Ext 1

NORTH SYDNEY BOYS HIGH SCHOOL

2009
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics
Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write on one side of the paper (with lines) in the booklet provided
- Write using blue or black pen
- Board approved calculators may be used
- All necessary working should be shown in every question
- Each new question is to be started on a **new page.**
- Attempt all questions

Class Teacher:
(Please tick or highlight)
- O Mr Barrett
- O Mr Ireland
- O Mr Lowe
- O Mr Fletcher
- O Mr Trenwith
- O Mr Weiss

Student Number:

(To be used by the exam markers only.)

Question No	1	2	3	4	5	6	7	Total	Total
Mark	/12	/12	/12	/12	/12	/12	/12	/84	/100

Question 1 (12 marks) **Marks**

(a) Find $\lim_{x \to 0} \frac{\sin 3x}{x}$ 1

(b) Differentiate $\log_e(\sin^3 x)$, writing the answer in simplified form. 2

(c) Find the range of the function $f(x) = x^2 - 6x + 10$. 2

(d) Find the acute angle between the lines $-x + 2y + 4 = 0$ and $3x + y + 1 = 0$. Give your answer to the nearest minute. 2

(e) Evaluate $\int_1^{\sqrt{3}} \frac{1}{\sqrt{4 - x^2}}$ 2

(f) i) Show that $x - 2$ is a factor of $x^3 - 4x^2 + 7x - 6$ 1

ii) Show why $x^3 - 4x^2 + 7x - 6 = 0$ has only 1 real root. 2

Question 2 (12 marks)

(a) Find the exact value of $\cos\left[\sin^{-1}\left(\frac{1}{3}\right)\right]$ 2

(b) Solve the equation $3x^3 - 17x^2 - 8x + 12 = 0$ given that the product of two of the roots is 4. 3

(c)

(i) Express $\sqrt{12}\sin x + 2\cos x$ in the form $R\cos(x - \alpha)$ where $R > 0$ and $0 \le x \le \frac{\pi}{2}$ 2

(ii) Hence solve $\sqrt{12}\sin x + 2\cos x = -3$ for $0 \le x \le 2\pi$ 2

(d) Solve the inequality $\frac{x^2 - 9}{x} \le 8$ 3

Question 3 (12 marks)

(a) (i) Show why the equation $ln\,(x+1)+x-1=0$ must have a root between $x=0$ and $x=1$. 1

(ii) Given that the solution of $ln\,(x+1)+x-1=0$ is approximately equal to $\frac{1}{2}$, use one application of Newton's method to get a better approximation. Write your answer correct to two decimal places. 2

(b) The acceleration of a particle P is given by $a=32x(x^2+1)$ where x cm is the displacement at time t sec. Initially P starts from the origin with velocity 4cm/s.

(i) Use the fact that $\ddot{x}=\frac{d}{dx}\left(\frac{1}{2}v^2\right)$ to show that $v=\pm 4(x^2+1)$ 2

(ii) Justify why the positive solution is the correct one. 1

(iii) Find x in terms of t. 2

(c) Prove using mathematical induction that, for all positive integers $n \geq 1$,

$$\frac{1}{1\times 5}+\frac{1}{5\times 9}+\frac{1}{9\times 13}+\cdots+\frac{1}{(4n-3)\times(4n+1)}=\frac{n}{4n+1}$$ 4

Question 4 (12 marks)

(a) Find $\int \cos^2 3x\,dx$ 2

(b) (i) Show that the maximum value of $2x(1-x)$ is $\frac{1}{2}$ 1

(ii) Find the range of the function $f(x)=\sin^{-1}\{2x(1-x)\}$, with domain $0 \leq x \leq 1$. 2

(c) (i) Show that $T=22+Ae^{kt}$ is a solution to the equation

$\frac{dT}{dt}=k(T-22)$. 1

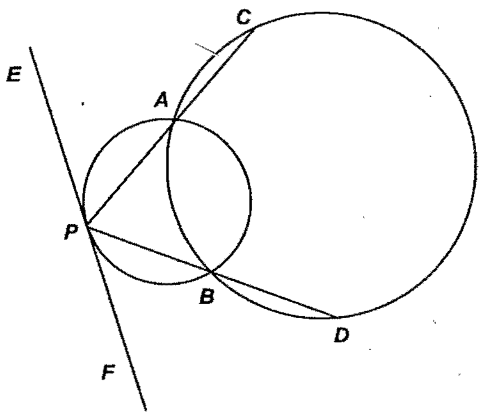
(ii) A wealthy industrialist is found murdered in his home. When police arrived on the scene at 11:00 pm, the temperature of the body was 31°C, and one hour later it was 30°C. The temperature of the room where the body was found was 22°C.

