SYDNEY GRAMMAR SCHOOL
MATHEMATICS DEPARTMENT
TRIAL EXAMINATIONS 2004

FORM VI

MATHEMATICS EXTENSION 1

Examination date

Tuesday 10th August 2004

Time allowed

2 hours (plus 5 minutes reading time)

Instructions

All seven questions may be attempted.
All seven questions are of equal value.
All necessary working must be shown.
Marks may not be awarded for careless or badly arranged work.
Approved calculators and templates may be used.
A list of standard integrals is provided at the end of the examination paper.

Collection

Write your candidate number clearly on each booklet.
Hand in the seven questions in a single well-ordered pile.
Hand in a booklet for each question, even if it has not been attempted.
If you use a second booklet for a question, place it inside the first.
Keep the printed examination paper and bring it to your next Mathematics lesson.

Checklist

SGS booklets: 7 per boy. A total of 1000 booklets should be sufficient.
Candidature: 121 boys.

Examiner

MLS

QUESTION ONE (12 marks) Use a separate writing booklet. **Marks**

(a) Solve the inequation $\dfrac{4}{5-x} \leq 1$. $\boxed{3}$

(b) For what value of p is the expression $4x^3 - x + p$ divisible by $x + 3$? $\boxed{2}$

(c) Expand $\left(a + \frac{1}{2}\right)^5$, expressing each term in its simplest form. $\boxed{2}$

(d) Given the points $A(1,4)$ and $B(5,2)$, find the co-ordinates of the point that divides the interval AB externally in the ratio $1:3$. $\boxed{2}$

(e) Find $\displaystyle\int x(1-x^2)^5\,dx$, using the substitution $u = 1 - x^2$, or otherwise. $\boxed{3}$

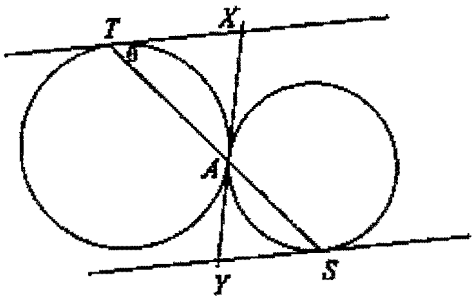
QUESTION TWO (12 marks) Use a separate writing booklet. **Marks**

(a) Consider the parabola $x = 4t$, $y = 2t^2$.

(i) Find the gradient of the parabola at the point where $t = 4$. $\boxed{1}$

(ii) Find the equation of the tangent to the parabola at $t = 4$. $\boxed{2}$

(b)



In the diagram above, two circles touch one another externally at the point A. A straight line through A meets one of the circles at T and the other at S. The tangents at T and S meet the common tangent at A at X and Y respectively.

Let $\theta = \angle XTA$.

(i) Explain why $\angle XAT$ is θ. $\boxed{1}$

(ii) Prove that $TX \parallel YS$. $\boxed{2}$

(c) (i) Write down the first three terms in the expansion of $(1 + mx)^n$. $\boxed{1}$

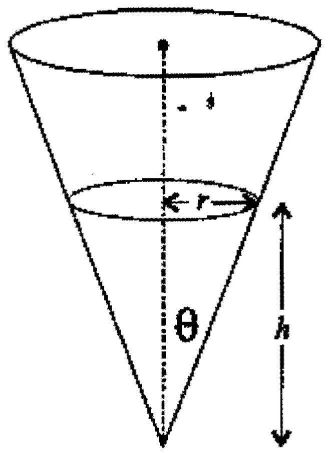
(ii) If $(1 + mx)^n \equiv 1 - 4x + 7x^2 - \cdots$, find the values of m and n. $\boxed{3}$

(d) Evaluate $\displaystyle\lim_{x \to 0} \frac{5x \cos 2x}{\sin x}$, showing your reasoning. $\boxed{2}$

Exam continues next page ...

QUESTION THREE (12 marks) Use a separate writing booklet. **Marks**

(a)



The diagram above shows a container in the shape of a right circular cone. The semi-vertical angle $\theta = \tan^{-1}\frac{1}{2}$.

Water is poured in at the constant rate of 10 cm^3 per minute.

