Ext 2

NORTH SYDNEY BOYS HIGH SCHOOL

2009
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

Examiner: B. Weiss

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write on one side of the paper (with lines) in the booklet provided
- Write using blue or black pen
- Board approved calculators may be used
- All necessary working should be shown in every question
- Each new question is to be started on a **new page.**

- Attempt all questions

Class Teacher:
(Please tick or highlight)
- O Mr Barrett
- O Mr Fletcher
- O Mr Weiss

Student Number

(To be used by the exam markers only.)

Question No	1	2	3	4	5	6	7	8	Total	Total
Mark	/15	/15	/15	/15	/15	/15	/15	/15	/120	/100

Question 1

(a) Find the following integrals:

(i) $\int \tan^3 x \, dx$ **3**

(ii) $\int \frac{dx}{x^2 - 6x + 13}$ **2**

(b) Evaluate

(i) $\int_0^1 \frac{x}{\sqrt{4 - x^2}} \, dx$ **3**

(ii) $\int_0^{\frac{\pi}{2}} x \cos^2 x \, dx$ **3**

(c) (i) Show that if $I_n = \int_0^1 x^n e^{-x} \, dx$, then $I_n = n \cdot I_{n-1} - \frac{1}{e}$ **2**

(ii) Hence find $\int_0^1 x^3 e^{-x} \, dx$. **2**

Question 2 (Start a new page)

(a) Find $\sqrt{6i - 8}$, and hence solve the equation $2z^2 - (3 + i)z + 2 = 0$. **4**

(b) Solve $3x^3 - 10x^2 + 7x + 10 = 0$ given that $x = 2 - i$ is a root of the equation. **3**

(c) The polynomial equation $P(x) = x^3 + px^2 + q = 0$ has roots α, β and γ.
Form the polynomial equation with roots given by

(i) $\frac{1}{\alpha}$, $\frac{1}{\beta}$ and $\frac{1}{\gamma}$ **2**

(ii) α^2, β^2 and γ^2 **2**

(d) Use the method of cylindrical shells to find the voume of the solid generated by rotating the region bounded by $y = \ln x$, the x-axis and the lines $x = 1$ and $x = e$, about the y-axis. **4**

Question 3 (Start a new page)

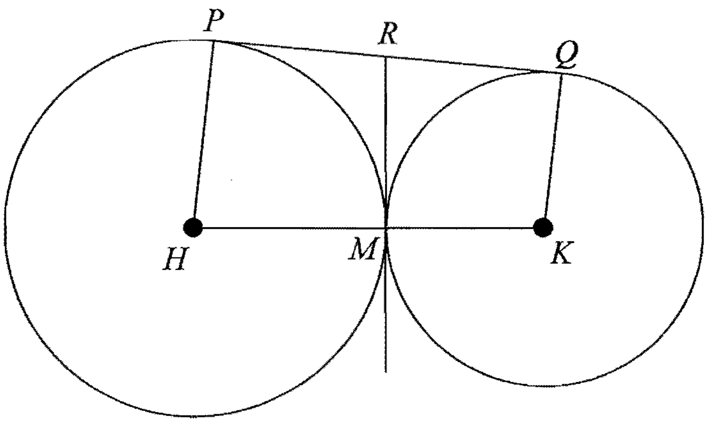
(a) (i) Prove that the equation of the tangent at the point $\left(t, \dfrac{1}{t}\right)$ to the hyperbola $xy = 1$ is $x + t^2y = 2t$. **2**

(ii) The tangent at a point P on the hyperbola $xy = 1$ meets the y-axis at A, and the normal at P meets the x-axis at B. Find the equation of the locus of the midpoint of AB as P moves on the hyperbola. (Draw a diagram) **3**

(b) $P(a\cos\alpha,\ b\sin\alpha)$ and $Q(a\cos\beta,\ b\sin\beta)$ are the endpoints of a focal chord of the curve $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. **3**

Show that $e = \dfrac{\sin(\alpha - \beta)}{\sin\alpha - \sin\beta}$.

(c) Shown below are two circles with centres H and K which touch at M. PQ and RM are common tangents.