Using Newton's law of cooling $\frac{dT}{dt}=k(T-22)$, and the fact that normal body temperature is 37°C, estimate the time that the murder occurred. 3

[Question 4 continued]

(d) Copy the diagram into your answer booklet. PAC, PBD are straight lines. EF is the tangent at P. 3

Prove $CD \parallel EF$



Question 5 (12 marks)

(a) Use the substitution $u = x - 1$ to evaluate $\int_2^4 \frac{x}{(x-1)^2}\,dx$ 3

(b) A particle moves along a straight line in such a way that its distance x cm from a fixed point O at time t seconds is $x = 2\cos 3t$.

(i) Show that the particle is moving in Simple Harmonic Motion. 2

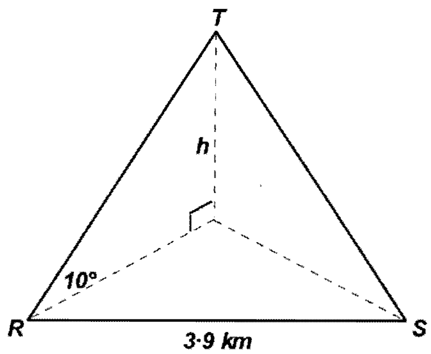
(ii) Write down the period of the motion. 1

(iii) Find the particle's speed when it is first 1 cm from O. 2

(c) Two boats, *Rascal* and *Sirocco*, are sailing near Ball's Pyramid, a giant pillar of rock that rises from the sea. 4

The boats are 3.9 km apart, and the angle of elevation of the top T of Ball's Pyramid from *Rascal* is 10°.

Given that $\angle TRS$ and $\angle TSR$ are 65° and 48° respectively, find the height h of Ball's Pyramid to the nearest metre.



Question 6 (12 marks)

(a) Water is being pumped into an empty inverted conical tank – that is, a tank with its apex down – at a rate of $300\ 000\ \text{cm}^3/\text{min}$. The tank is 6 metres high and the diameter at the top is 4 metres. Let h be the height of the water and r the radius of the water surface at time t minutes.

Calculate the rate at which the water level is rising at the instant the water level reaches 2 metres. (Write your answer to 1 decimal place). **4**

(b) $P(2ap, ap^2)$ is a point on the parabola $x^2 = 4ay$, which has focus S.

(i) Derive the equation of the tangent to the parabola at P. Hence show that the tangent meets the x-axis at the point $T(ap, 0)$. **3**

(ii) Find the coordinates of M, the point that divides ST externally in the ratio 2 : 1 **2**

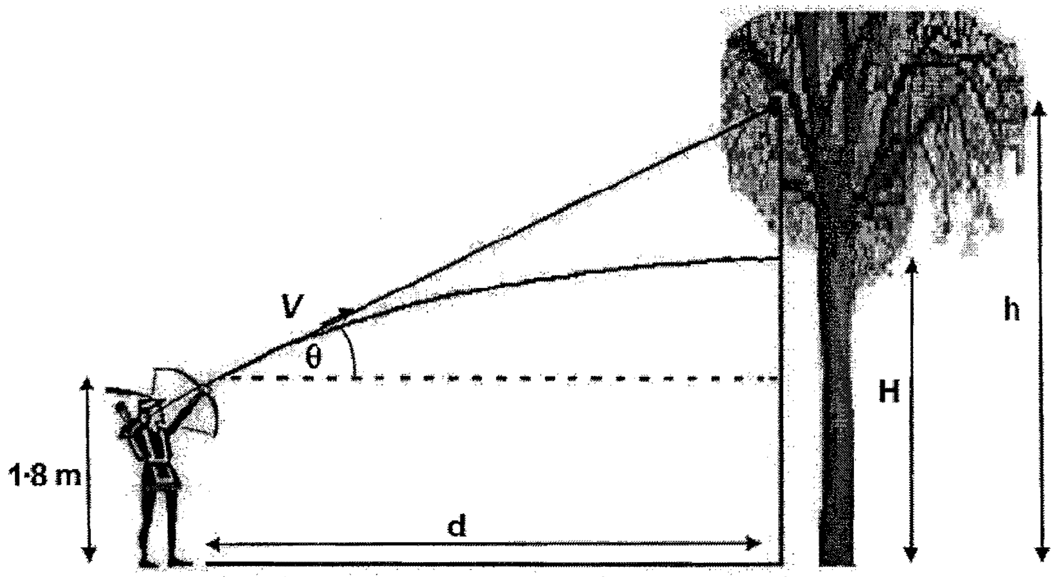
(iii) Describe geometrically the locus of M as P moves on the parabola. **1**

