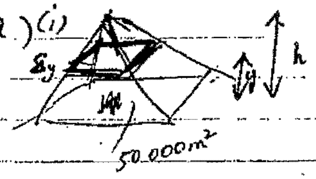
$h = \dfrac{g}{\omega^2}$ as can be seen h depends on g a constant and ω which is the same as before $\therefore$ h remains the same

$\therefore$ double mass $\Rightarrow$ no effect on θ

(2)

Question 7 Solutions

(a) (i)



Since a square pyramid

If $A_1 : A_2$ then $l_1^2 : l_2^2$

$\dfrac{A(y)}{A} = \dfrac{(h-y)^2}{h^2}$ similar areas (all □'s are similar)

$\therefore A(y) = 5\times10^4 \times \dfrac{(h-y)^2}{h^2}$ (1)

(ii) Volume $= \sum_0^h A(y)\,\delta y$ ✓ 1

$= \int_0^h \dfrac{A(h-y)^2}{h^2}\,dy$

$= \left[-\dfrac{A}{3h^2}(h-y)^3\right]_0^h$ ✓ 1

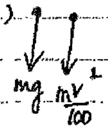
$= -\dfrac{A}{3h^2}\left(0-h^3\right)$

$= \dfrac{Ah}{3}$ ✓ 1

$= \dfrac{50\,000 \times 150}{3}$

$= 2\,500\,000\ m^3$ ✓ 1

(b) (i)



note $m = 1$kg

$m\ddot{x} = -10m - \dfrac{mv^2}{100}$ ($m = 1$ kg)

$\therefore \ddot{x} = -\left(10 + \dfrac{v^2}{100}\right)$

$v\dfrac{dv}{dx} = -\left(\dfrac{1000+v^2}{100}\right)$

$\dfrac{dv}{dx} = -\left(\dfrac{1000+v^2}{100v}\right)$

$\therefore \dfrac{dx}{dv} = -\dfrac{100v}{1000+v^2}$

$\therefore x = -50\ln(1000+v^2) + c$

when $x = 0$ $v = 100$

$\therefore c = 50\ln 11000$

$\therefore x = 50\ln\left(\dfrac{11000}{1000+v^2}\right)$

when $v = 0$ (highest pt)

$x = 50\ln 11$ (4)

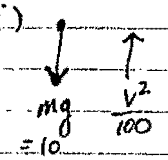
(ii) The downward velocity will less than 100 m/s (its original). For example consider a speargun being fired upwards in water from low in the ocean then after reaching its highest point, slowly drifting back downwards.

OR more everyday

A football being kicked from a boot directly upwards high velocity at impact, highest point zero velocity. person is able to catch it on return (lower velocity at return, otherwise unable to catch it.)

(2)

(iii)



$$m\ddot{x} = 10 - \frac{v^2}{100} \qquad (m=1)$$

$$\ddot{x} = \frac{1000-v^2}{100}$$

$$v\frac{dv}{dx} = \frac{1000-v^2}{100}$$

$$\frac{dv}{dx} = \frac{1000-v^2}{100v}$$

$$\frac{dx}{dv} = \frac{100v}{1000-v^2}$$

$$x = -50\ln(1000-v^2) + c \qquad (\text{at top } x=0\ v=0)$$

$$\therefore C = 50\ln 1000$$

$$x = 50\ln\frac{1000}{1000-v^2}$$

when $x = 50\ln 11$
(distance of return from highest pt)

$$11 = \frac{1000}{1000-v^2}$$

$$11\,000 - 11v^2 = 1000$$

$$10\,000 = 11v^2$$

$$v^2 = \frac{10\,000}{11} \qquad v = \frac{100}{\sqrt{11}}\text{ m/s} = \frac{100\sqrt{11}}{11}\text{ m/s.} \qquad (4)$$

Q8. a) i)

$$\tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$$

let $x = \tan\theta$ and $\tan 4\theta = \frac{1}{\sqrt{3}}$

$$\therefore \frac{4x-4x^3}{1-6x^2+x^4} = \frac{1}{\sqrt{3}}$$

$$\therefore x^4 + 4\sqrt{3}x^3 - 6x^2 - 4\sqrt{3}x + 1 = 0$$

$$\tan 4\theta = \frac{1}{\sqrt{3}}$$

$$4\theta = n\pi + \frac{\pi}{6}$$

$$\theta = \frac{n\pi}{4} + \frac{\pi}{24}, \quad n = 0, 1, 2, 3$$

$n=0$, $\theta = \frac{\pi}{24}$ so $x = \tan\frac{\pi}{24}$

$n=1$, $\theta = \frac{\pi}{4} + \frac{\pi}{24} = \frac{7\pi}{24}$, $x = \tan\frac{7\pi}{24}$

$n=2$, $\theta = \frac{\pi}{2} + \frac{\pi}{24} = \frac{13\pi}{24}$, $x = \tan\frac{13\pi}{24}$ (or $-\tan\frac{11\pi}{24}$)

$n=3$, $\theta = \frac{3\pi}{4} + \frac{\pi}{24} = \frac{19\pi}{24}$, $x = \tan\frac{19\pi}{24}$ (or $-\tan\frac{5\pi}{24}$)

ii) (1) from (i) $\tan\frac{\pi}{24} + \tan\frac{7\pi}{24} - \tan\frac{5\pi}{24} - \tan\frac{11\pi}{24} = -4\sqrt{3}$ (sum of roots)

$$\therefore \tan\frac{\pi}{24} + \tan\frac{7\pi}{24} + 4\sqrt{3} = \tan\frac{5\pi}{24} + \tan\frac{11\pi}{24}$$

