**SYDNEY BOYS HIGH SCHOOL**
**MOORE PARK, SURRY HILLS**

**2003**
**TRIAL HIGHER SCHOOL CERTIFICATE**

# Mathematics Extension 2

**General Instructions**

- Reading Time - 5 Minutes
- Working time - 3 hours
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators may be used.
- All necessary working should be shown in every question.

**Total Marks – 120**

- Attempt all questions.
- All questions are of equal value.
- Each section is to be answered in a separate bundle, labeled Section A (Questions 1, 2, 3), Section B (Questions 4, 5, 6) and Section C (Questions 7 and 8).

Examiner: *C.Kourtesis*

*Note:* *This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate.*



**Section A**
**Start a new answer sheet**

**Question 1. (Start a new answer sheet.)** (15 marks)

| | | Marks |
|---|---|---|
| (a) | Find $\int \frac{dx}{\sqrt{4-9x^2}}$. | 2 |
| (b) | Find $\int \frac{4}{(x-1)(2-x)}dx$ | 3 |
| (c) | Use integration by parts to find $\int te^{\frac{t}{4}}\,dt$ | 3 |
| (d) | Use the substitution $u=2+\cos\theta$ to show that $\int_0^{\frac{\pi}{2}} \frac{\sin 2\theta}{2+\cos\theta}d\theta = 2+4\log_e\left(\frac{2}{3}\right)$ | 4 |
| (e) | Evaluate $\int_0^{2\pi} \lvert\sin x\rvert\,dx$ | 2 |
| (f) | Determine whether the following statement is True of False, and give a brief reason for your answer. $\int_{-1}^{4} \frac{dx}{x^3} = \frac{15}{32}$ | 1 |

**Question 2.** (15 marks)

**Marks**

(a) (i) Express $w=-1-i$ in modulus-argument form. **2**

(ii) Hence express $w^{12}$ in the form $x+iy$ where $x$ and $y$ are real numbers. **2**

(b) Find the equation, in Cartesian form, of the locus of the point $z$ if **2**

$$|z-i|=|z+3|.$$

(c) Sketch the region in the Argand diagram that satisfies the inequality **3**

$$\mathrm{Re}\left(\frac{1}{z}\right)\le\frac{1}{2}.$$

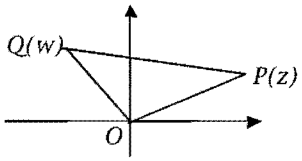
(d) (i) On the Argand diagram draw a neat sketch of the locus specified by **1**

$$\arg(z+1)=\frac{\pi}{3}.$$

(ii) Hence find $z$ so that $|z|$ is a minimum. **2**

(e) Points $P$ and $Q$ represent the complex numbers $z$ and $w$ respectively in the Argand Diagram.

If $\Delta OPQ$ (where $O$ is the origin) is equilateral




(i) Explain why $wz=z^2 cis\frac{\pi}{3}$. **1**

(ii) Prove that $z^2+w^2=zw$. **2**



**Question 3.** (15 marks)

**Marks**

(a) Sketch the following curves on separate diagrams, for $-\frac{3\pi}{2}\le x\le\frac{3\pi}{2}$.
[Note: There is no need to use calculus.]

(i) $y=\tan x$ **1**

(ii) $y=|\tan x|$ **1**

(iii) $y=\tan|x|$ **1**

(iv) $y=\tan^2 x$ **2**

(b) Consider the function $f(x)=\frac{x}{\ln x}$, $x>0$

(i) Determine the domain and write down the equations of any asymptotes. **2**

(ii) Show that there is a minimum turning point at $(e, e)$. **3**

(iii) Show that there is a point of inflexion at $x=e^2$. **3**

(iv) Sketch the graph of $y=f(x)$. **2**

## Section B
**Start a new booklet.**

**Question 4** (15 marks)

**Marks**

(a) (i) By solving the equation $z^3 = 1$ find the three cube roots of 1. **2**

(ii) Let $w$ be a cube root of 1 where $w$ is not real. Show that $1 + w + w^2 = 0$. **1**

(iii) Find the quadratic equation, with integer coefficients, that has roots $4 + w$ and $4 + w^2$. **3**

(b) A monic cubic polynomial, when divided by $x^2 + 4$ leaves a remainder of $x + 8$ and when divided by $x$ leaves a remainder of $-4$. Find the polynomial in expanded form. **3**

(c) Consider the polynomial $P(z) = z^3 + az^2 + bz + c$ where $a$, $b$ and $c$ are all real.

If $P(\theta i) = 0$ where $\theta$ is real and non-zero:

(i) Explain why $P(-\theta i) = 0$ **1**

(ii) Show that $P(z)$ has one real zero. **1**

(iii) Hence show that $c = ab$, where $b > 0$. **4**



**Question 5** (15 marks)

**Marks**

(a) A particle of mass $m$ falls vertically from rest at a height of $H$ metres above the Earth's surface, against a resistance $mkv$ when its speed is $v$ m/s. ($k$ is a positive constant).
Let $x$ m be the distance the particle has fallen, and $v$ m/s its speed at $x$. Let $g$ m/s$^2$ be the acceleration due to gravity.