(c) A function is given by the rule $f(x) = log_e \frac{1+x}{1-x}$. Find the rule for the inverse function, $f^{-1}(x)$. **2**

Question 7 (12 marks)

(a) Robin Hood aims his arrow at an acorn which is on an oak tree d metres away. The acorn is h metres above the ground, and Robin releases the arrow from a point 1·8 metres above the ground, at an angle of elevation of θ degrees. Robin can vary the initial velocity V of the arrow.

At the instant Robin releases the arrow, the acorn begins to fall vertically downwards under gravity, with acceleration g. (Neglect air resistance).



[Question 7 continued]

(i) Taking the ground at Robin's feet as the origin of coordinates, show that at time t the x- and y-coordinates of the arrow's tip are

$$x = Vt\cos\theta \quad \text{and} \quad y = Vt\sin\theta - \frac{gt^2}{2} + 1.8$$ **3**

(ii) Show that, when the arrow reaches the tree, its vertical height above the ground is given by **2**

$$H = d\tan\theta - \frac{gd^2\sec^2\theta}{2V^2} + 1.8$$

(iii) Robin's arrow hits the acorn as it falls. Show mathematically why this will in fact always be the case in this situation, no matter what the initial velocity of the arrow. **3**

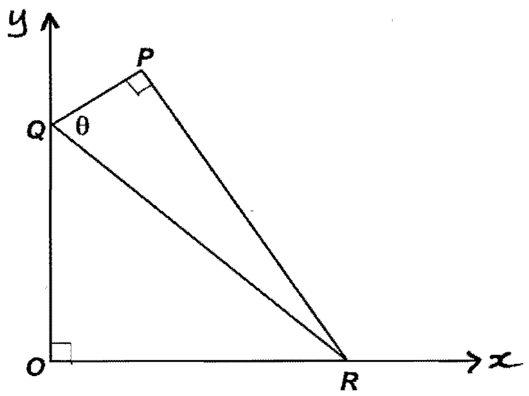
(Provided, of course, that it is great enough to reach the tree!)

(b) On a toss of two dice, John throws a total of 5. Find the probability that he will throw another 5 before he throws 7. **2**

(c) A triangle PQR, right angled at P and with $\angle PQR = \theta$, slides on a horizontal floor with its vertices Q and R in contact with perpendicular walls.

(i) State why $OQPR$ is a cyclic quadrilateral. **1**

(ii) Derive the equation of the locus of P. **1**



2009 Extension 1 TRIAL – Suggested Solutions

QUESTION 1

a) $\lim\limits_{x\to 0} \frac{\sin 3x}{x} = 3 \cdot \lim\limits_{x\to 0} \frac{\sin 3x}{3x}$
$= 3 \cdot 1$
$= 3$ ✓

(b) $\frac{d}{dx} \ln(\sin^3 x) = \frac{3\sin^2 x \cdot \cos x}{\sin^3 x}$ ✓
$= \frac{3\cos x}{\sin x}$
$= 3\cot x$ ✓

(c) $f(x) = x^2 - 6x + 10$
$= (x-3)^2 + 1$
The graph of $y = f(x)$ is a concave up parabola, vertex $(3, 1)$. $\therefore R: y \geq 1$ ✓✓

[Alternatively,
$f'(x) = 2x - 6$
$= 0$ when $x = 3$.
Since $f''(x) = 2 > 0$ ∴ minimum turning point at $x = 3$, $y = 1$. $\therefore R: y \geq 1$]

(d) $m_1 = \frac{1}{2}$, $m_2 = -3$
$\therefore \tan\theta = \left|\frac{\frac{1}{2} - (-3)}{1 + \frac{1}{2}(-3)}\right|$ ✓
$= 7$
$\therefore \theta = 81°52'$ (nearest minute) ✓