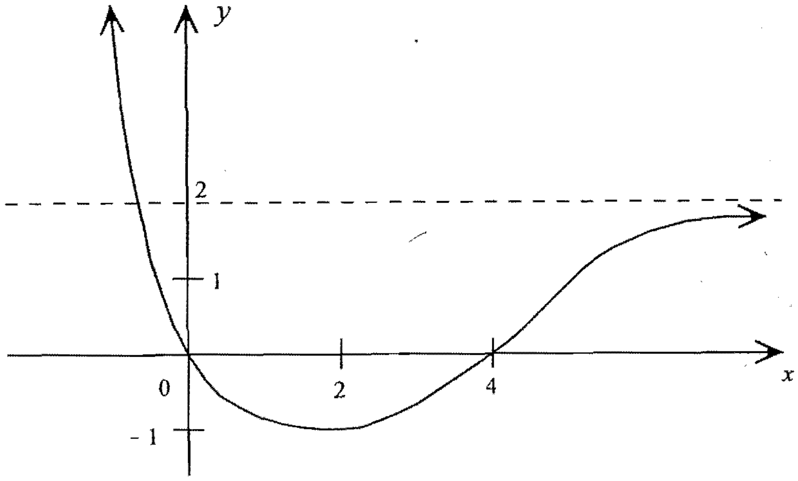
(i) Show that quadrilaterals $HPRM$ and $MRQK$ are cyclic. **2**

(ii) Prove tha triangles PRM and MKQ are similar. **2**

(d) Show that the polynomial equation $4x^3 + 20x^2 - 23x + 6 = 0$ has a double root, and find the value of each of its roots. **3**

Question 4

(a) The diagram shows the graph of $y = f(x)$.



Sketch on separate diagrams, the following curves, indicating clearly any turning points and asymptotes.

(i) $y = \dfrac{1}{f(x)}$ **1**

(ii) $y = [f(x)]^2$ **2**

Draw neat sketches of the following:

(b) $y = x \sin x$ **2**

(c) $y = \sin^{-1}(\sin x)$ **2**

(d) $y = x^2 - \dfrac{1}{x}$ **3**

(e) (i) $f(x) = \dfrac{x^2 - 4}{x - 3}$ **3**

(ii) $[f(x)]^2 = \dfrac{x^2 - 4}{x - 3}$ **2**

Question 5 (Start a new page)

(a) (i) Express the complex number $z = -\sqrt{3} + i$ in mod-arg form. **1**

(ii) Hence, or otherwise, show that $z^7 + 64z = 0$. **2**

(b) Find the equation, in Cartesian form, of the locus of the point z if

$$\text{Re}\left[\frac{z-4}{z}\right] = 0$$ **3**

(c) Sketch the region S in the complex plane, where

$$S = \left\{ |z| \le 1 \quad \text{and} \quad 0 \le \arg z < \frac{\pi}{3} \right\}$$ **2**

(d) (i) Use de Moivre's theorem to express $\cos 5\theta$ and $\sin 5\theta$ in terms of $\sin\theta$ and $\cos\theta$. **3**

(ii) Hence express $\tan 5\theta$ as a rational function of t, where $t = \tan\theta$. **2**

(iii) Find $\tan\frac{\pi}{5} \cdot \tan\frac{2\pi}{5} \cdot \tan\frac{3\pi}{5} \cdot \tan\frac{4\pi}{5}$ **2**

Question 6 (Start a new page)

(a) A particle of mass 1 kg is projected upwards with initial speed $10\ \text{ms}^{-1}$. **5**
The air resistance is given by $R = \frac{1}{10}V^2$.
Take the acceleration due to gravity to be $10\ \text{ms}^{-2}$.
Find the maximum height reached, and the time taken to reach this height.

(b) Find the largest coefficient in the expansion of $(2x + 3)^{21}$. **3**

(c) If $x^m y^n = k$, where k is a constant, show that $\frac{dy}{dx} = -\frac{my}{nx}$. **3**

(d) Use the expansion of $(1 + x)^{2n}$ to show that

(i) $\binom{2n}{1} + \binom{2n}{2} + \binom{2n}{3} + \ldots + \binom{2n}{2n} = 4^n - 1$ **2**

(ii) Use the identity $(1 + x)^{2n} \equiv (1 + x)^n (1 + x)^n$ to show that **2**

$$\binom{2n}{2} = 2 \cdot \binom{n}{2} + \binom{n}{1}^2$$

Question 7 (Start a new page)

(a) Use the process of mathematical induction to show that

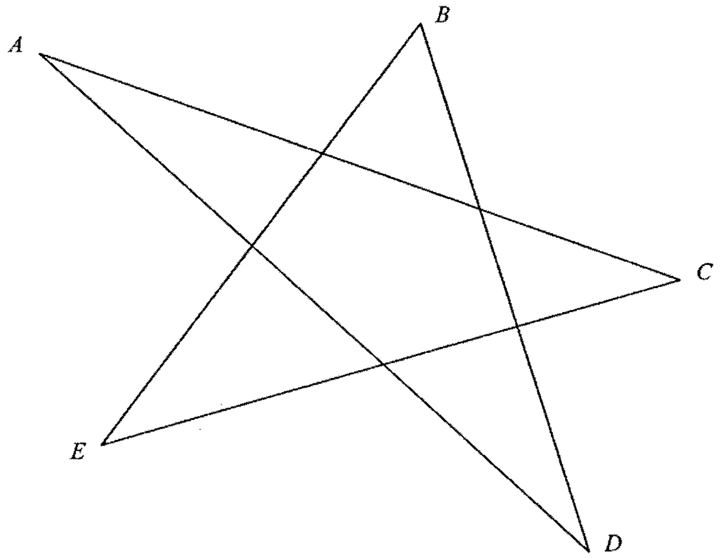
$$\sum_{k=1}^{n} \frac{k}{(k+1)!} = 1 - \frac{1}{(n+1)!}$$ **5**