(i) Show that the equation of motion is given by **1**

$$v\frac{dv}{dx} = g - kv$$

(ii) If the particle reaches the surfaced of the Earth with speed $V_0$, show that **4**

$$\ln\left(1 - \frac{kV_0}{g}\right) + \frac{kV_0}{g} + \frac{k^2H}{g} = 0.$$

(iii) Show that the time $T$ taken to reach the Earth's surface is given by **3**

$$T = \frac{1}{k}\ln\left(\frac{g}{g - kV_o}\right).$$

(iv) Show that $V_0 = Tg - kH$. **2**

(v) Hence prove that $T < \frac{1}{k} + \frac{kH}{g}$. **1**

(b) The letters A, B, C, D, E, F, I, O are arranged in a circle. In how many ways can this be done if at least two of the vowels are together? **2**

(c) A man has five friends. In how many ways can he invite one or more of them to dinner? **2**

**Question 6** (15 marks)

**Marks**

(a) (i) Expand $(\cos\theta + i\sin\theta)^3$ and hence express $\cos 3\theta$ and $\sin 3\theta$ in terms of $\cos\theta$ and $\sin\theta$ respectively. **2**

(ii) Show that $\cot 3\theta = \dfrac{t^3 - 3t}{3t^2 - 1}$ where $t = \cot\theta$. **2**

(iii) Solve $\cot 3\theta = 1$ for $0 \le \theta \le 2\pi$. **2**

(iv) Hence show that $\cot\dfrac{\pi}{12}.\cot\dfrac{5\pi}{12}.\cot\dfrac{9\pi}{12} = -1$. **2**

(v) Write down a cubic equation with roots $\tan\dfrac{\pi}{12}$, $\tan\dfrac{5\pi}{12}$, $\tan\dfrac{9\pi}{12}$. **1**

[Express your answer as a polynomial equation with integer coefficients.]

(b) (i) Draw a sketch showing that if $f(x)$ and $g(x)$ are continuous functions and $f(x) > g(x) > 0$ for $a \le x \le b$ then **2**

$$\int_a^b f(x)\,dx > \int_a^b g(x)\,dx.$$

(ii) Show that $y = \tan x$ is an increasing function for $\dfrac{\pi}{4} \le x \le \dfrac{\pi}{3}$. **1**

(iii) Prove that $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\tan x}{x}\,dx > \log_e\left(\frac{4}{3}\right)$. **3**



**Section C**
**Start a new booklet**

**Question 7** (15 marks)

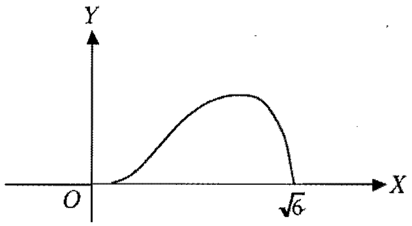
**Marks**

(a) (i) If $I_n = \displaystyle\int_1^e x(\ln x)^n\,dx$ (where $n$ is a non-negative integer) **3**

show that $I_n = \dfrac{e^2}{2} - \dfrac{n}{2}I_{n-1}$ (where $n \ge 1$).

(ii) Hence evaluate $I_3$. **2**

(b) **4**

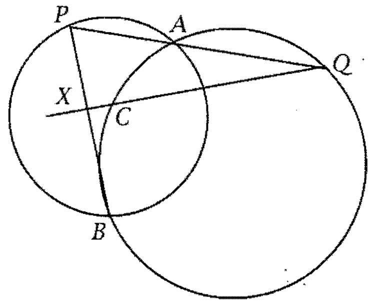



The diagram shows the graph of $y = x^2(6 - x^2)$ for $0 \le x \le \sqrt{6}$. The area bounded by this curve and the $x$-axis is rotated through one revolution about the $y$-axis.

Use the method of cylindrical shells to find the volume of the solid that is generated.

**Question continued .....**

(c)




The two circles intersect at $A$ and $B$. The larger circle passes through the centre $C$ of the smaller circle. $P$ and $Q$ are points on the circles such that $PQ$ passes through $A$. $QC$ is produced to meet $PB$ at $X$.

Let $\angle QAB = \theta$.

(i) Make a neat copy of the diagram on your answer sheet.

(ii) Show that $\angle BCX = 180^0 - \theta$. **2**

(iii) Prove that $\angle PXC = 90^0$. **4**

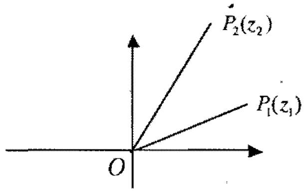


**Question 8** (15 marks) **Marks**

(a) Two of the roots of $x^3 + ax^2 + bx + c = 0$ are $\alpha$ and $\beta$. **4**

Prove that $\alpha\beta$ is a root of $x^3 - bx^2 + acx - c^2 = 0$.