Let the height of the water at time t seconds be h cm, let the radius of the water surface be r cm, and let the volume of water be V cm^3.

(i) Show that $r = \frac{1}{2}h$. $\boxed{1}$

(ii) Show that $V = \frac{1}{12}\pi h^3$. $\boxed{1}$

(iii) Find the exact rate at which h is increasing when the height of the water in the cone is 50 cm. $\boxed{2}$

(b) Show that there is no term independent of x in the expansion of $\left(2x^2 - \dfrac{1}{4x}\right)^{11}$. $\boxed{3}$

(c) Evaluate $\displaystyle\int_{-1}^{0} x\sqrt{1+x}\,dx$, using the substitution $u = 1 + x$. $\boxed{4}$

(d) Find $\displaystyle\int \sin x \cos^3 x\,dx$. $\boxed{1}$

Exam continues overleaf ...

QUESTION FOUR (12 marks) Use a separate writing booklet. **Marks**

(a) If $y = \frac{1}{200}te^{-t}$, show that $\frac{dy}{dt} = \frac{1}{200}(1-t)e^{-t}$. **1**

(b) Fred has recently consumed three standard alcoholic drinks. Immediately after he has finished his last drink, his blood alcohol level is measured over a four-hour period.

Let his blood alcohol level at any time t be A, where t is the time in hours after his last drink.

It is found that the rate of change $\frac{dA}{dt}$ of his blood alcohol content is given by

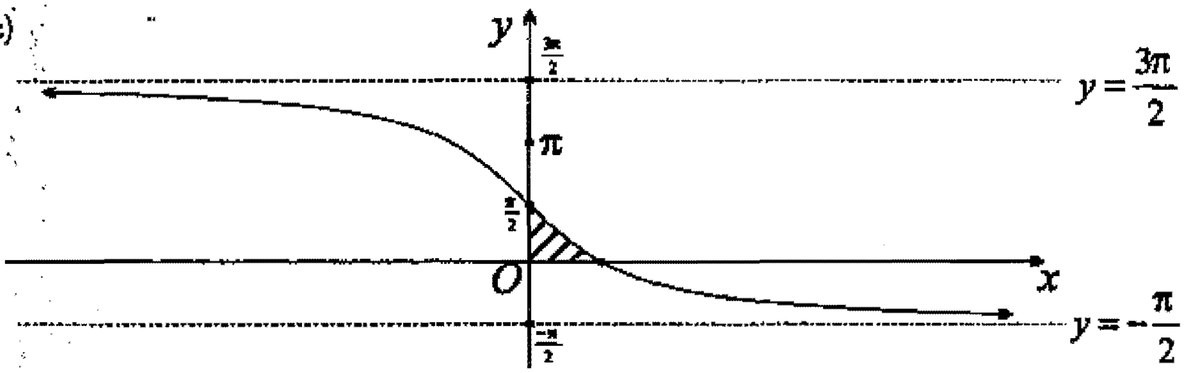
$$\frac{dA}{dt} = \frac{1}{200}(1-t)e^{-t}, \quad \text{where } 0 \le t \le 4.$$

(i) Show that his blood alcohol content increases during the first hour and decreases after the first hour. **2**

(ii) Initially his blood alcohol content was 0·0005. Find A as a function of t. You will need to use part (a). **2**

(iii) Determine his maximum alcohol content during the four-hour period. Give your answer correct to four decimal places. **1**

(c)



The graph of the curve $y = \frac{\pi}{2} - 2\tan^{-1}x$ is drawn above. It cuts the y-axis at $(0, \frac{\pi}{2})$.

(i) Write down the domain of the inverse function of $y = \frac{\pi}{2} - 2\tan^{-1}x$. **1**

(ii) Find the equation of the inverse function of $y = \frac{\pi}{2} - 2\tan^{-1}x$. **1**

(iii) Find the volume generated when the shaded region is rotated about the y-axis. **4**

Exam continues next page ...