(b) With the aid of a diagram, show that the area enclosed by the ellipse **4**

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is given by $A = \frac{4b}{a}\int_0^a \sqrt{a^2 - x^2}\,dx$.

Hence show that the area of the ellipse is πab.

(c) **2**



Prove that $\angle A + \angle B + \angle C + \angle D + \angle E = 180°$

(d) The base of a solid is the region bounded by the parabolas $x = y^2$ and $x = 4 - 3y^2$, and the cross-sections perpendicular to the x-axis are squares.

(i) Draw a neat sketch of this solid. **1**

(ii) Find the volume of the solid. **3**

Question 8 (Start a new page)

(a) If a and b are positive numbers such that $a + b = 1$, prove that

(i) $a + b \geq 2\sqrt{ab}$ **1**

(ii) $\dfrac{1}{a} + \dfrac{1}{b} \geq 4$ **2**

(iii) $a^2 + b^2 \geq \dfrac{1}{2}$ **2**

(iv) $\left(1 + \dfrac{1}{a}\right)\left(1 + \dfrac{1}{b}\right) \geq 9$ **2**

(b) A particle is projected from ground level so that it just clears two poles of height h at distances of b and c metres from the point of projection.
If v m/s is the velocity of projection, and θ is the angle of projection to the horizontal:

(i) Show that $y = x \tan\theta - \dfrac{gx^2}{2v^2} \cdot \sec^2\theta$ **2**

(ii) Show that $v^2 = \dfrac{(b + c)g \sec^2\theta}{2 \tan\theta}$ **3**

(iii) Hence or otherwise show that $\tan\theta = \dfrac{h(b+c)}{bc}$ **3**

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Q1) a) i/ $\int \tan^3 x \, dx$

$= \int \tan x (\sec^2 x - 1)\, dx$ 1

$= \frac{\tan^2 x}{2} - \log(\cos x) + c$ 1

ii/ $\int \frac{dx}{x^2 - 6x + 13}$

$= \int \frac{dx}{(x-3)^2 + 4}$ 1

$= \frac{1}{2} \tan^{-1}\left(\frac{x-3}{2}\right) + c$ 1

b) i/ $\int_0^1 \frac{x}{\sqrt{4-x^2}}\, dx$

Let $u = 4 - x^2$

$du = -2x\, dx$ 1

$= -\frac{1}{2}\int_4^3 u^{-1/2}\, du$ 1

$= \frac{1}{2}\int_3^4 u^{-1/2}\, du$

$= \frac{1}{2}\left[2u^{1/2}\right]_3^4$

$= 2 - \sqrt{3}$ 1

ii/ $\int_0^{\pi/2} x \cos^2 x \, dx$

$= \int_0^{\pi/2} x\left(\frac{1}{2} + \frac{\cos 2x}{2}\right) dx$ 1

$= \left.\frac{x^2}{4}\right]_0^{\pi/2} + \int_0^{\pi/2} \frac{x \cos 2x}{2}\, dx$

$= \frac{\pi^2}{16} + \left. x\frac{\sin 2x}{4}\right]_0^{\pi/2} - \int_0^{\pi/2} \frac{\sin 2x}{4}\, dx$ 1

