NORTH SYDNEY
GIRLS HIGH SCHOOL

2009
TRIAL HIGHER SCHOOL CERTIFICATE

Mathematics Extension 2

Student Number:________________________________**Teacher:** _______________

Student Name:__________________________________

General Instructions

- Reading time – 5 minutes.
- Working time – 3 hours.
- Write using black or blue pen.
- Board approved calculators may be used.
- All *necessary* working should be shown in every question if full marks are to be awarded.
- Marks may **NOT** be awarded for untidy or badly arranged work.
- Start each **NEW** question in a separate answer booklet.

Total Marks - 120 Marks

- Attempt Questions 1-8
- All questions are of equal value.

At the end of the examination, place your solution booklets in order and put this question paper on top.
Submit one bundle.
The bundle will be separated before marking commences so that anonymity will be maintained.

Question	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**Total**
Mark									**/120**

Total marks – 120
Attempt Questions 1 - 8
All questions are of equal value

Answer each section in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Find $\int \cos x \sin^6 x \, dx$ **1**

(b) Evaluate $\int_0^1 \frac{2+6x}{\sqrt{4-x^2}} \, dx$, leaving your answer in exact form **3**

(c) Use integration by parts to evaluate $\int_0^{\sqrt{3}} x \tan^{-1} x \, dx$ **4**

(d) (i) Find real constants A and B such that $\frac{3}{(x-2)(2x-1)} = \frac{A}{x-2} + \frac{B}{2x-1}$ **2**

(ii) Hence find $\int \frac{3\,dx}{(x-2)(2x-1)}$ **2**

(e) Using the substitution $t = \tan\frac{\theta}{2}$ and the results of (d), find **3**

$$\int \frac{5}{4-3\sin\theta} \, d\theta$$

Question 2 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Let $z = \dfrac{3-6i}{2+i}$, find

(i) $|z|$ **2**

(ii) $\arg z$ **2**

(b) Find real values p and q where $\dfrac{p-5qi}{1+i} = \overline{1-4i}$ **3**

(c) Let $u = \dfrac{7\sqrt{2}}{2}(1+i)$, $v = r\cos\theta + ir\sin\theta$ and $uv = 42\left(\cos\dfrac{\pi}{20} + i\sin\dfrac{\pi}{20}\right)$

(i) Write u in modulus-argument form. **2**

(ii) Find r and θ. **2**

(d) z lies on the locus defined by $|z+2| = 2$ and let $\arg z = \theta$

(i) By use of an appropriate diagram, show that $\arg(z+2) = 2\theta - \pi$ **2**

(ii) Hence, or otherwise, find $\arg(z^2 + 6z + 8)$ **2**

Question 3 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Consider the rectangular hyperbola $x^2 - y^2 = 4$.

(i) Sketch the curve, showing the coordinates of the foci S and S' and the equations of the directrices and asymptotes. **3**

(ii) The point $P(2\sec\theta, 2\tan\theta)$ lies on the curve. **2**
Show that the tangent at P has equation $x\sec\theta - y\tan\theta = 2$.

(iii) The tangent meets the x-axis at Q. **3**
Show that the locus of the midpoint M of PQ is given by $x^2 - y^2 - 3 = \dfrac{1}{y^2+1}$

(b) The polynomial $u(x) = mx^7 + nx^6 + 1$ is divisible by $(x+1)^2$.

(i) Show that $7m = 6n$. **1**

(ii) Find the values of m and n, where m and n are real numbers. **2**

(c) Given that α, β and γ are the roots of the equation $x^3 + 3x + 1 = 0$

(i) Find a polynomial equation of smallest degree that has α^2, β^2 and γ^2 as roots. **2**

(ii) Hence find $\alpha^2 + \beta^2 + \gamma^2$ **1**

(d) Which one of the following diagrams below could represent the location of the roots of $z^5 + z^2 - z + c = 0$ in the complex plane, where c is a real number. Without any calculations, justify your answer. **1**