Q1 cont'd

(e) $\int_1^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\,dx = \left[\sin^{-1}\frac{x}{2}\right]_1^{\sqrt{3}}$ ✓ correct primitive
$= \sin^{-1}\frac{\sqrt{3}}{2} - \sin^{-1}\frac{1}{2}$
$= \frac{\pi}{3} - \frac{\pi}{6}$
$= \frac{\pi}{6}$ ✓

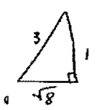
(f) (i) Let $x^3 - 4x^2 + 7x - 6 = P(x)$
Then $P(2) = 8 - 16 + 14 - 6 = 0$ ✓ shows $P(2) = 0$
$\therefore (x-2)$ is a factor of $P(x)$

(ii)
$$\begin{array}{r} x^2 - 2x + 3 \\ x-2 \,\big)\, x^3 - 4x^2 + 7x - 6 \\ x^3 - 2x^2 \\ \hline -2x^2 + 7x \\ -2x^2 + 4x \\ \hline 3x - 6 \\ 3x - 6 \\ \hline 0 \end{array}$$

$\therefore P(x) = (x-2)(x^2 - 2x + 3)$ ✓
But $x^2 - 2x + 3$ has $\Delta = (-2)^2 - 4(1)(3)$
$= -8$
< 0
and thus has no real roots. ✓ correct reasoning with Δ
$\therefore x = 2$ is the only real root.

QUESTION 2

(a) $\cos\left[\sin^{-1}\left(\tfrac{1}{3}\right)\right] = \frac{\sqrt{8}}{3}$ $\left(= \frac{2\sqrt{2}}{3}\right)$ ✓ relevant working ✓ correct answer



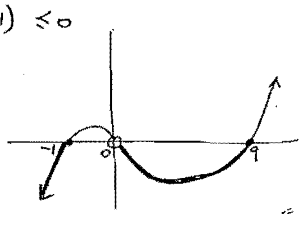
(b) $3x^3 - 17x^2 - 8x + 12 = 0$
Let roots be α, β, γ, and
Suppose $\alpha\beta = 4$ ✓
Now $\alpha\beta\gamma = -\frac{12}{3} = -4$ $\therefore \gamma = -1$
Also $\alpha + \beta + \gamma = \frac{17}{3}$
$\therefore \alpha + \frac{4}{\alpha} - 1 = \frac{17}{3}$ (since $\alpha\beta = 4$)
$\therefore 3\alpha^2 - 20\alpha + 12 = 0$
$(3\alpha - 2)(\alpha - 6) = 0$
$\therefore \alpha = \frac{2}{3}$ or 6
ie. roots are $-1, \frac{2}{3}, 6$. ✓✓

(c) (i) Let $\sqrt{12}\sin x + 2\cos x = R\cos(x - \alpha)$
$= R\cos x\cos\alpha + R\sin x\sin\alpha$
$\therefore$ $R\cos\alpha = 2$, $R\sin\alpha = \sqrt{12}$ $\Big\}$ $\therefore R = 4$ ✓ $R = 4$
$\tan\alpha = \frac{\sqrt{12}}{2} = \sqrt{3}$ $\therefore \alpha = \pi/3$ ✓ $\alpha = \frac{\pi}{3}$
$\therefore$ expression $= 4\cos(x - \pi/3)$

(ii) $4\cos(x - \frac{\pi}{3}) = -3$, $0 \leq x \leq 2\pi$
$\therefore \cos(x - \frac{\pi}{3}) = -\frac{3}{4}$ ✓ (1 mark each answer.)
2.864

Q2 cont'd

(d) $\frac{x^2 - 9}{x} \leq 8$
$\therefore x(x^2 - 9) \leq 8x^2$ $(x \neq 0)$ ✓
$\therefore x[x^2 - 9 - 8x] \leq 0$
$x(x-9)(x+1) \leq 0$ ✓



$\therefore x \leq -1$ or $0 < x \leq 9$. ✓

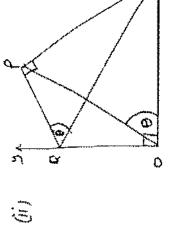
Q7 CONTINUED

✓ correct reason ✓ correct answer

cont'd

(i) OQPR is a cyclic quadrilateral since opposite angles $\angle QPR$ and $\angle QOR$ are supplementary (each = 90°).