$= \frac{\pi^2}{16} + \left.\frac{\cos 2x}{8}\right]_0^{\pi/2}$

$= \frac{\pi^2}{16} + \frac{-1}{8} - \frac{1}{8}$

$= \frac{\pi^2 - 4}{16}$

c) i/ $I_n = \int_0^1 x^n e^{-x}\, dx$

$= -x^n e^{-x}\Big]_0^1 + \int_0^1 n x^{n-1} e^{-x}\, dx$ 1

$= -e^{-1} + n\int_0^1 x^{n-1} e^{-x}\, dx$

$\therefore I_n = n I_{n-1} - \frac{1}{e}$ 1

ii/ $I_0 = \int_0^1 e^{-x}\, dx$

$= -\frac{1}{e^x}\Big]_0^1$

$= 1 - \frac{1}{e}$

$I_1 = I_0 - \frac{1}{e}$

$= 1 - \frac{1}{e} - \frac{1}{e}$

$= 1 - \frac{2}{e}$ 1

$I_2 = 2I_1 - \frac{1}{e}$

$= 2\left(1 - \frac{2}{e}\right) - \frac{1}{e}$

$= 2 - \frac{5}{e}$

$I_3 = 3I_2 - \frac{1}{e}$

$= 3\left(2 - \frac{5}{e}\right) - \frac{1}{e}$

$= 6 - \frac{16}{e}$

$\therefore \int_0^1 x^3 e^{-x}\, dx$

$= 6 - \frac{16}{e}$ 1

Q2) a) $\sqrt{6i - 8} = a + ib$

$a^2 - b^2 = -8$

$2ab = 6$

$a = \frac{b}{3}$ 1

$\frac{b^2}{9} - b^2 = -8$

$-\frac{8b^2}{9} = 8$

$b = \pm 3$

$\therefore a = \pm 1$ 1

$2z^2 - (3+i)z + 2 = 0$

$x = \frac{(3+i) \pm \sqrt{9 + 6i - 1 - 4.2.2}}{4}$ 1

$= \frac{3 + i \pm \sqrt{6i - 8}}{4}$

$= \frac{3 + i \pm (1 + 3i)}{4}$

$= 1 + i$ or $\frac{1-i}{2}$ 1

b) If $2 - i$ is a root so is $2 + i$

$(x - (2+i))(x - (2-i))$

$= x^2 - (2-i)x - (2+i)x + 5$

$= x^2 - 4x + 5$ 1

$$\begin{array}{r} 3x + 2 \\ x^2 - 4x + 5 \overline{)\,3x^3 - 10x^2 + 7x + 10} \\ 3x^3 - 12x^2 + 15x \\ 2x^2 - 8x + 10 \end{array}$$

$\therefore$ roots are $x = (2-i), (2+i), -\frac{2}{3}$ 1

c) $P(x) = x^3 + px^2 + q = 0$

$P\left(\frac{1}{x}\right) = \frac{1}{x^3} + \frac{p}{x^2} + \frac{q}{1} = 0$ 1

$= 1 + px + qx^3 = 0$

$\therefore qx^3 + px^2 + 1 = 0$ 1

ii) $P(x) = x^3 + px^2 + q = 0$

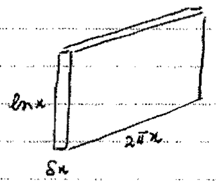
$P(\sqrt{x}) = x^{3/2} + px + q = 0$ 1

$\therefore x^{3/2} = -(px + q)$

$x^3 = p^2x^2 + 2pqx + q^2$

$\therefore x^3 - p^2x^2 - 2pqx - q^2 = 0$ 1

d)



1

Volume of slice $2\pi x\, \delta x \ln x$

Volume $= 2\pi \int_1^e x \ln x \, dx$ 1

$= 2\pi\left[\frac{x^2}{2}\ln x\right]_1^e - \frac{1}{2}.2\pi\int_1^e x\, dx$ 1

$= 2\pi\left[\frac{e^2}{2}\right] - \pi\left[\frac{x^2}{2}\right]_1^e$

$= \pi e^2 - \frac{\pi e^2}{2} + \frac{\pi}{2}$

$= \frac{\pi}{2}(e^2 + 1)\ u^3$ 1

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Q3) a) i/ $y = \frac{1}{x}$

$$\frac{dy}{dx} = \frac{-1}{x^2}$$

at $x = t$ $\frac{dy}{dx} = -\frac{1}{t^2}$ 1

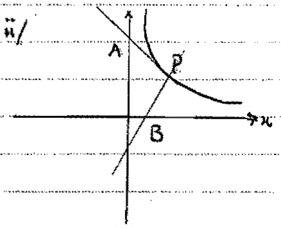
∴ eqn of tangent

$$y - \frac{1}{t} = \frac{-1}{t^2}(x - t)$$

$$t^2y - t = -x + t$$

$$x + t^2y = 2t$$ 1

ii/



tangent meets y axis at

$$\left(0, \frac{2}{t}\right)$$

normal meets x axis at

$$t^2x = y + t^3 - \frac{1}{t}$$

$$\left(t - \frac{1}{t^3}, 0\right)$$ 1

Mid point $\left(\frac{t^4 - 1}{2t^3}, \frac{1}{t}\right)$ 1

$$y = \frac{1}{t}$$

$$x = \frac{t^4 - 1}{2t^3}$$

$$= \frac{\frac{1}{y^4} - 1}{\frac{2}{y^3}}$$

$$\therefore x = \frac{1 - y^4}{2y}$$

$$2xy + y^4 - 1 = 0$$ 1

d) $P(x) = 4x^3 + 20x^2 - 23x + 6$

$$P'(x) = 12x^2 + 40x - 23$$

$$= (6x + 23)(2x - 1)$$

$$\therefore (6x + 23)(2x - 1) = 0$$

when $x = \frac{1}{2}$ or $-\frac{23}{6}$ 1

$$P\left(\tfrac{1}{2}\right) = 0 = P'\left(\tfrac{1}{2}\right)$$

∴ $x = \frac{1}{2}$ is the double root 1

Using product of roots

$$\frac{1}{2} \cdot \frac{1}{2} \cdot \alpha = -\frac{6}{4}$$

$$\alpha = -6$$

∴ roots are $\frac{1}{2}, \frac{1}{2}, -6.$ 1

Q3)