QUESTION FIVE (12 marks) Use a separate writing booklet. **Marks**

(a) Evaluate $\displaystyle\int_0^4 \frac{1}{3+\sqrt{x}}\,dx$, using the substitution $x = (u-3)^2$. [3]

(b) (i) Write down the expansion of $(1+x)^n$ in ascending powers of x. Then differentiate both sides of your identity. [1]

(ii) Make an appropriate substitution for x to show that [1]

$$\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + 4\binom{n}{4} + \cdots + n\binom{n}{n} = n(2^{n-1}).$$

(iii) Hence find an expression for [1]

$$2\binom{n}{1} + 3\binom{n}{2} + 4\binom{n}{3} + 5\binom{n}{4} + \cdots + (n+1)\binom{n}{n}.$$

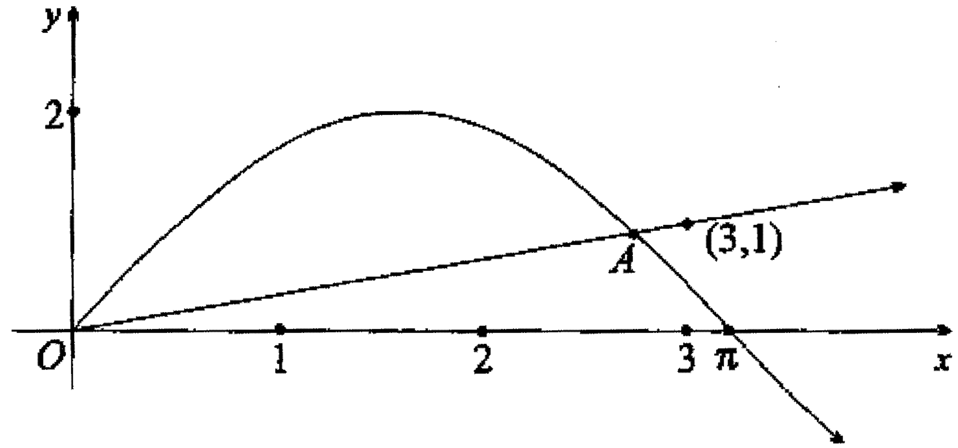
(c) Find values for R and α if $\sqrt{3}\sin\theta - \cos\theta = R\cos(\theta+\alpha)$, where R and α are positive constants and $0 < \alpha < 2\pi$. [2]

(d) Use the method of mathematical induction to prove that [4]

$$\frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{n}{(n+1)!} = \frac{(n+1)!-1}{(n+1)!}, \quad \text{for all positive integers } n.$$

QUESTION SIX (12 marks) Use a separate writing booklet. **Marks**

(a)



The sketch above shows the curve $y = 2\sin x$ and the line $x - 3y = 0$. The graphs meet at the point A in the first quadrant.

(i) Write down an equation whose solution gives the x-coordinate of A. [1]

(ii) An approximate value for the x-coordinate of A is $x = 3$. Apply Newton's method once to find a closer approximation for this value. Give your answer correct to one decimal place. [2]

(b) Newton's law of cooling states that a body cools according to the equation

$$\frac{dT}{dt} = -k(T - S),$$

where T is the temperature of the body at time t, S is the temperature of the surroundings and k is a constant.

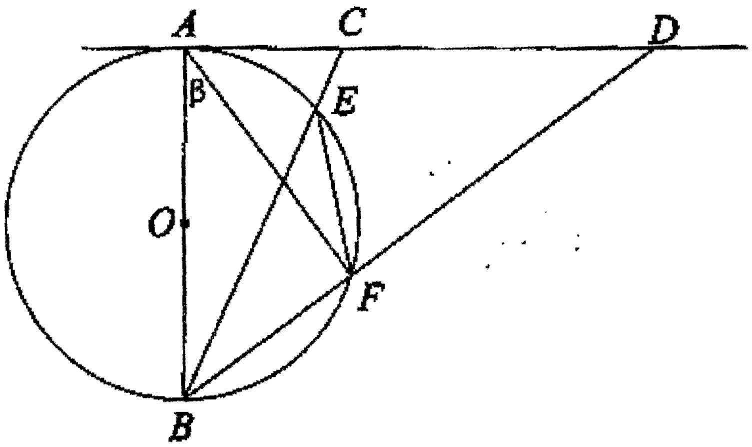
(i) Show that $T = S + Ae^{-kt}$ satisfies the equation, where A is a constant. **1**

(ii) A metal rod has an initial temperature of 470°C and cools to 250°C in 10 minutes. The surrounding temperature is 30°C.