(2) $\left(\tan\frac{\pi}{24}\right)\left(\tan\frac{7\pi}{24}\right)\left(-\tan\frac{5\pi}{24}\right)\left(-\tan\frac{11\pi}{24}\right) = 1$ (product of roots)

$$\tan\frac{\pi}{24}\tan\frac{5\pi}{24} = \frac{1}{\tan\frac{7\pi}{24}\cdot\tan\frac{11\pi}{24}}$$

$$\therefore \tan\frac{\pi}{24}\tan\frac{5\pi}{24} = \cot\frac{7\pi}{24}\cot\frac{11\pi}{24}$$

iii) let $\alpha = \tan\frac{\pi}{24}$

$$\frac{1}{\alpha^2} = \left(\cot\frac{\pi}{24}\right)^2$$

let $\frac{1}{\alpha^2} = x$ so $\alpha = \frac{1}{\sqrt{x}}$

$$\left(\frac{1}{\sqrt{x}}\right)^4 + 4\sqrt{3}\left(\frac{1}{\sqrt{x}}\right)^3 - 6\left(\frac{1}{\sqrt{x}}\right)^2 - 4\sqrt{3}\left(\frac{1}{\sqrt{x}}\right) + 1 = 0$$

$$\frac{1}{x^2} + \frac{4\sqrt{3}}{x\sqrt{x}} - \frac{6}{x} - \frac{4\sqrt{3}}{\sqrt{x}} + 1 = 0$$

$$x^2 - 4\sqrt{3}x\sqrt{x} - 6x + 4\sqrt{3}\sqrt{x} + 1 = 0$$

$$4\sqrt{3}\sqrt{x}(x-1) = x^2 - 6x + 1$$

$$48x(x-1)^2 = (x^2-6x+1)^2$$

$$48x^3 - 96x^2 + 48x = x^4 - 12x^3 + 2x^2 + 36x^2 - 12x + 1$$

ie $\quad x^4 - 60x^3 + 134x^2 - 60x + 1 = 0$

28 b) i) $I_n = \int_0^1 x(x^2-1)^n \, dx$

$$= \left[\frac{x^2}{2}(x^2-1)^n \right]_0^1 - \int_0^1 \frac{x^2}{2} \cdot n(x^2-1)^{n-1} \cdot 2x \, dx$$

$$= \left[\tfrac{1}{2} \times 0 - 0\right] - n\int_0^1 x^3 (x^2-1)^{n-1} \, dx$$

$$= -n\int_0^1 \frac{x^3(x^2-1)^n}{x^2-1} \, dx$$

$$= -n\int_0^1 \left(x + \frac{x}{x^2-1}\right)(x^2-1)^n \, dx$$

$$= -n\left[\int_0^1 x(x^2-1)^n \, dx + \int_0^1 \frac{x}{x^2-1}(x^2-1)^n \, dx\right]$$

$$= -n\int_0^1 x(x^2-1)^n \, dx - n\int_0^1 x(x^2-1)^{n-1} \, dx$$

$\therefore I_n = -nI_n - nI_{n-1}$

$(1+n)I_n = -nI_{n-1}$

$I_n = \frac{-n}{(n+1)} I_{n-1}$ for $n \geq 1$

ii) "Hence": $I_n = \frac{-n}{n+1} I_{n-1}$

$$= \frac{-n}{n+1} \cdot \frac{-n+1}{n} \cdot \frac{-n+2}{n-1} \cdot \cdots \frac{-3}{4} \cdot \frac{-2}{3} \cdot \frac{-1}{2} I_0$$

$$I_0 = \int_0^1 x(x^2-1)^0 \, dx$$

$$= \left[\frac{x^2}{2}\right]_0^1$$

$$= \frac{1}{2}$$

So $I_n = (-1)^n \frac{n}{n+1} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n-1} \cdots \frac{3}{4} \cdot \frac{2}{3} \cdot \frac{1}{2} \cdot \frac{1}{2}$

$$= \frac{(-1)^n}{2(n+1)}, \quad n \geq 0$$

"Otherwise": $I_n = \int_0^1 x(x^2-1)^n \, dx$

$$= \frac{1}{2}\int_0^1 2x(x^2-1)^n \, dx$$

$$= \frac{1}{2}\left[\frac{(x^2-1)^{n+1}}{n+1}\right]_0^1$$

$$= \frac{1}{2(n+1)}\left(0 - (-1)^{n+1}\right)$$

$$= \frac{(-1)^n}{2(n+1)}, \quad n \geq 0$$

iii) $I_0 = \frac{1}{2}$, $I_1 = \frac{-1}{4}$, $I_2 = \frac{1}{6}$, $I_3 = \frac{-1}{8}$, $I_4 = \frac{1}{10}$, $I_5 = \frac{-1}{12}$

ie $I_{2n} > 0$ and $I_{2n+1} < 0$

so $I_{2n} > I_{2n+1}$

OR: $I_{2n} = \frac{(-1)^{2n}}{2(2n+1)}$

$$= \frac{1}{2(2n+1)}$$

$$> 0$$

$$I_{2n+1} = \frac{(-1)^{2n+1}}{2((2n+1)+1)}$$

$$= \frac{-1}{4(n+1)}$$

$$< 0$$

So $I_{2n} > I_{2n+1}$

iv) from (iii), if n is even, $I_n > I_{n+2}$ ie $I_2 = \frac{1}{6} > I_4 = \frac{1}{10} > I_6 = \frac{1}{14}$

if n is odd, $I_1 = \frac{-1}{4} < I_3 = \frac{-1}{8} \leq I_5 = \frac{-1}{12}$

$\therefore I_n \not> I_{n+2}$ for all $n \geq 0$