b)

grad of m $= \frac{b(\sin\alpha - \sin\beta)}{a(\cos\alpha - \cos\beta)}$

eqn of PQ

$$y - b\sin\alpha = \frac{b}{a}\left(\frac{s\alpha - s\beta}{c\alpha - c\beta}\right)(x - a\cos\alpha)$$

Now focal chord ⇒ $(ae, 0)$ 1

satisfies eqn

$$-\cancel{b}\sin\alpha = \frac{\cancel{b}(s\alpha - s\beta)}{\cancel{a}(c\alpha - c\beta)}(\cancel{a}e - \cancel{a}\cos\alpha)$$

$$\therefore -\sin\alpha\frac{(c\alpha - c\beta)}{\sin\alpha - \sin\beta} = e - \cos\alpha$$

$$-\frac{(\sin\alpha\cos\alpha - \sin\alpha\cos\beta)}{\sin\alpha - \sin\beta} + \cos\alpha = e$$

$$\therefore e = \frac{-\cancel{\sin\alpha\cos\alpha} + \sin\alpha\cos\beta + \cancel{\sin\alpha\cos\alpha} - \cos\alpha\sin\beta}{\sin\alpha - \sin\beta}$$ 2

$$= \frac{\sin(\alpha - \beta)}{\sin\alpha - \sin\beta}$$

c) i/ In HPRM

∠ HPR = ∠ RMH (angle between radius + tangent) 1

∴ HPRM is a cyclic quad (opp angles add to 180°) 1

Similarly MRQK

ii/ Join PM and QM

In △ PRM and MQK

∠ PRM = ∠ MKQ (ext angle of a cyclic quad)

PR = RM (tangents from an ext pt) 1

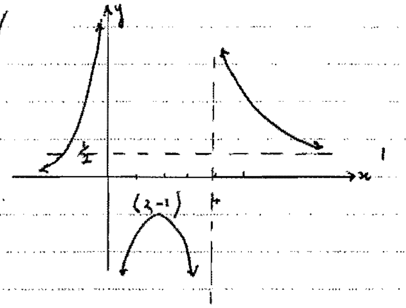
KM = KQ (radii)

∴ △' isos

∠ RPM = ∠ RMP = ∠ KMQ 1

∴ △ PRM ||| △ MQK (equiangular)

Q4) a) i/



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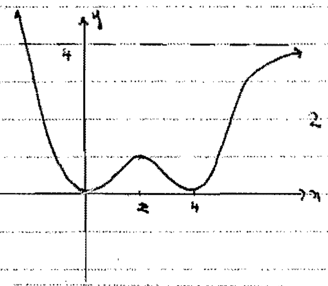
ii/



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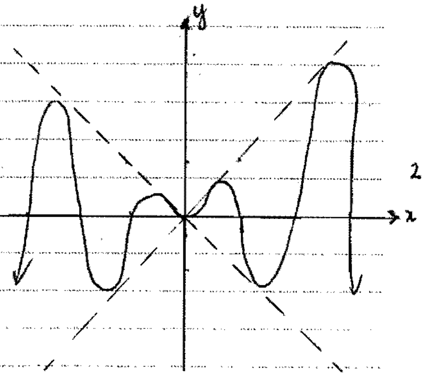
b)



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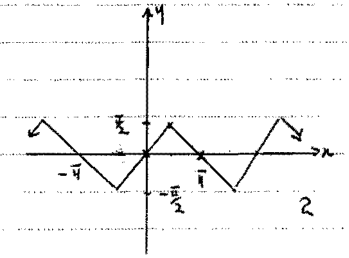
c)



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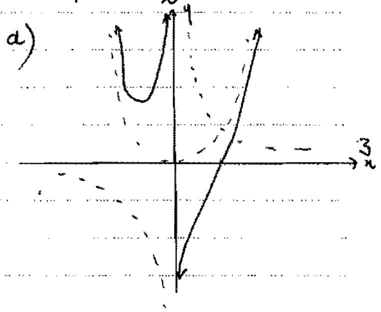
d) $y = x^2 - \frac{1}{x}$