(α) Find the value of A and show that $k = \frac{1}{10}\log_e 2$. **2**

(β) Find how much longer it will take the rod to cool to 70°C, giving your answer correct to the nearest minute. **2**

(c)



In the diagram above, the straight line ACD is a tangent at A to the circle with centre O. The interval AOB is a diameter of the circle. The intervals BC and BD meet the circle at E and F respectively.

Let $\angle BAF = \beta$.

Copy or trace this diagram into your answer booklet.

(i) Explain why $\angle ABF = \frac{\pi}{2} - \beta$. **1**

(ii) Prove that the quadrilateral $CDFE$ is cyclic. **3**

Exam continues next page ...

QUESTION SEVEN (12 marks) Use a separate writing booklet. **Marks**

(a) Car A and car B are travelling along a straight level road at constant speeds V_A and V_B respectively. Car A is behind car B, but is travelling faster.

When car A is exactly D metres behind car B, car A applies its brakes, producing a constant deceleration of $k\,\text{m/s}^2$.

(i) Using calculus, find the speed of car A after it has travelled a distance x metres under braking. **2**

(ii) Prove that the cars will collide if $V_A - V_B > \sqrt{2kD}$. **4**

(b) A particle is moving in simple harmonic motion of period T about a centre O. Its displacement at any time t is given by $x = a \sin nt$, where a is the amplitude.

(i) Draw a neat sketch of one period of this displacement–time equation, showing all intercepts. **1**

(ii) Show that $\dot{x} = \dfrac{2\pi a}{T} \cos \dfrac{2\pi t}{T}$. **1**

(iii) The point P lies D units on the positive side of O. Let V be the velocity of the particle when it first passes through P. **4**

Show that the time between the first two occasions when the particle passes through P is $\dfrac{T}{\pi} \tan^{-1} \dfrac{VT}{2\pi D}$.

END OF EXAMINATION

Question 1   2004 Trial Ext I

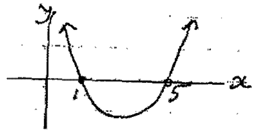
(a) $\frac{4}{5-x} \leq 1$

$4(5-x) \leq (5-x)^2$ ✓

$4(5-x) - (5-x)^2 \leq 0$

$(5-x)(4-(5-x)) \leq 0$

$(5-x)(x-1) \leq 0$ ✓



Solution: $x \leq 1$ or $x > 5$. ✓

(b) Let $P(x) = 4x^3 - x + p$.

If $P(x)$ is divisible by $(x+3)$ then $P(-3) = 0$

$P(-3) = -108 + 3 + p$ ✓

$-108 + 3 + p = 0$

$p = 105$ ✓

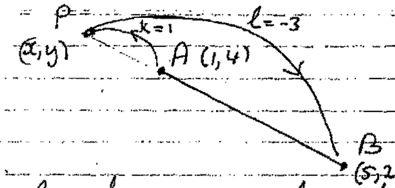
(c) $(a+\frac{1}{2})^5 = a^5 + 5a^4\frac{1}{2} + 10a^3(\frac{1}{2})^2 + 10a^2(\frac{1}{2})^3 + 5a(\frac{1}{2})^4 + (\frac{1}{2})^5$

$= a^5 + \frac{5}{2}a^4 + \frac{5}{2}a^3 + \frac{5}{4}a^2 + \frac{5}{16}a + \ldots$

✓ for correct bin coeff 1, 5, 10, 10, 5, 1

✓ for all correct.

(d)



$x = \frac{lx_1 + kx_2}{l+k}$, $y = \frac{ly_1 + ky_2}{l+k}$

$= \frac{-3+5}{-2}$, $y = \frac{-12+2}{-2}$

$= -1$, $= 5$

The point is $(-1, 5)$ ✓ ✓

Give ✓ if they find the pt that divides AB internally.