(b) The points $P_1$ and $P_2$ represent the complex numbers $z_1$ and $z_2$ on the Argand diagram.




(i) Prove that $|z_1 - z_2| \geq |z_1| - |z_2|$ **2**

(ii) If $\left|z - \frac{4}{z}\right| = 2$ prove that the maximum value of $|z|$ is $\sqrt{5} + 1$. **3**

(c) (i) Prove that if the polynomial equation $P(x) = 0$ has a root of multiplicity $n$, then the derived polynomial equation $P'(x) = 0$ has the same root with multiplicity $n - 1$. **2**

(ii) If the equation $x^3 + 3px^2 + 3qx + r = 0$ has a repeated root, show that this root is $\frac{r - pq}{2(p^2 - q)}$, where $p^2 \neq q$. **4**

**This is the end of the paper.**

# SOLUTIONS : SBHS.

## Question 1.

(a) $\int \frac{dx}{\sqrt{4-9x^2}} = \frac{1}{3}\int \frac{dx}{\sqrt{\frac{4}{9}-x^2}}$ ②

$= \frac{1}{3}\sin^{-1}\frac{3x}{2} + c$

(b) Let $\frac{4}{(x-1)(2-x)} = \frac{A}{x-1} + \frac{B}{2-x}$

$\therefore 4 = A(2-x) + B(x-1)$

If $x=2$: $4 = B$

If $x=1$: $4 = A$. ③

$\therefore \int \frac{4\,dx}{(x-1)(2-x)} = 4\left(\int \frac{dx}{x-1} - \int \frac{dx}{x-2}\right)$

$= 4\left(\ln(x-1) - \ln(x-2)\right) + c$

$= 4\ln\left(\frac{x-1}{x-2}\right) + c$

(c) $\int t e^{t/4}\,dt$ $\quad u = t \quad v = 4e^{t/4}$; $\quad u' = 1 \quad v' = e^{t/4}$

$= 4te^{t/4} - \int 4e^{t/4}\,dt$ ③

$= 4te^{t/4} - 16e^{t/4} + c$

(d) $\int_0^{\pi/2} \frac{\sin 2\theta}{2+\cos\theta}\,d\theta$ $\quad$ Let $u = 2+\cos\theta$

$= \int_0^{\pi/2} \frac{2\sin\theta\cos\theta}{2+\cos\theta}\,d\theta$ $\quad du = -\sin\theta\,d\theta$

If $\theta = 0$ $u = 3$

$= 2\int_3^2 \frac{u-2}{u}(-du)$ $\quad$ If $\theta = \pi/2$ $u = 2$

$= 2\int_2^3 \left(1 - \frac{2}{u}\right)du$ ④

$= 2\left[u - 2\ln u\right]_2^3$

$= 2\left\{[3 - 2\ln 3] - [2 - 2\ln 2]\right\}$

$= 2\left\{1 + 2\ln 2 - 2\ln 3\right\}$

$= 2\left\{1 + 2\ln\frac{2}{3}\right\}$

$= 2 + 4\ln\frac{2}{3}$

(e) $\int_0^{2\pi} |\sin x|$

$= 2\int_0^{\pi} \sin x\,dx$

$= 2\left[-\cos x\right]_0^{\pi}$

$= 2\{[-(-1)] - [-1]\}$

$= 2\times 2$

$= 4$

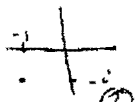
(f) $\int_{-4}^{4} \frac{dx}{x^3}$

Integral is not defined due to at $x = 0$.

## Question 2:

(a) (i) $w = -1 - i$

$= \sqrt{2}\,\text{cis}\left(-\frac{3\pi}{4}\right)$ ②



(ii) $w^{12} = \left(\sqrt{2}\,\text{cis}\left(-\frac{3\pi}{4}\right)\right)^{12}$

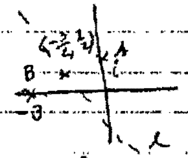
$= 2^6\,\text{cis}\left(-\frac{36\pi}{4}\right)$

$= 64\,\text{cis}(-9\pi)$

$= 64\,\text{cis}\,\pi$ ②

$= -64$

(b) $|z - i| = |z + 3|$



$m_{AB} = \frac{1}{3}$

$m_\ell = -3$

$\therefore$ Locus is $y - \frac{1}{2} = -3\left(x + \frac{3}{2}\right)$

$y - \frac{1}{2} = -3x - \frac{9}{2}$

$y = -3x - 4$ ②

(c) $Re\left(\frac{1}{z}\right) \le \frac{1}{2}$ $\quad (z \ne 0)$

$Re\left(\frac{1}{x+iy}\right) \le \frac{1}{2}$

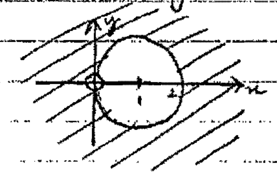
$Re\left(\frac{x-iy}{x^2+y^2}\right) \le \frac{1}{2}$

$\therefore \frac{x}{x^2+y^2} \le \frac{1}{2}$

$\therefore 2x \le x^2 + y^2$

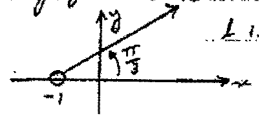
$\therefore x^2 - 2x + 1 + y^2 \ge 1$

$\therefore (x-1)^2 + y^2 \ge 1$ $\quad (z \ne 0)$



③

(d) (i) $\arg(z+1) = \frac{\pi}{3}$ $\ell$



$\ell$ is $y - 0 = \tan\frac{\pi}{3}(x+1)$

$y = \sqrt{3}(x+1)$ ①

(ii) $|z|$ is a minimum at $A$ where $OA \perp \ell$.

