WORKSHEETS ON MATHEMATICAL INDUCTION

## Example	
Follow carefully the steps shown below: **Show that** $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{1}{6}n(n+1)(2n+1)$, for n a positive integer.	*Now try this question:* **Show that** $1 + 2 + 3 + \ldots + n = \frac{n}{2}(n+1)$, for n a positive integer.
Step 1 Testing $n = 1$, LHS $= 1$ RHS $= \frac{1}{6} \times 1 \times (1+1) \times (2+1)$ $= 1$ $\therefore$ result is true for $n = 1$.	**Step 1**
Step 2 Assume the result is true for $n = k$, that is, assume $S_k = \frac{1}{6}k(k+1)(2k+1)$.	**Step 2**
Step 3 Hence show the result is true for $n = k + 1$ (the next term), that is, show $S_{k+1} = \frac{1}{6}(k+1)(k+2)(2k+3)$ $S_{k+1} = S_k + T_{k+1}$ $= \frac{1}{6}k(k+1)(2k+1) + (k+1)^2$ $= (k+1)\left[\frac{1}{6}k(2k+1) + (k+1)\right]$ $= \frac{1}{6}(k+1)(2k^2 + k + 6k + 6)$ $= \frac{1}{6}(k+1)(2k^2 + 7k + 6)$ $= \frac{1}{6}(k+1)(2k+3)(k+2)$. Hence, if the result is true for $n = k$, then it is true for $n = k + 1$.	**Step 3**
Step 4 Since the result is true for $n = 1$, from Step 3 it is true for $n = 1 + 1 = 2$, and then for $n = 3$, and so on for all positive integral values of n.	**Step 4**
Some additional questions of this type: Prove, for n a positive integer: $1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{1}{4}n^2(n+1)^2$ $1 + 3 + 5 + \ldots + (2n-1) = n^2$ $1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 = \frac{1}{3}n(2n-1)(2n+1)$ $1 + 2 + 2^2 + \ldots + 2^{n-1} = 2^n - 1$	**1978 HSC 3U Questions:** Write down the formula for the sum of the first n positive odd integers. Explain the method of mathematical induction and use it to prove this formula.

Example **Prove that** $1.2.3 + 2.3.4 + \ldots + n(n+1)(n+2) = \frac{n}{4}(n+1)(n+2)(n+3)$ for n a positive integer.	**Prove** $1.2 + 2.3 + 3.4 + \ldots + (n+1)$ $= \frac{1}{3}n(n+1)(n+2)$ for n a positive integer.
Step 1 Testing $n = 1$, LHS $= 1.2.3 = 6$ RHS $= \frac{1}{6}(1+1)(1+2)(1+3)$ $= 6$ $\therefore$ result is true for $n = 1$.	**Step 1**
Step 2 Assume the result is true for $n = k$, that is, assume $S_k = \frac{1}{4}k(k+1)(k+2)(k+3)$.	**Step 2**
Step 3 Hence show the result is true for $n = k+1$, that is, show $S_{k+1} = \frac{1}{4}(k+1)(k+1+1)(k+1+2)(k+1+3)$ $= \frac{1}{4}(k+1)(k+2)(k+3)(k+4)$ $S_{k+1} = S_k + T_{k+1}$ $= \frac{1}{4}k(k+1)(k+2)(k+3) + (k+1)(k+2)(k+3)$ $= (k+1)(k+2)(k+3)(\frac{1}{4}k + 1)$ $= (k+1)(k+2)(k+3)(\frac{k+4}{4})$ $= \frac{1}{4}(k+1)(k+2)(k+3)(k+4)$. Hence, if the result is true for $n = k$, then it is true for $n = k+1$.	**Step 3**
Step 4 Since the result is true for $n = 1$, from Step 3 it is true for $n = 1 + 1 = 2$, and then for $n = 2 + 1 = 3$, and so on for all positive integral values of n.	**Step 4**
Exercise: Prove: $1.3 + 2.4 + 3.5 + \ldots + n(n+2)$ $= \frac{1}{6}n(n+1)(2n+7)$.	**1980 HSC 3U Questions** If $S_n = 1.2 + 2.3 + \ldots + n(n+1)$, use the Principle of Mathematical Induction to show that $S_n = \frac{n}{3}(n+1)(n+2)$ for all positive integers n.

Example

Show

$$\frac{1}{1.4}+\frac{1}{4.7}+\ldots+\frac{1}{(3n-2)(3n+1)}=\frac{n}{3n+1}$$

for n a positive integer.

Step 1

Testing $n=1$,

$\text{LHS}=\frac{1}{1.4}=\frac{1}{4}$

$\text{RHS}=\frac{1}{3\times 1+1}=\frac{1}{4}=1$

$\therefore$ result is true for $n=1$.

Step 2

Assume the result is true for $n=k$, that is,

$$S_k=\frac{k}{3k+1}.$$

Step 3

Hence show the result is true for

$n=k+1$, that is,

show $S_{k+1}=\frac{k+1}{3(k+1)+1}=\frac{k+1}{3k+4}$

$$\begin{aligned}S_{k+1}&=S_k+T_{k+1}\\&=\frac{k}{3k+1}+\frac{1}{(3k+1)(3k+4)}\\&=\frac{k(3k+4)+1}{(3k+1)(3k+4)}\\&=\frac{3k^2+4k+1}{(3k+1)(3k+4)}\\&=\frac{(3k+1)(k+1)}{(3k+1)(3k+4)}\\&=\frac{k+1}{3k+4}.\end{aligned}$$

Hence, if the result is true for $n=k$,

then it is true for $n=k+1$.

Step 4

Since the result is true for $n=1$,

from Step 3 it is true for $n=1+1=2$,

and then for $n=2+1=3$, and so on for all

positive integral values of n.

Show

$$\frac{2}{1.3}+\frac{2}{3.5}+\ldots+\frac{2}{(2n-1)(2n+1)}=1-\frac{1}{2n+1}$$

for n a positive integer.

Step 1

Step 2

Step 3

Step 4

Exercise

Prove:

$$\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\ldots+\frac{1}{(2n-1)(2n+1)}$$
$$=\frac{n}{2n+1}$$

for n a positive integer.

HSC 3U Questions

1974 $\frac{1}{1.2}+\frac{1}{2.3}+\ldots+\frac{1}{n(n+1)}=1-\frac{1}{n+1}$

1975 $\frac{1}{x-1}-\frac{1}{x}-\frac{1}{x^2}-\frac{1}{x^3}-\ldots-\frac{1}{x^n}$

$=\frac{1}{x^n(x-1)}$, $x\neq 0$ or 1, n a positive integer.