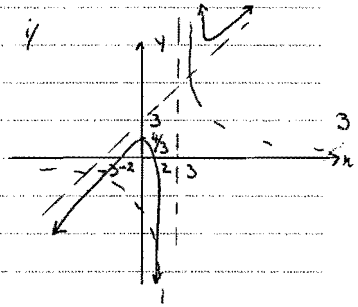


3

e) $f(x) = \frac{x^2 - 4}{x - 3}$

$= x + 3 + \frac{5}{x-3}$

i/



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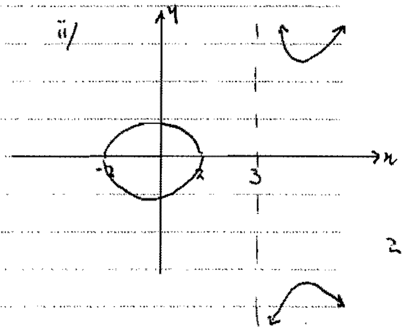
ii/



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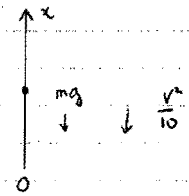
Q6)

a/



$m\ddot{x} = -mg - \frac{v^2}{10}$ 1

$\ddot{x} = -g - \frac{v^2}{10}$ ($\because m = 1$)

$\frac{dv}{dt} = -g - \frac{v^2}{10}$

$\therefore t = \int \frac{dv}{-g - \frac{v^2}{10}}$ 1

$= -10 \int \frac{dv}{100 + v^2}$

$= -\tan^{-1}\left(\frac{v}{10}\right) + c_1$

when $v = 10$, $t = 0 \Rightarrow c_1 = \frac{\pi}{4}$

$\therefore \tan^{-1}\frac{v}{10} = \frac{\pi}{4} - t$ 1

$\frac{v}{10} = \tan\left(\frac{\pi}{4} - t\right)$

when $v = 0$ $t = \frac{\pi}{4}$

$\therefore x = \int 10 \tan\left(\frac{\pi}{4} - t\right) dt$ 1

$= 10 \ln \cos\left(\frac{\pi}{4} - t\right) + c_2$

when $t = 0$, $x = 0$

$\therefore c_2 = -10 \ln \frac{1}{\sqrt{2}} = 5 \ln 2$

$\therefore$ when $x = \frac{\pi}{4}$ $x = 5 \ln 2$ metres.

1

b) $(2x+3)^{21}$

$\frac{T_{r+1}}{T_r} = \frac{{}^{21}C_r (2x)^{21-r} 3^r}{{}^{21}C_{r-1} (2x)^{22-r} 3^{r-1}}$ 1