$A$ is $(a, \sqrt{3}(a+1))$

$m_{OA} = \frac{\sqrt{3}(a+1)}{a} = -\frac{1}{\sqrt{3}}$

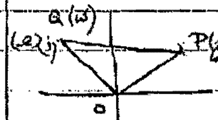
$\therefore 3(a+1) = -a$

$\therefore 4a + 3 = 0$

$\therefore a = -\frac{3}{4}$ ②

$\therefore A$ is $\left(-\frac{3}{4}, \frac{\sqrt{3}}{4}\right)$,

i.e. $z = -\frac{3}{4} + i\frac{\sqrt{3}}{4}$ is the required solution



$w = z\,\text{cis}\frac{\pi}{3}$

as $|w| = |z|$ and

$\angle QOP = \frac{\pi}{3}$ (equilateral)

$\therefore wz = \left(z\,\text{cis}\frac{\pi}{3}\right).z$ ①

$= z^2\,\text{cis}\frac{\pi}{3}$

(ii) $z^2 + w^2 = z^2 + z^2\,\text{cis}\frac{2\pi}{3}$

$= z^2\left(1 + \text{cis}\frac{2\pi}{3}\right)$

$= z^2\left(1 + \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

$= z^2\left(1 - \frac{1}{2} + i.\frac{\sqrt{3}}{2}\right)$

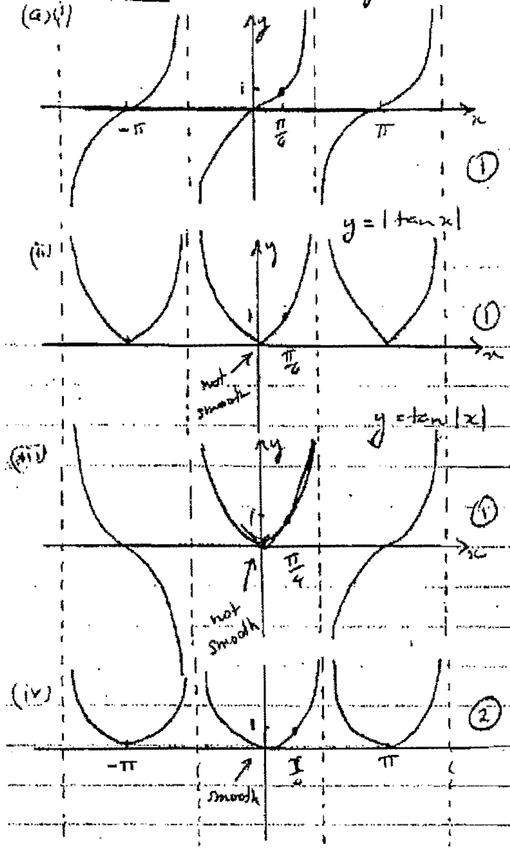
$= z^2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$

$= z^2\,\text{cis}\frac{\pi}{3}$ ②

$= wz$

## Question 3

(a)(i)




(b) $f(x) = \dfrac{x}{\ln x}, \quad x > 0$

(i) Domain $0 < x < 1, \ x > 1$ (2)

Asymptote $x = 1$

(ii) $f'(x) = \dfrac{\ln x \cdot 1 - x \cdot \frac{1}{x}}{(\ln x)^2}$

$= \dfrac{\ln x - 1}{(\ln x)^2}$

$f''(x) = \dfrac{(\ln x)^2 \cdot \frac{1}{x} - (\ln x - 1) \cdot 2\ln x \cdot \frac{1}{x}}{(\ln x)^4}$

$= \dfrac{\ln x - 2\ln x + 2}{x(\ln x)^3}$

$= \dfrac{2 - \ln x}{x(\ln x)^3}$

For turning point: $\ln x - 1 = 0$

$\therefore \ln x = 1$

$\therefore x = e$

$\therefore y = \dfrac{e}{\ln e} = e$

$y'' = \dfrac{2 - \ln e}{e(\ln e)^3}$

$= \dfrac{2-1}{e \cdot 1}$

$= \dfrac{1}{e} > 0$ (3)