(ii)



Join PO.
From (i), PR is a chord of a circle.
So $\angle POR = \angle PQR$ (angles in the same segment)
$= \theta$
But this is true for all positions of P as Q, R slide along the walls.
∴ the gradient of PO is always $\tan\theta$
Since PO goes through the origin, P's locus is the straight line $y = (\tan\theta)x$

QUESTION 3.

(a) (i) Let $f(x) = \ln(x+1) + x - 1$

Then $f(0) = \ln 1 + 0 - 1 = -1 < 0$

and $f(1) = \ln 2 + 1 - 1 = \ln 2 > 0$

$f(0)$ and $f(1)$ have opposite signs, hence a root lies between 0 and 1. ✓

(ii) $f'(x) = \frac{1}{x+1} + 1$

Let $x_1 = \frac{1}{2}$

Then $x_2 = \frac{1}{2} - \frac{f(\frac{1}{2})}{f'(\frac{1}{2})}$

$= \frac{1}{2} - \frac{(\ln 1.5 - 0.5)}{\left(\frac{1}{1.5} + 1\right)}$ ✓ *correct sub. into correct formula*

$\doteqdot 0.5567209$

$= 0.56$ (to 2 d.p.). ✓ *correct answer*

(b) (i) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 32x\,(x^2+1)$

$= 32x^3 + 32x$

$\therefore \frac{1}{2}v^2 = 8x^4 + 16x^2 + C$ ✓

But $t=0,\ x=0,\ v=4$

$\therefore \frac{1}{2}\cdot 4^2 = 0 + C \qquad \therefore C = 8$

$\therefore v^2 = 16x^4 + 32x^2 + 16$

$= 16(x^4 + 2x^2 + 1)$

$= 16(x^2+1)^2$

$\therefore v = \pm 4(x^2+1)$ ✓ *authentic simplification*

Q3 cont'd

(b) (ii) At $t=0$, $x=0$ and $v=4$.

So particle is moving to the right. But since $a>0$ when $x>0$, it will keep moving to the right. ie $V>0$ ✓ *refers to $a>0$ since $x>0$*

Thus $v = 4(x^2+1)$

(iii) $v = \frac{dx}{dt} = 4(x^2+1)$

$\therefore \frac{dt}{dx} = \frac{1}{4}\cdot\frac{1}{x^2+1}$ ✓

$t = \frac{1}{4}\tan^{-1}x + C$

But at $t=0, x=0 \quad \therefore C=0$

$\therefore 4t = \tan^{-1}x$

$\therefore x = \tan 4t$ ✓

(c) We prove that, for $n \geq 1$,

$$\frac{1}{1\times5} + \frac{1}{5\times9} + \dots + \frac{1}{(4n-3)(4n+1)} = \frac{n}{4n+1}$$

When $n=1$

$LHS = \frac{1}{1\times5} = \frac{1}{5}$

$RHS = \frac{1}{4(1)+1} = \frac{1}{5}$

$\therefore$ it is true for $n=1$ ✓ *proves for $n=1$*

Assume it is true for $n=k$ $(k>1)$

ie $\frac{1}{1\times5} + \frac{1}{5\times9} + \dots + \frac{1}{(4k-3)(4k+1)} = \frac{k}{4k+1}$ (*)

Thus

$\frac{1}{1\times5} + \frac{1}{5\times9} + \dots + \frac{1}{(4k-3)(4k+1)} + \frac{1}{[4(k+1)-3][4(k+1)+1]}$

$= \frac{k}{4k+1} + \frac{1}{[4(k+1)-3][4(k+1)+1]}$ by (*) ✓ *correct start*

3 cont'd

$= \frac{k}{4k+1} + \frac{1}{(4k+1)(4k+5)}$

$= \frac{k(4k+5)+1}{(4k+1)(4k+5)}$

$= \frac{4k^2+5k+1}{(4k+1)(4k+5)}$

$= \frac{(4k+1)(k+1)}{(4k+1)(4k+5)}$

$= \frac{(k+1)}{4(k+1)+1}$ ✓ *correct working/Simplifying*

Thus if true for $n=k$, it is also true for $n=k+1$.