(e) $\int x(1-x^2)^5\,dx$

$= \int -\frac{1}{2}u^5\,du$ ✓

$= -\frac{1}{2}\frac{u^6}{6} + c$

$= -\frac{1}{12}(1-x^2)^6 + c$ ✓

$u = 1-x^2$

$du = -2x\,dx$

$-\frac{1}{2}du = x\,dx$ ✓

Question 2.

a) (i) $x = 4t$ and $y = 2t^2$

$\frac{dx}{dt} = 4$ $\quad$ $\frac{dy}{dt} = 4t$

so $\frac{dy}{dx} = \frac{4t}{4}$

$= t$

so at $t = 4$, the gradient is 4 ✓

(ii) When $t = 4$, $x = 16$; $y = 32$ ✓

Tangent equation is $y - 32 = 4(x - 16)$

$y - 32 = 4x - 64$

$y = 4x - 32$ ✓

(b)

(i) Tangents to a circle from an external point are equal ✓

$\therefore$ $TX = XA$

So ΔTXA is isosceles and $\angle XTA = \angle XAT = \theta$

(base angles of isosceles triangle)

(ii) Similarly, ΔAYS is isosceles with base angles $\angle YAS$ and $\angle YSA$ equal

But $\angle YAS = \angle TAX = \theta$ (vertically opposite) ✓

So $\angle YSA = \theta$ and $\angle XTA = \theta$

But these are alternate

So $TX \parallel YS$ ✓

(c) (i) $(1+mx)^n = 1 + nmx + \frac{n(n-1)}{2}(mx)^2 + \ldots$ ✓

(ii) $1 - 4x + 7x^2 - \ldots = 1 + nmx + \frac{n(n-1)}{2}(mx)^2 + \ldots$

equate coefficients of x: $-4 = nm$ ①

equate coefficients of x^2: $7 = \frac{n(n-1)}{2} m^2$ ② ✓

from ①, $m = -\frac{4}{n}$, substitute this in ②.

$14 = n(n-1)\frac{16}{n^2}$

$14 = \frac{16(n-1)}{n}$

✓ for a sensible method

$14n = 16n - 16$

$2n = 16$

$n = 8$

$m = -\frac{4}{8}$

$= -\frac{1}{2}$

$n = 8$ and $m = -\frac{1}{2}$ ✓

(d) $\lim_{x \to 0} \frac{5x \cos 2x}{\sin x} = \lim_{x \to 0} \frac{5x}{\sin x} \times \lim_{x \to 0} \cos 2x$

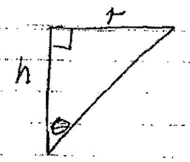
$= 5 \lim_{x \to 0} \frac{x}{\sin x} \times 1$ ✓

$= 5 \times 1$

$= 5$ ✓

Q3

a) (i)



$\tan\theta = \frac{r}{h}$

$\frac{1}{2} = \frac{r}{h}$

$r = \frac{1}{2}h$

(ii) $V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi (\frac{1}{2}h)^2 h$

$= \frac{1}{12}\pi h^3$

(iii) find $\frac{dh}{dt}$, given $\frac{dV}{dt} = 10$

$\frac{dh}{dt} = \frac{dh}{dV}\frac{dV}{dt}$

now $V = \frac{1}{12}\pi h^3$

$\frac{dV}{dh} = \frac{\pi h^2}{4}$ so $\frac{dh}{dV} = \frac{4}{\pi h^2}$

so $\frac{dh}{dt} = \frac{4}{\pi h^2} \times 10$

$= \frac{4}{\pi \times 50^2} \times 10$ when $h = 50$

$= \frac{4\cancel{0}}{\pi \times 5\cancel{0} \times 50}$

$= \frac{4}{250\pi}$ cm per minute

(b) The general term is $\binom{11}{r}(2x^2)^{11-r}(-4x)^{-r}$

the index of x is $22-2r-r=0$ if the term is independent of x.

$3r = 22$

$r = \frac{22}{3}$

since r must be an integer, there is no term independent of x.