$\therefore$ Min turning point at $(e, e)$

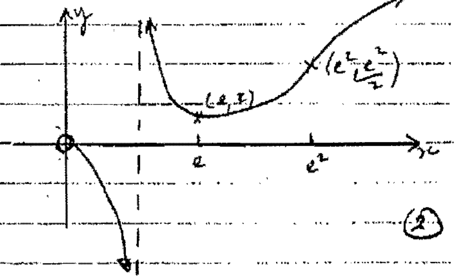
(iii)

| $x$ | $e^2 - \epsilon$ | $e^2$ | $e^2 + \epsilon$ |
|---|---|---|---|
| $y''$ | $\dfrac{2 - \ln(e^2-\epsilon)}{(e^2-\epsilon)\ln(e^2-\epsilon)} = \dfrac{+ve}{+ve \cdot +ve} = +ve$ | $\dfrac{2-\ln e^2}{e^2(\ln e^2)^3} = 0$ | $\dfrac{2-\ln(e^2+\epsilon)}{(e^2+\epsilon)\ln(e^2+\epsilon)} = \dfrac{-ve}{+ve \cdot +ve} = -ve$ |
| concavity | up | — | down |

(3)

$\therefore$ Change of concavity at $x = e^2$

$\therefore$ Pt of inflexion at $\left(e^2, \dfrac{e^2}{2}\right)$




[2] 4. (a) (i)

$z^3 - 1 = 0,$

$(z-1)(z^2+z+1) = 0,$

$\therefore z = 1 \text{ or } \dfrac{-1 \pm \sqrt{1-4}}{2},$

$= 1 \text{ or } \dfrac{-1 \pm \sqrt{3}i}{2}.$

[1] (ii) $(\omega - 1)(\omega^2 + \omega + 1) = 0$ from (i).
Now $\omega \neq 1$, $\therefore \omega^2 + \omega + 1 = 0$.

[3] (iii) $\alpha + \beta = 4 + \omega + 4 + \omega^2,$
$= 7 + 1 + \omega + \omega^2,$
$= 7.$
$\alpha\beta = 16 + 4\omega + 4\omega^2 + \omega^3,$
$= 12 + 4(1 + \omega + \omega^2) + 1,$
$= 13.$
$\therefore z^2 - 7z + 13 = 0.$

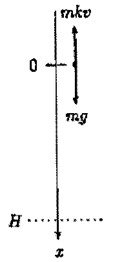
[3] (b) $P(x) = x^3 + ax^2 + bx + c.$
$P(0) = c = 4.$
$P(x) = (x^2+4)(x+\alpha) + x + 8.$
$P(0) = 4\alpha + 8 = -4,$
$\alpha = -3.$
$(x^2+4)(x-3) + 8 = x^3 - 3x^2 + 4x - 12 + x + 8.$
$\therefore P(x) = x^3 - 3x^2 + 5x - 4.$

[1] (c) (i) As the polynomial has real coefficients, if $(z - i\theta)$ is a factor then $(z + i\theta)$ is also a factor (conjugate root theorem).
i.e., $P(-i\theta) = 0.$

[1] (ii) $(z^2 + \theta^2)$ is a factor of $P(x)$. Let $(z - \alpha)$ be the last factor.
Sum of roots is $\theta i - \theta i + \alpha = -a.$
i.e., $\alpha = -a$ which is real,
$\therefore \alpha$ is real and there is one real root.

[4] (iii) Taking roots two at a time,
$b = \theta^2 + \theta i a - \theta i a,$
$= \theta^2.$
$\therefore b > 0$ as $\theta \in \mathbb{R}.$
Product of roots, $-c = -\theta^2 i^2(-a),$
$c = \theta^2 a,$
$= ab.$

[1] 5. (a) (i)




$\sum F_x = ma = mg - mkv,$
$a = g - kv.$
i.e. $v\dfrac{dv}{dx} = g - kv.$

[4] (ii) $\int dx = \int \frac{v\,dv}{g-kv}$,

$$= -\frac{1}{k}\int \frac{g-kv}{g-kv}dv + \frac{-g}{k^2}\int \frac{-k\,dv}{g-kv}.$$

$$x = -\frac{v}{k} - \frac{g}{k^2}\ln(g-kv) + c.$$

When $x = 0$, $v = 0$, $\therefore$ $c = \frac{g}{k^2}\ln g$.

$$x = \frac{g}{k^2}\ln\left(\frac{g}{g-kv}\right) - \frac{v}{k}.$$

When $x = H$, $v = V_0$,

$$H = \frac{g}{k^2}\ln\left(\frac{g}{g-kV_0}\right) - \frac{V_0}{k}.$$

Rearranging, $\ln\left(\frac{g-kV_0}{g}\right) + \frac{kV_0}{g} + \frac{k^2H}{g} = 0$,

i.e., $\ln\left(1-\frac{kV_0}{g}\right) + \frac{kV_0}{g} + \frac{k^2H}{g} = 0$.

[3] (iii) $\frac{dv}{dt} = g - kv$,

$$\int dt = \frac{-1}{k}\int \frac{-k\,dv}{g-kv},$$

$$t = -\frac{1}{k}\ln(g-kv) + c.$$

When $t = 0$, $v = 0$, $\therefore$ $c = \frac{1}{k}\ln g$,

So $t = \frac{1}{k}\ln\left(\frac{g}{g-kv}\right)$.