$= \dfrac{\frac{21!}{r!(21-r)!} \times 3}{\frac{21!}{r!(22-r)!} \times 2}$

$= \frac{22-r}{r} \times \frac{3}{2}$

Now $\frac{66-3r}{2r} \geq 1$ 1

$66 - 3r \geq 2r$

$r \leq 13\frac{1}{5}$

$\therefore r = 13$

$\therefore T_{14} = {}^{21}C_{13}\, 2^8\, 3^{13}$ 1

c) $x^m y^n = k$

$\therefore x^m n y^{n-1} \frac{dy}{dx} + m x^{m-1} y^n = 0$ 1

$x^m n y^{n-1} \frac{dy}{dx} = -m x^{m-1} y^n$ 1

$\therefore \frac{dy}{dx} = \frac{-m x^{m-1} y^n}{n x^m y^{n-1}}$

$= \frac{-my}{nx}$ 1

d) i/ $(1+x)^{2n} = {}^{2n}C_0 + {}^{2n}C_1 x + \ldots + {}^{2n}C_{2n} x^{2n}$

when $x = 1$

$2^{2n} = 1 + {}^{2n}C_1 + \ldots + {}^{2n}C_{2n}$

$\therefore 4^n - 1 = {}^{2n}C_1 + \ldots + {}^{2n}C_{2n}$

ii/ $(1+x)^{2n} = (1+x)^n (1+x)^n$

term in x^2 on RHS

$= {}^nC_2 x^2 + {}^nC_1 x\, {}^nC_1 x + {}^nC \ldots$

equating coeff of x^2

Q5) a) i) $z = -\sqrt{3} + i$ 1

$= 2 \operatorname{cis}\left(\frac{5\pi}{6}\right)$

ii) $z^7 = 2^7 \operatorname{cis}\left(\frac{35\pi}{6}\right)$

$= 128\left(\cos -\frac{\pi}{6} + i \sin -\frac{\pi}{6}\right)$

$= 128\left(\cos \frac{\pi}{6} - i \sin \frac{\pi}{6}\right)$ 1

$64z = 2.64 \operatorname{cis} \frac{5\pi}{6}$

$= 128\left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6}\right)$

$= 128\left(-\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}\right)$ 1

$= -128 \cos \frac{\pi}{6} + 128 \sin \frac{\pi}{6}$

$\therefore z^7 + 64z = 0.$ 1

b) $Re\left(\frac{z-4}{z}\right) = 0$

$\frac{z-4}{z} = \frac{x-4+iy}{x+iy} \times \frac{x-iy}{x-iy}$ 1

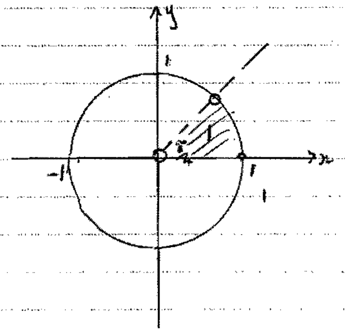
$= \frac{x^2-4x+y^2+i(xy-xy+4)}{x^2+y^2}$ 1

$\therefore x^2 - 4x + y^2 = 0$

$(x-2)^2 + y^2 = 4$

circle centre $(2,0)$ $r = 2$ 1

c)



Q5)

d) i) $(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i \sin 5\theta$

$= \cos^5\theta + 5i\cos^4\theta\sin\theta + 10\cos^3\theta(i\sin\theta)^2 + 10\cos^2\theta(i\sin\theta)^3 + 5\cos\theta(i\sin\theta)^4 + (i\sin\theta)^5$

$= \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta$

$\therefore \cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

$\sin 5\theta = 5\sin\theta\cos^4\theta - 10\sin^3\theta\cos^2\theta + \sin^5\theta$ 3

ii) Now $\tan 5\theta = \frac{\sin 5\theta}{\cos 5\theta} = \frac{5c^4s - 10c^2s^3 + s^5}{c^5 - 10c^3s^2 + 5s^4c}$ $\div c^5$

$= \frac{5t - 10t^3 + t^5}{1 - 10t^2 + 5t^4}$ let $\tan\theta = t$ 2

iii) Let $\tan 5\theta = 0$

then $5\theta = 0, \pi, 2\pi, 3\pi, 4\pi, 5\pi, 6\pi \ldots$

$\therefore \theta = 0, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}, \pi, \ldots$

$\tan 5\theta = 0$ has an infinite no. of soln^s

but $\frac{5t - 10t^3 + t^5}{1 - 10t^2 + 5t^4} = 0$

has only 5 roots

$t(5 - 10t^2 + t^4) = 0$

$\tan\theta = 0$ $\tan\frac{\pi}{5}, \tan\frac{2\pi}{5}, \tan\frac{3\pi}{5}, \tan\frac{4\pi}{5}$ 2

product of roots is 5

$\therefore \tan\frac{\pi}{5} \cdot \tan\frac{2\pi}{5} \tan\frac{3\pi}{5} \tan\frac{4\pi}{5} = 5$

Q7) a) $\sum_{K=1}^{n} \frac{K}{(K+1)!} = 1 - \frac{1}{(n+1)!}$

Step 1 Let $n=1$

$L.H.S. = \frac{1}{2!} = \frac{1}{2}$ $RHS = 1 - \frac{1}{2!} = \frac{1}{2}$ 1

$\therefore$ true for $n=1$

Step 2. Assume true for $n=K$

Prove true for $n=K+1$

$\sum_{K=1}^{n+1} \frac{K}{(K+1)!} = 1 - \frac{1}{(n+1)!} + \frac{n+1}{(n+2)!}$

$= 1 - \frac{n+2}{(n+2)!} + \frac{n+1}{(n+2)!}$

$= 1 - \frac{n+2-n-1}{(n+2)!}$ 3

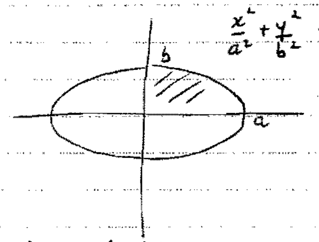
$= 1 - \frac{1}{(n+2)!}$

$= RHS.$

So if true for n it is true for $n+1$

Step 3 It is true for $n=1$ and as it works for n and $n+1$ then it is true by math induction 1

b)



1

Area of ellipse = 4 × shaded area

$A = 4\int_0^a y\,dx$ 1

$= 4\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx$

as $y^2 = \frac{b^2}{a^2}(a^2-x^2)$

$y = \frac{b}{a}\sqrt{a^2-x^2}$ 1

$A = \frac{4b}{a}\int_0^a \sqrt{a^2-x^2}\,dx$

$= \frac{4b}{a} \times \frac{\pi a^2}{4}$ ($\frac{1}{4}$ of circle)