(c) $\int_{-1}^{0} x\sqrt{1+x}\,dx$

$u = 1+x$

$du = dx$

when $x=0$, $u=1$

when $x=-1$, $u=0$

$= \int_0^1 (u-1)u^{\frac{1}{2}}\,du$

$= \int_0^1 (u^{3/2} - u^{\frac{1}{2}})\,du$

$= \left[\frac{2u^{5/2}}{5} - \frac{2u^{3/2}}{3}\right]_0^1$

$= (\frac{2}{5} - \frac{2}{3}) - (0)$

$= -\frac{4}{15}$

(d) $\int \sin.\cos^3 x\,dx$

$= -\frac{1}{4}\cos^4 x + c$

Question 4

a) $y = \frac{1}{200} \, t \cdot e^{-t}$

$\frac{dy}{dt} = \frac{1}{200}\left[t \times (-e^{-t}) + e^{-t} \right]$

$= \frac{1}{200} e^{-t}(1-t).$

b)(i) $\frac{dA}{dt} = \frac{1}{200}(1-t)e^{-t}$

Now, $\frac{1}{200}(1-t)e^{-t} > 0$ when $1-t>0$
or $t<1$

so, for $0<t<1$, $\frac{dA}{dt} > 0$ and A is increasing.

and $\frac{1}{200}(1-t)e^{-t} < 0$ when $1-t<0$
$t>1$

so, for $t>1$, $\frac{dA}{dt} < 0$ and A is decreasing

(ii) $t=0$, $A = 0.0005$

$A = \int \frac{1}{200}(1-t)e^{-t} \, dt$

$= \frac{1}{200} t e^{-t} + c$ from (a).

when $t=0$, $0.0005 = 0 + c$ so $c = 0.0005$

$A = \frac{1}{200} t e^{-t} + 0.0005$

(iii) from (i), the maximum A is when $t=1$.

so, if $t=1$, $A = \frac{1}{200} e^{-1} + 0.0005$

$\doteq 0.001839 + 0.0005$

$\doteq 0.002339$

$\doteq 0.0023$

(c) (i) $-\frac{\pi}{2} < x < \frac{3\pi}{2}$

(ii) $x = \frac{\pi}{2} - 2\tan^{-1} y$

$2\tan^{-1} y = \frac{\pi}{2} - x$

$\tan^{-1} y = \frac{1}{2}\left(\frac{\pi}{2} - x\right)$

$y = \tan\left(\frac{\pi}{4} - \frac{1}{2}x\right), \quad -\frac{\pi}{2} < x < \frac{3\pi}{2}$

(iii) $V = \pi \int x^2 \, dy$

or, using the inverse function

$V = \pi \int y^2 \, dx$

$= \pi \int_0^{\frac{\pi}{2}} \tan^2\left(\frac{\pi}{4} - \frac{1}{2}x\right) dx$

$= \pi \int_0^{\frac{\pi}{2}} \sec^2\left(\frac{\pi}{4} - \frac{1}{2}x\right) - 1 \quad dx$

$= \pi \left[-2\tan\left(\frac{\pi}{4} - \frac{1}{2}x\right) - x \right]_0^{\frac{\pi}{2}}$

$= \pi \left\{ \left(-2\tan 0 - \frac{\pi}{2}\right) - \left(-2\tan\frac{\pi}{4} - 0\right) \right\}$

$\pi\left(2 - \frac{\pi}{2}\right)$ ...

uestion 5

1 $\displaystyle\int_0^4 \frac{1}{3+\sqrt{x}}\,dx$

$x = (u-3)^2$

$dx = 2(u-3)\,du$

when $x=4$, $u=5$

when $x=0$, $u=3$.

$$= \int_3^5 \frac{2(u-3)}{u}\,du$$

$$= 2\int_3^5 \left(1 - \frac{3}{u}\right) du$$

$$= 2\left[u - 3\log_e u\right]_3^5$$

$$= 2\left[(5 - 3\log 5) - (3 - 3\log 3)\right]$$

$$= 2\left(2 - 3\log 5 + 3\log 3\right)$$

$$= 2\left(2 + \log \tfrac{3}{5}\right)$$

(b)(i) $(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \cdots \binom{n}{n}x^n$

$n(1+x)^{n-1} = \binom{n}{1} + 2\binom{n}{2}x + 3\binom{n}{3}x^2 + \cdots n\binom{n}{n}x^{n-1}$

(ii) let $x=1$

$LHS = n(2)^{n-1}$

$RHS = \binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots n\binom{n}{n}$