When $t = T$, $v = V_0$,

$\therefore$ $T = \frac{1}{k}\ln\left(\frac{g}{g-kV_0}\right)$.

[2] (iv) $\ln\left(1-\frac{kV_0}{g}\right) = -kT$ from (iii).

Substitute in (ii),

$$-kT + \frac{kV_0}{g} + \frac{k^2H}{g} = 0,$$

$$\frac{kV_0}{g} = kT - \frac{k^2H}{g},$$

$$V_0 = gT - kH.$$

[1] (v) Terminal velocity occurs when $\ddot{x} = 0$,

i.e. $V_T = \frac{g}{k}$.

Now $V_0 < V_T$,

$\therefore$ $V_0 < \frac{g}{k}$

$$T = \frac{V_0}{g} + \frac{kH}{g} \text{ from (iv)},$$

$$T < \frac{g}{k} \times \frac{1}{g} + \frac{kH}{g},$$

$$T < \frac{1}{k} + \frac{kH}{g}.$$

[2] (b) At least two together $\Longrightarrow$ not all separate.

Total number of arrangements in a circle $= 7!$

Number of arrangements where separated $= 3!4!$

$\therefore$ Ways with at least two together $= 7! - 3!4!$

$= 4896$.

[2] (c) Number of ways $= \binom{5}{1} + \binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5}$,

$= 2^5 - 1$,

$= 31$.

[2] 6. (a) (i) $\text{cis}\,3\theta = (\text{cis}\,\theta)^3$, by De Moivre's Theorem.

i.e., $\cos 3\theta + i\sin 3\theta = \cos^3\theta + 3i\sin\theta\cos^2\theta + 3i^2\sin^2\theta\cos\theta + i^3\sin^3\theta$.

Equating real and imaginary parts,

$$\begin{aligned}\cos 3\theta &= \cos^3\theta - 3\sin^2\theta\cos\theta,\\ &= \cos^3\theta - 3(1-\cos^2\theta)\cos\theta,\\ &= \cos^3\theta - 3\cos\theta + 3\cos^3\theta,\\ &= 4\cos^3\theta - 3\cos\theta.\end{aligned}$$

$$\begin{aligned}\sin 3\theta &= 3\sin\theta\cos^2\theta - \sin^3\theta,\\ &= 3\sin\theta(1-\sin^2\theta) - \sin^3\theta,\\ &= 3\sin\theta - 3\sin^3\theta - \sin^3\theta,\\ &= 3\sin\theta - 4\sin^3\theta.\end{aligned}$$

[2] (ii)

$$\begin{aligned}\cot 3\theta &= \frac{\cos 3\theta}{\sin 3\theta},\\ &= \frac{4\cos^3\theta - 3\cos\theta}{3\sin\theta - 4\sin^3\theta},\\ &= \frac{4\cot^3\theta - 3\cot\theta.\sec^2\theta}{3\sec^2\theta - 4},\\ &= \frac{4\cot^3\theta - 3\cot\theta.(1+\cot^2\theta)}{3(1+\cot^2\theta) - 4},\\ &= \frac{4\cot^3\theta - 3\cot\theta - 3\cot^3\theta}{3 + 3\cot^2\theta - 4},\\ &= \frac{\cot^3\theta - 3\cot\theta)}{3\cot^2\theta - 1},\\ &= \frac{t^3 - 3t}{3t^2 - 1}, \text{ using } t = \cot\theta.\end{aligned}$$

[2] (iii) Now $\cot 3\theta = 1$, $\quad 0 \le \theta \le 2\pi$

$3\theta = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}$, $\quad 0 \le 3\theta \le 6\pi$ $\left(\frac{24\pi}{4}\right)$

$\frac{13\pi}{4}, \frac{17\pi}{4}, \frac{21\pi}{4}$

$\therefore$ $\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{9\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}, \frac{21\pi}{12}$.

[2] (iv) $\frac{t^4 - 3t}{3t^2 - 1} = 1$.

$\therefore$ $t^3 - 3t^2 - 3t + 1 = 0$.

As $\cot\theta = \cot(\pi + \theta)$,

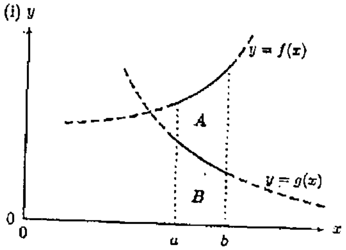
$\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{9\pi}{12}$ are the only distinct values from (iii) above.

So $t = \cot\theta = \cot\frac{\pi}{12}$, $\cot\frac{5\pi}{12}$, $\cot\frac{9\pi}{12}$ are the roots.

Product of the roots, $-1 = \cot\frac{\pi}{12}.\cot\frac{5\pi}{12}.\cot\frac{9\pi}{12}$.