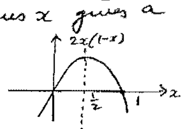
$\therefore$ Since it is true for $n=1$, it is true for $n=1+1=2$, and thus also for $n=2+1=3$, and so on for all integers $n\geq1$. ✓

QUESTION 4

(a) $\int \cos^2 3x\,dx = \int\left(\frac{1}{2} + \frac{1}{2}\cos 6x\right)dx$ ✓ *Correct use of trig relation*

$= \frac{1}{2}x + \frac{1}{12}\sin 6x + C$ ✓ *correct answer*

(b) (i) Graphing $2x(1-x)$ versus x gives a concave-down parabola with axis of symmetry $x=\frac{1}{2}$.



$\therefore$ maximum value is $2\left(\frac{1}{2}\right)\left(1-\frac{1}{2}\right) = \frac{1}{2}$ ✓ *a correct demonstration*

[Alternatively, Let $f(x) = 2x(1-x)$

$\therefore f'(x) = 2 - 4x$, $f''(x) = -4$.

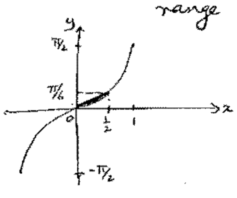
for max., $f'(x) = 0 \quad \therefore x = \frac{1}{2}$

$f(x) = \frac{1}{2}$

Since $f''(x) < 0 \quad \therefore \frac{1}{2}$ is a maximum value.]

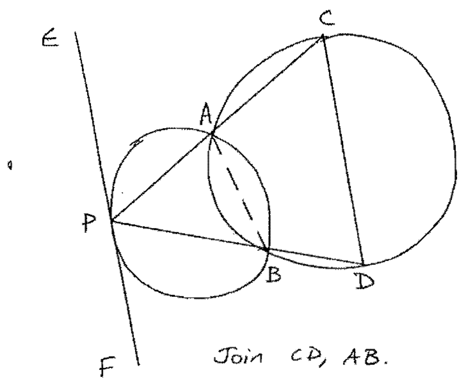
(ii) If $0 \leq x \leq 1$ then $0 \leq 2x(1-x) \leq \frac{1}{2}$ [from (i)] ✓ *relevant working*

Thus $\sin^{-1}\{2x(1-x)\}$ has range $R: 0 \leq y \leq \pi/6$. ✓ *correct answer*



Q4 cont'd

(d)



Join CD, AB.

$\angle EPA = \angle ABP$ (alternate segment theorem) ✓

$\angle ABP = \angle ACD$ (exterior angle of cyclic quad. ACDB) ✓

$\therefore$ $EF \parallel CD$ (alternate angles $\angle EPA$ and $\angle ACD$ are equal) ✓

Q4 cont'd

(c) (i) Let $T = 22 + Ae^{kt}$

Then $\frac{dT}{dt} = k \cdot Ae^{kt}$

$= k(T-22)$

So $T = 22 + Ae^{kt}$ is a solution. ✓ Shows it's a Solution

(ii) Measure time t (in hours) from 11:00pm.

Thus at $t=0$, $T=31$

$t=1$, $T=30$.

That is, $31 = 22 + Ae^{0}$ $\therefore A = 9$

and $\therefore 30 = 22 + 9e^{k}$

$\therefore \frac{8}{9} = e^{k}$

$\therefore k = \ln\left(\frac{8}{9}\right)$ $(\doteqdot -0.117783)$ ✓ Correct k value

$\therefore$ $37 = 22 + 9e^{\ln\left(\frac{8}{9}\right)t}$

$\therefore \frac{15}{9} = e^{\ln\left(\frac{8}{9}\right)t}$

$\therefore t = \frac{\ln\left(\frac{15}{9}\right)}{\ln\left(\frac{8}{9}\right)}$ ✓ Solves for t in correct expression

$= -4.337$

ie he was murdered at $(11 - 4.337)$ pm.

$= 6:40$ pm (nearest minute). ✓ correct answer

QUESTION 5

(a) $\int_2^4 \frac{x}{(x-1)^2}\,dx$. Let $u = x-1$

$\therefore du = dx$

$x = u+1$

$x=2 \rightarrow u=1$

$x=4 \rightarrow u=3$

$\therefore \int = \int_1^3 \frac{u+1}{u^2}\,du$ ✓ Correctly set-up for u

$= \int_1^3 \left(\frac{1}{u} + u^{-2}\right) du$

$= \left[\ln u - \frac{1}{u}\right]_1^3$ ✓ Correct primitive

$= \left(\ln 3 - \frac{1}{3}\right) - (\ln 1 - 1)$

$= \ln 3 + \frac{2}{3}$ ✓ correct answer

$(\doteqdot 1.765)$.