(iii) let $x=1$ in expansion of $(1+x)^n$

then $2^n = 1 + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots \binom{n}{n}$

and $n2^{n-1} = \binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots n\binom{n}{n}$

adding $2^n + n2^{n-1} = 1 + 2\binom{n}{1} + 3\binom{n}{2} + 4\binom{n}{3} + \cdots (n+1)\binom{n}{n}$

so $2^n + n2^{n-1} - 1 = 2\binom{n}{1} + 3\binom{n}{2} + 4\binom{n}{3} + \cdots (n+1)\binom{n}{n}$

(c) $\sqrt{3}\sin\theta - \cos\theta = R\cos(\theta+\alpha)$

$= R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$

so $-1 = R\cos\alpha$ and $\sqrt{3} = -R\sin\alpha$

$R = \sqrt{3+1}$

$= 2$

and $\alpha = \frac{4\pi}{3}$

(d) If $n=1$

$LHS = \frac{1}{2!} = \frac{1}{2}$

$RHS = \frac{2!-1}{2!}$

$= \frac{1}{2}$

so the statement is true when $n=1$.

Now suppose the statement is true for some value of n, k.

ie $\frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{k}{(k+1)!} = \frac{(k+1)!-1}{(k+1)!}$

We now prove the result for $n = k+1$

That is prove that $\frac{1}{2!} + \frac{2}{3!} + \cdots \frac{k}{(k+1)!} + \frac{k+1}{(k+2)!} = \frac{(k+2)!-1}{(k+2)!}$

Now, $LHS = \frac{(k+1)!-1}{(k+1)!} + \frac{k+1}{(k+2)!}$ using induction hypothesis

$LHS = \frac{(k+1)! - 1}{(k+1)!} + \frac{k+1}{(k+2)!}$

$= \frac{(k+2)(k+1)! - (k+2) + (k+1)}{(k+2)!}$

$= \frac{(k+2)! - 1}{(k+2)!}$

$= RHS.$

So, the statement is true for $k+1$ as long as it is true for k. Hence, by the principle of mathematical induction, it is true for all positive integers n.

Question 6.

(a) i) At A, $y = 2\sin x$ and $y = \frac{1}{3}x$
so we want $2\sin x = \frac{1}{3}x$
or $2\sin x - \frac{1}{3}x = 0$

ii) Let $f(x) = 2\sin x - \frac{1}{3}x$
$f'(x) = 2\cos x - \frac{1}{3}$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}, \quad x_0 = 3$

$= 3 - \frac{2\sin 3 - 1}{2\cos 3 - \frac{1}{3}}$

$\doteqdot 3 - 0.3102$

$\doteqdot 2.7$

(b) (i) $T = S + Ae^{-kt}$

$\frac{dT}{dt} = -kAe^{-kt}, \quad Ae^{-kt} = T - S$

$= -k(T-S)$

(ii) (a) $T = 30° + Ae^{-kt}$

when $t = 0$, $470° = 30° + Ae^{0}$

$A = 440°$

when $t = 10$, $250 = 30 + 440e^{-10k}$

$440e^{-10k} = 220$

$e^{-10k} = \frac{1}{2}$

$-10k = \log_e \frac{1}{2}$

$k = -\frac{1}{10} \log \frac{1}{2}$

$= \frac{1}{10} \log 2$

(β) Find t when $T = 70°$

$70 = 30 + 440e^{-kt}$, $\quad k = \frac{1}{10}\ln 2$.

$e^{-kt} = \frac{40}{440}$

$= \frac{1}{11}$

$-kt = \log_e \frac{1}{11}$

$t = \dfrac{\log_e \frac{1}{11}}{-\frac{1}{10}\log_e 2}$

$\doteqdot 35$ min

(c) (i) $\angle AFB = \frac{\pi}{2}$ (the angle in a semicircle is a right angle).

So $\angle ABF = \pi - (\frac{\pi}{2} + \beta)$ (the angle sum of ΔAFB is π)

$= \frac{\pi}{2} - \beta$

(ii)

$\angle BAD = \frac{\pi}{2}$ (angle between tangent and radius is $\frac{\pi}{2}$).

So $\angle ADB = \pi - (\frac{\pi}{2} + (\frac{\pi}{2} - \beta))$ (angle sum of ΔADB is π).