1 (v) $\frac{1}{x^3} - \frac{3}{x^2} - \frac{3}{x} + 1 = 0,$
$x^3 - 3x^2 - 3x + 1 = 0.$

2 (b) (i)




$\int_a^b f(x)\,dx$ is shown by $A + B$ and
$\int_a^b g(x)\,dx$ is shown by $B$.
It is clear that $A + B > B$,
*i.e.*, $\int_a^b f(x)\,dx > \int_a^b g(x)\,dx$.

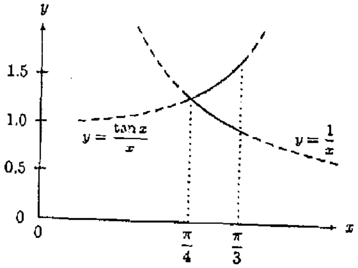
1 (ii) $y = \tan x,$
$y' = \sec^2 x > 1 \;\forall\, x.$
$\therefore$ $\tan x$ is an increasing function, $\frac{\pi}{4} \le x \le \frac{\pi}{3}$ (discontinuities at $\pm\frac{\pi}{2}$ are outside the range).

3 (iii) When $x = \frac{\pi}{4}$, $\tan x = 1$,
and for $\frac{\pi}{4} < x \le \frac{\pi}{3}$, $\tan x > 1$ as $\tan x$ is an increasing function.
$\therefore$ $\frac{\tan x}{x} > \frac{1}{x}$ as $x > 0$.

$\therefore$ by part (i): $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\tan x}{x}dx > \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{dx}{x}$.

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{dx}{x} = [\ln x]_{\frac{\pi}{4}}^{\frac{\pi}{3}},$$
$$= \ln\left(\frac{\pi}{3}\cdot\frac{4}{\pi}\right),$$
$$= \ln\frac{4}{3}.$$

*i.e.*, $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\tan x}{x}dx > \ln\frac{4}{3}$. [See sketch.]




QUESTION 7

(a) (i) $I_n = \int_1^e x(\ln x)^n\,dx$

$$= \int_1^e \frac{d}{dx}\left(\tfrac{1}{2}x^2\right)\cdot(\ln x)^n\,dx.$$
$$= \left[\tfrac{1}{2}x^2(\ln x)^n\right]_1^e - \int_1^e \tfrac{1}{2}x^2\cdot n(\ln x)^{n-1}\cdot\tfrac{1}{x}\cdot dx.$$
$$= \tfrac{1}{2}e^2(\ln e)^n - \tfrac{1}{2}\cdot 1^2(\ln 1)^n - \tfrac{n}{2}\int_1^e x(\ln x)^{n-1}\,dx$$
$$= \tfrac{1}{2}e^2 - \tfrac{1}{2}\cdot 0 - \tfrac{n}{2}I_{n-1}$$
$$\therefore\ \boxed{I_n = \tfrac{1}{2}e^2 - \tfrac{n}{2}I_{n-1}}.$$

(ii) $I_3 = \frac{e^2}{2} - \frac{3}{2}I_2$

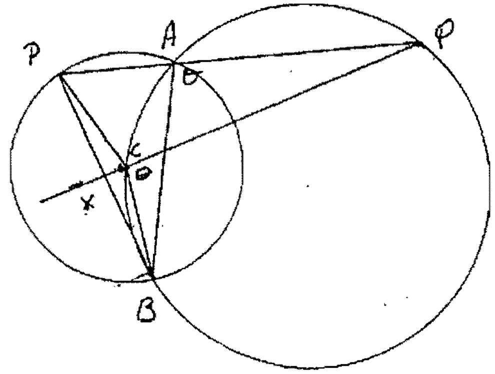
$$= \frac{e^2}{2} - \frac{3}{2}\left[\frac{e^2}{2} - I_1\right]$$
$$= \frac{e^2}{2} - \frac{3e^2}{4} + \frac{3}{2}\left[\frac{e^2}{2} - \frac{1}{2}I_0\right]$$
$$= \frac{e^2}{2} - \frac{3e^2}{4} + \frac{3e^2}{4} - \frac{3}{4}I_0 \quad ; \quad I_0 = \int_1^e x\,dx = \left[\frac{x^2}{2}\right]_1^e = \frac{e^2}{2} - \frac{1}{2}.$$
$$= \frac{e^2}{2} - \frac{3e^2}{4} + \frac{3e^2}{4} - \frac{3}{4}\left(\frac{e^2}{2} - \frac{1}{2}\right)$$
$$= \frac{e^2}{2} - \frac{3e^2}{8} + \frac{3}{8}$$
$$\therefore\ \boxed{I_3 = \frac{e^2}{8} + \frac{3}{8}.}$$

(b) $V = \lim_{\delta x \to 0} \sum_{x=0}^{\sqrt{6}} 2\pi x y\,\delta x.$

$$= \int_0^{\sqrt{6}} 2\pi x y\,dx.$$
$$= 2\pi\int_0^{\sqrt{6}} x^3(6 - x^2)\,dx.$$
$$= 2\pi\int_0^{\sqrt{6}} (6x^3 - x^5)\,dx$$
$$= 2\pi\left[\frac{3x^4}{2} - \frac{x^6}{6}\right]_0^{\sqrt{6}}$$
$$= 2\pi\left[54 - 36\right]$$
$$\boxed{= 36\pi\ u^3}$$