(b) (i) $x = 2\cos 3t$

$\therefore \dot{x} = -6\sin 3t$

$\therefore \ddot{x} = -18\cos 3t$ ✓ shows $\ddot{x} = -9x$

$\therefore \ddot{x} = -9x$ ✓ Conclusion

This is in the form $\ddot{x} = -n^2x$, hence it's S.H.M. ✓

(ii) $T = \frac{2\pi}{3}$ seconds

(iii) when $x=1$, $1 = 2\cos 3t$

$\therefore 3t = \frac{\pi}{3}$

$\therefore t = \frac{\pi}{9}$ seconds ✓ correct time

At $t = \frac{\pi}{9}$, $|\dot{x}| = \left|-6\sin 3\left(\frac{\pi}{9}\right)\right| = 3\sqrt{3}$ cm/s. ✓ correct speed

Q5 cont'd

[(b)(iii) Alternatively, $v^2 = n^2(a^2 - x^2)$ ✓ correct use of formula

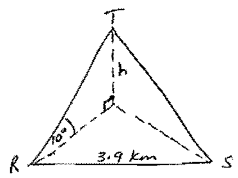
$= 9(4 - 1^2)$

$= 27$

$\therefore v = \pm\sqrt{27}$

$\therefore$ speed $= |v| = 3\sqrt{3}$ cm/s] ✓ correct speed

(c)



$\angle RTS = 180° - (65° + 48°)$ ✓ Finds $\angle R$

$= 67°$

$\therefore \frac{TR}{\sin 48°} = \frac{3.9}{\sin 67°}$

$\therefore TR = \frac{3.9 \sin 48°}{\sin 67°}$ ✓ correct expression for TR

But $\frac{h}{TR} = \sin 10°$

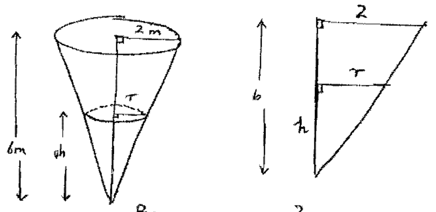
$\therefore h = \frac{3.9\sin 48°}{\sin 67°} \cdot \sin 10°$ ✓ correct expression h

$\doteqdot 0.5467$ km

$= 547$ m. (nearest metre). ✓ correct answer

QUESTION 6

(a)



By similar Δ's, $\frac{r}{h} = \frac{2}{6}$ $\therefore r = \frac{h}{3}$ — ✓ expresses r in terms of h

Thus $V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi\left(\frac{h}{3}\right)^2 h$

$\therefore V = \frac{\pi h^3}{27}$ — ✓ correct expression for V

Now $\frac{dV}{dt} = \frac{dV}{dh}\cdot\frac{dh}{dt}$, (where $\frac{dV}{dh} = \frac{\pi h^2}{9}$)

$\therefore 300\,000 \frac{cm^3}{min} = \frac{\pi}{9}(200)^2\, cm^2 \cdot \frac{dh}{dt}$ — ✓ correct use of Chain Rule

$\therefore \frac{dh}{dt} = \frac{9}{\pi(200)^2}\cdot 300\,000$ cm/min

$\doteqdot 21.4859$

$= 21.5$ cm/min (1 d.p.) — ✓ correct answer

Q6 cont'd

(b) (i) $y = \frac{x^2}{4a}$ $\therefore \frac{dy}{dx} = \frac{x}{2a}$

$\therefore$ at P, $\frac{dy}{dx} = \frac{2ap}{2a} = p$ — ✓ derives $\frac{dy}{dx}$

$\therefore$ tangent is $y - ap^2 = p(x - 2ap)$

$\therefore y = px - ap^2$ — ✓ derives equation of tangent

At T, $y = 0$ $\therefore px - ap^2 = 0$

$x = \frac{ap^2}{p}$

$\therefore x = ap$

Hence $T = (ap, 0)$ — ✓ shows $T = (ap, 0)$

(ii) $S(0, a)$, $T(ap, 0)$, Let $k : l = 2 : -1$

Then $M = \left[\frac{(2)(ap) + (-1)(0)}{2 + (-1)}, \frac{2(0) + (-1)(a)}{2 + (-1)}\right]$ — ✓ correct use of division formula

$\therefore M = (2ap, -a)$ — ✓ correct coordinates of M

(iii) As P moves on the parabola, M moves along the line $y = -a$; ie M moves along the directrix of the parabola. — ✓ geometric answer.