$= \pi ab \ u^2$. 1

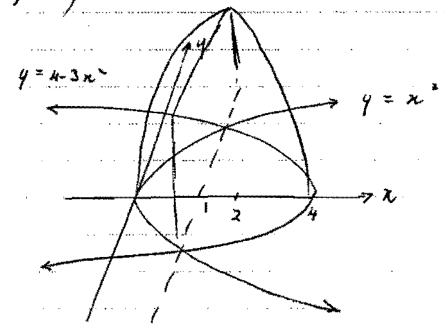
c) $\hat{F} = D\hat{F}C = \hat{A} + \hat{C}$ (ext angle of △)

$\hat{G} = D\hat{G}F = \hat{B} + \hat{E}$ (" " ") 1

$\hat{D} + \hat{F} + \hat{G} = 180°$ (angle sum of △)

$\therefore \hat{A} + \hat{B} + \hat{C} + \hat{D} + \hat{E} = 180$ 1

Q7) d)



Curves meet at $x=1$ 1

From $x=0$ to 1 the cross-section with co-ord x has $A = (2y)^2 = 4y^2$
$= 4x$

$\therefore V = \int_0^1 4x\,dx$

$= 2$ 1

from $x=1$ to 4 the cross section co-ord x has $A = (2y)^2 = 4y^2$
$= \frac{4}{3}(4-x)$

$\therefore V = \frac{4}{3}\int_1^4 4-x\,dx$

$= \frac{4}{3}\left[4x - \frac{x^2}{2}\right]_1^4$

$= \frac{4}{3}\left(16 - 8 - 4 + \frac{1}{2}\right)$

$= 6$ 1

$\therefore$ total volume is $8\ u^3$

1

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Q8) a) ii/ $\frac{1}{a}+\frac{1}{b} = \frac{b+a}{ab}$

as $a+b=1$

$1 = a+b \geq 2\sqrt{ab}$

$\frac{1}{2} \geq \sqrt{ab}$

$ab \leq \frac{1}{4}$

$\therefore \frac{1}{ab} \geq 4$

$a+b \geq 4$ 2

i/ $(\ldots)^2 \geq 0$ for $a, b > 0$

$a+b \geq 2\sqrt{ab}$

$(a+b)^2 \geq 4ab$ 1

$a^2 - 2ab + b^2 \geq 0$

$(a-b)^2 \geq 0$

true

iii/ $a^2+b^2 = a^2 + (1-a)^2$

$= 2a^2 - 2a + 1$

$= 2(a-\frac{1}{2})^2 + \frac{1}{2} \geq \frac{1}{2}$

equality holds when $a = \frac{1}{2}$ 2

iv/ $\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right) = \left(1+\frac{1}{a}\right)\left(1+\frac{1}{1-b}\right)$

$= 1 + \frac{1}{a} + \frac{1}{1-a} + \frac{1}{a(1-a)}$

$= 1 + \frac{2}{a(1-a)}$

Now $a(1-a) = -a^2+a = -(a-\frac{1}{2})^2 + \frac{1}{4} \leq \frac{1}{4}$

and $1 + \frac{2}{a(1-a)} \geq 1 + \frac{2}{\frac{1}{4}}$

$\geq 9.$ 2

Q8)

b)

i/ from $x = Vt\cos\theta$ & $y = Vt\sin\theta - \frac{1}{2}gt^2$ 1

$\therefore t = \frac{x}{V\cos\theta}$

$y = V\sin\theta \cdot \frac{x}{V\cos\theta} - \frac{1}{2}g\frac{x^2}{V^2\cos^2\theta}$

$= x\tan\theta - \frac{gx^2\sec^2\theta}{2V^2}$ 1

ii/ passes thru (b, h) and (c, h)

$\therefore h = b\tan\theta - \frac{gb^2}{2V^2}\sec^2\theta$ (1)

and $h = c\tan\theta - \frac{gc^2}{2V^2}\sec^2\theta$ (2) 1

$\therefore b\tan\theta - \frac{gb^2}{2V^2}\sec^2\theta = c\tan\theta - \frac{gc^2}{2V^2}\sec^2\theta$

$(b-c)\tan\theta = (b^2-c^2)\frac{g}{2V^2}\sec^2\theta$ 1

$\therefore \tan\theta = (b+c)\frac{g}{2V^2}\sec^2\theta$

$\therefore V^2 = \frac{(b+c)g\sec^2\theta}{2\tan\theta}$ 1

iii/ Subst into (1) 1

$h = b\tan\theta - \frac{gb^2\sec^2\theta}{2} \cdot \frac{2\tan\theta}{(b+c)g\sec^2\theta}$

$= b\tan\theta - \frac{b^2\tan\theta}{b+c}$ 1

$h(b+c) = (b^2+bc)\tan\theta - b^2\tan\theta$

$= bc\tan\theta$

$\therefore \tan\theta = \frac{h(b+c)}{bc}$ 1