Q7 (cont'd)

(c)




(ii) now $B\hat{C}Q = B\hat{A}Q = \theta$ (angles in the same segment standing on the same arc, are equal)

$\therefore B\hat{C}X = 180^\circ - \theta$ (angles are supplementary)

(iii) now $P\hat{A}B = 180^\circ - \theta$ (supplementary angles)

$P\hat{C}B = 360^\circ - 2\theta$ (angle at the centre is double the angle at the circumference standing on same arc)

$\therefore P\hat{C}X = X\hat{C}B$ $(180^\circ - \theta + 180^\circ - \theta = 360^\circ - 2\theta)$

$CX$ is common

$PC = BC$ (equal radii)

$\therefore \triangle PCX \equiv \triangle BCX$ (SAS)

$\therefore P\hat{X}C = C\hat{X}B$ (corresponding angles of congruent $\triangle$'s)

now $P\hat{X}C + C\hat{X}B = 180^\circ$ (supplementary angles)

$\therefore P\hat{X}C = 90^\circ$.

8 (c) (i) If $P(x) = 0$ has a root of multiplicity $n$, say $\alpha$.

then $P(x) = (x-\alpha)^n \cdot Q(x)$

$P'(x) = n(x-\alpha)^{n-1} Q(x) + (x-\alpha)^n \cdot Q'(x)$.

$= (x-\alpha)^{n-1}[n\,Q(x) + (x-\alpha)\,Q'(x)]$

now since $Q'(x)$ is a polynomial

$n\,Q(x) + (x-\alpha)\,Q'(x)$ is a polynomial say $T(x)$.

$\therefore P'(x) = (x-\alpha)^{n-1}\,T(x)$.

which has a root $\alpha$ of multiplicity $n-1$.

(ii) given $x^3 + 3px^2 + 3qx + r = 0$ has a multiple root. (say $\alpha$)

$\therefore \alpha^3 + 3p\alpha^2 + 3q\alpha + r = 0$ — (1)

& $3\alpha^2 + 6p\alpha + 3q = 0$ (2) from (i).

$\therefore \alpha^2 + 2p\alpha + q = 0$ (2a)

hence $\alpha^3 + 2p\alpha^2 + q\alpha = 0$ — (3)

From (1) & (3)

$p\alpha^2 + 2q\alpha + r = 0$ — (4)

(2a) × p

$p\alpha^2 + 2p^2\alpha + pq = 0$ (2b)

2(b) − (4)

$2(p^2 - q)\alpha + pq - r = 0$

$\therefore 2(p^2-q)\alpha = r - pq$

$$\alpha = \frac{r - pq}{2(p^2-q)}$$

QUESTION 2.

(a) The roots of $x^3+ax^2+bx+c=0$ (A) are $\alpha, \beta$ & $\gamma$ (NB must have 3 roots)

need to establish equation with roots $\alpha\beta, \alpha\gamma$ & $\beta\gamma$.

now $\alpha\beta\gamma = -c$.

$\alpha\beta = \frac{-c}{\gamma}$ ∴ use $X = \frac{-c}{x}$ ie $x = -\frac{c}{X}$ in (A)

$$\left(\frac{-c}{X}\right)^3 + a\left(\frac{-c}{X}\right)^2 + b\left(\frac{-c}{X}\right) + c = 0$$

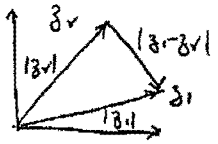
$$\frac{-c^3}{X^3} + \frac{ac^2}{X^2} - \frac{bc}{X} + c = 0$$

$$-c^3 + ac^2X - bcX^2 + cX^3 = 0$$

$$\Rightarrow X^3 - bX^2 + acX - c^2 = 0$$

OR $x^3 - bx^2 + acx - c^2 = 0$

(b) (i)




From triangle inequality, "the sum of the lengths of any two sides exceeds the third side".

ie $|z_1 - z_2| + |z_2| \geq |z_1|$

∴ $|z_2 - z_1| \geq |z_1| - |z_2|$.

(ii) Since $\left|z - \frac{4}{z}\right| = 2$ and $\left|z - \frac{4}{z}\right| \geq |z| - \left|\frac{4}{z}\right|$

then $|z| - \frac{4}{|z|} \leq 2$.

$|z|^2 - 4 \leq 2|z|$

$|z|^2 - 2|z| - 4 \leq 0$

$|z|^2 - 2|z| + 1 \leq 5$

$(|z| - 1)^2 \leq 5$

∴ $-\sqrt{5} \leq |z| - 1 \leq \sqrt{5}$

$-\sqrt{5} + 1 \leq |z| \leq \sqrt{5} + 1$

∴ $|z| \leq \sqrt{5} + 1 \Rightarrow$ max. value of $|z| = \sqrt{5} + 1$.