(c) Let $y = f(x)$, ie, $y = \log_e\left(\frac{1+x}{1-x}\right)$

For inverse, $x = \log_e\left(\frac{1+y}{1-y}\right)$

$\therefore e^x = \frac{1+y}{1-y}$ — ✓ correct to here

$e^x - e^x y = 1 + y$

$\therefore e^x - 1 = y(e^x + 1)$

Hence $y = \frac{e^x - 1}{\ldots}$ — ✓ correct answer

QUESTION 7

(a) (i)

$\ddot{x} = 0$

$\dot{x} = c$

at $t = 0$, $\dot{x} = V\cos\theta$

$\therefore \dot{x} = V\cos\theta$

$\therefore x = Vt\cos\theta + C$

at $t = 0$, $x = 0$ $\therefore C = 0$

$\therefore x = Vt\cos\theta$

$\ddot{y} = -g$

$\dot{y} = -gt + C$

at $t = 0$, $\dot{y} = V\sin\theta$ $\therefore C = V\sin\theta$

$\therefore \dot{y} = -gt + V\sin\theta$

$\therefore y = -\frac{gt^2}{2} + Vt\sin\theta + C$

at $t = 0$, $y = 1.8$ $\therefore C = 1.8$

$\therefore y = -\frac{gt^2}{2} + Vt\sin\theta + 1.8$

Correct derivation + use of initial conditions ✓✓✓

(ii) "Arrow reaches tree" means $x = d$

$\therefore d = Vt\cos\theta$ $\therefore t = \frac{d}{V\cos\theta}$ — ✓ finds t

$\therefore$ at this time,

$y = -\frac{g}{2}\left(\frac{d}{V\cos\theta}\right)^2 + V\left(\frac{d}{V\cos\theta}\right)\sin\theta + 1.8$

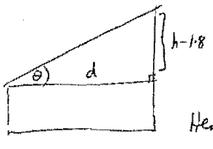
ie $H = -\frac{gd^2\sec^2\theta}{2V^2} + d\tan\theta + 1.8$ — ✓ correct substitution + simplifying

(iii) The acorn's equations of motion are

$x = d$, $y = h - \frac{gt^2}{2}$ — ✓ correct equations for acorn

Thus at $t = \frac{d}{V\cos\theta}$, $y = h - \frac{g}{2}\left(\frac{d}{V\cos\theta}\right)^2$

$y = h - \frac{gd^2\sec^2\theta}{2V^2}$



But $\frac{h - 1.8}{d} = \tan\theta$

$\therefore h = d\tan\theta + 1.8$ — ✓ writes h in terms of d

Hence $y = d\tan\theta + 1.8 - \frac{gd^2\sec^2\theta}{2V^2}$

$= H$ — ✓ correct conclusion

ie. the acorn is the same height above the ground as the arrow arrives at the tree

Q7 cont'd

(b) John will succeed if he throws a total of 5 on the next toss; or should he not throw a 5 or 7 on this toss but throws 5 on the next; or should he not throw a 5 or 7 on either of these tosses but throw 5 on the next; etc.

$\therefore$ P(throws 5 before 7)

$= \frac{4}{36} + \frac{26}{36}\cdot\frac{4}{36} + \left(\frac{26}{36}\right)^2\cdot\frac{4}{36} + \ldots$ — ✓ correct logic + series

$= \frac{\frac{4}{36}}{1 - \frac{26}{36}}$ $\left(\text{since } S_\infty = \frac{a}{1-r}\right)$

$= \frac{2}{5}$ — ✓ correct answer

[Alternatively,

John keeps throwing until he gets either a 5 or a 7. We need to find the probability the 5 comes up before the 7.

So the question is, given he gets a 5 or a 7, what is the probability that it is a 5?

There are 10 ways to throw a 5 or a 7, and of these four give a 5.

So the probability he gets a 5 before a 7 is just $\frac{4}{10} = \frac{2}{5}$. ✓✓

(no marks for just finding P(7) or P(5) on a single throw).

CONTINUED ON Page 1.