$= \beta$

Now, $\angle BAF = \angle BEF$ (both subtended at the circumference by arc BF)

$= \beta$

So, $\angle BEF = \angle CDF$

So, $CDFE$ is cyclic (exterior angle equals interior opposite angle)

Question 2.

(a) (i) For car A

$\ddot{x} = -k$, since the car is decelerating

$\frac{1}{2}v^2 = -kx + c$

When $x = 0$, $v = V_A$

so $\frac{1}{2}V_A^2 = 0 + c$ making $c = \frac{1}{2}V_A^2$

$v^2 = -2kx + V_A^2$

and speed $= \sqrt{V_A^2 - 2kx}$.

(ii) For car A:

$\ddot{x} = -k$

Integrating, $\dot{x} = -kt + c_1$

When $t = 0$, $\dot{x} = V_A$ making $c_1 = V_A$

So $\dot{x} = -kt + V_A$

Integrating, $x = -\frac{1}{2}kt^2 + tV_A + C_2$

When $t = 0$, $x = 0$, taking the origin of displacement at car A, so $C_2 = 0$

We have $x = tV_A - \frac{1}{2}kt^2$

For car B:

$\ddot{x} = 0$

Integrating, $\dot{x} = C_3$

When $t = 0$, $\dot{x} = V_B$, making $C_3 = V_B$

So $\dot{x} = V_B$

Integrating, $x = tV_B + c_4$

When $t = 0$, $x = D$, car B is D metres in front of car A, making $C_4 = V_B$

So $x = tV_B + D$.

When the cars collide their displacements are equal, so we have

$tV_B + D = tV_A - \frac{1}{2}kt^2$

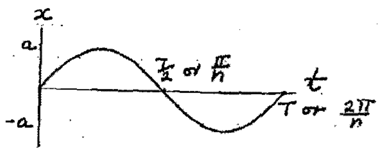
This is a quadratic in t.
For t to have a real value, the discriminant must be positive

$$\tfrac{1}{2}kt^2 - tV_A + tV_B + D = 0$$
$$kt^2 - 2t(V_A - V_B) + 2D = 0$$

$$\Delta = 4(V_A - V_B)^2 - 8kD$$
$$4(V_A - V_B)^2 - 8kD > 0$$
$$(V_A - V_B)^2 > 2kD$$
$$V_A - V_B > \sqrt{2kD}$$

since $V_A > V_B$ and so $V_A - V_B$ is positive.

(b) (i)

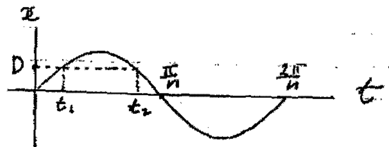


$T = \frac{2\pi}{n}$ so $n = \frac{2\pi}{T}$

(ii)

$$x = a\sin nt$$
$$\dot{x} = na\cos nt$$
$$= \frac{2\pi a}{T}\cos\frac{2\pi}{T}t$$

(iii)



At P we have $D = a\sin\frac{2\pi}{T}t$ ①

And $V = \frac{2\pi a}{T}\cos\frac{2\pi}{T}t$ ②

① ÷ ② $\frac{D}{V} = \frac{aT}{2\pi a}\tan\frac{2\pi t}{T}$

$$\frac{D2\pi}{VT} = \tan\frac{2\pi t}{T}$$

Let t_1 and t_2 be the first two times when the particle is at P

Then $\frac{2\pi t_1}{T} = \tan^{-1}\frac{2\pi D}{VT}$

$$t_1 = \frac{T}{2\pi}\tan^{-1}\frac{2\pi D}{VT}$$

And $t_2 = \frac{T}{2} - \frac{T}{2\pi}\tan^{-1}\frac{2\pi D}{VT}$

So the difference in times is

$$t_2 - t_1 = \frac{T}{2} - \frac{2T}{2\pi}\tan^{-1}\frac{2\pi D}{VT}$$
$$= \frac{T}{\pi}\left(\frac{\pi}{2} - \tan^{-1}\frac{2\pi D}{VT}\right)$$
$$= \frac{T}{\pi}\tan^{-1}\frac{VT}{2\pi D},$$ using complementary angles.