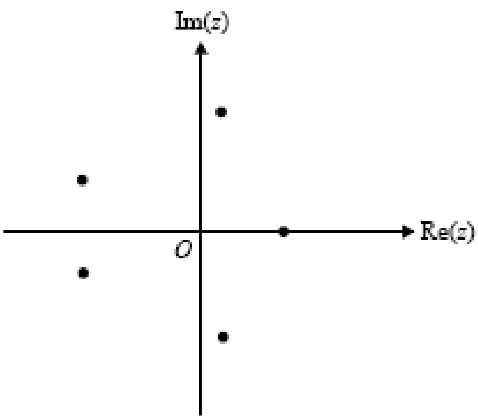


Diagram A

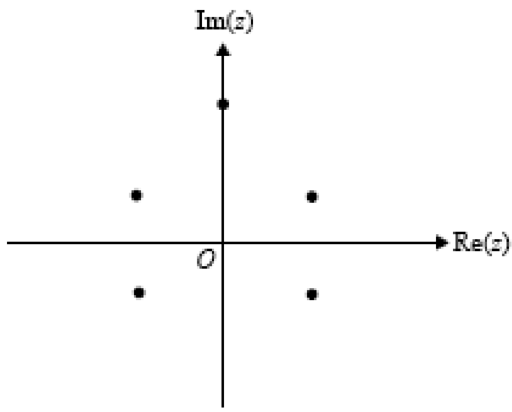
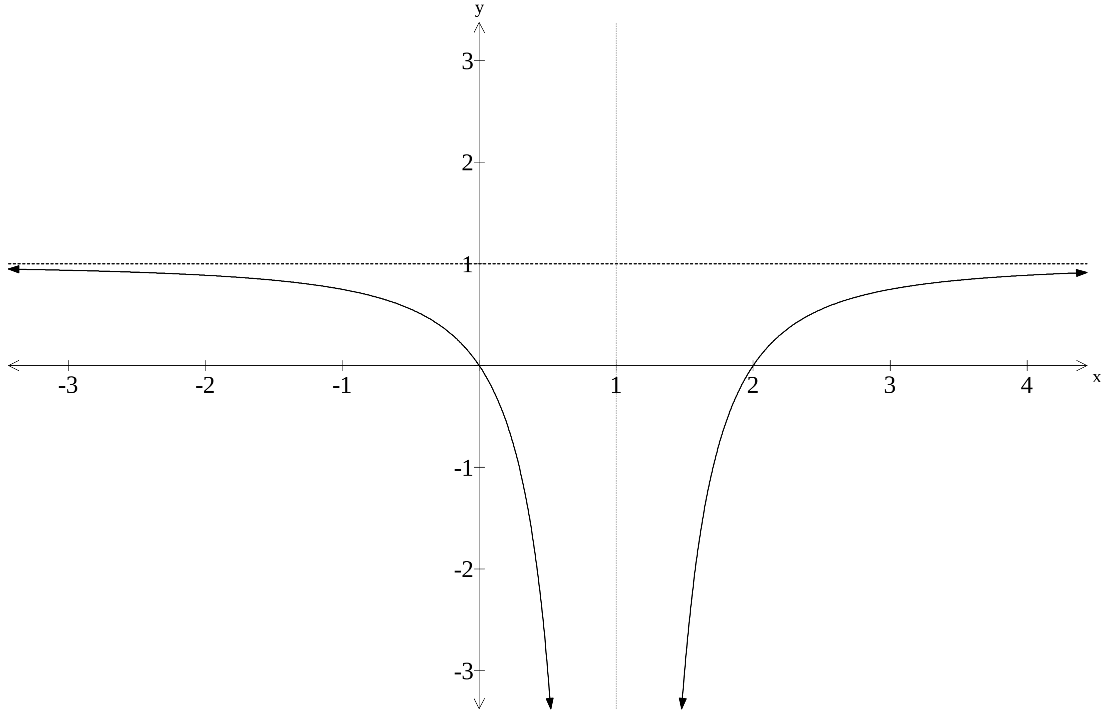


Diagram B

Question 4 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) The graph below shows a function that has $\tilde{x}$-intercepts at $x = 0$ and $x = 2$.
There is a vertical asymptote at $x = 1$ and a horizontal asymptote of $y = 1$.
The graph is symmetrical about the line $x = 1$.



Without using calculus, sketch the following graphs on the ANSWER sheet provided on page 15, clearly showing any asymptotes and intercepts.

(i) $y = f(x-1)$ **1**

(ii) $y = \left[f(x)\right]^2$ **2**

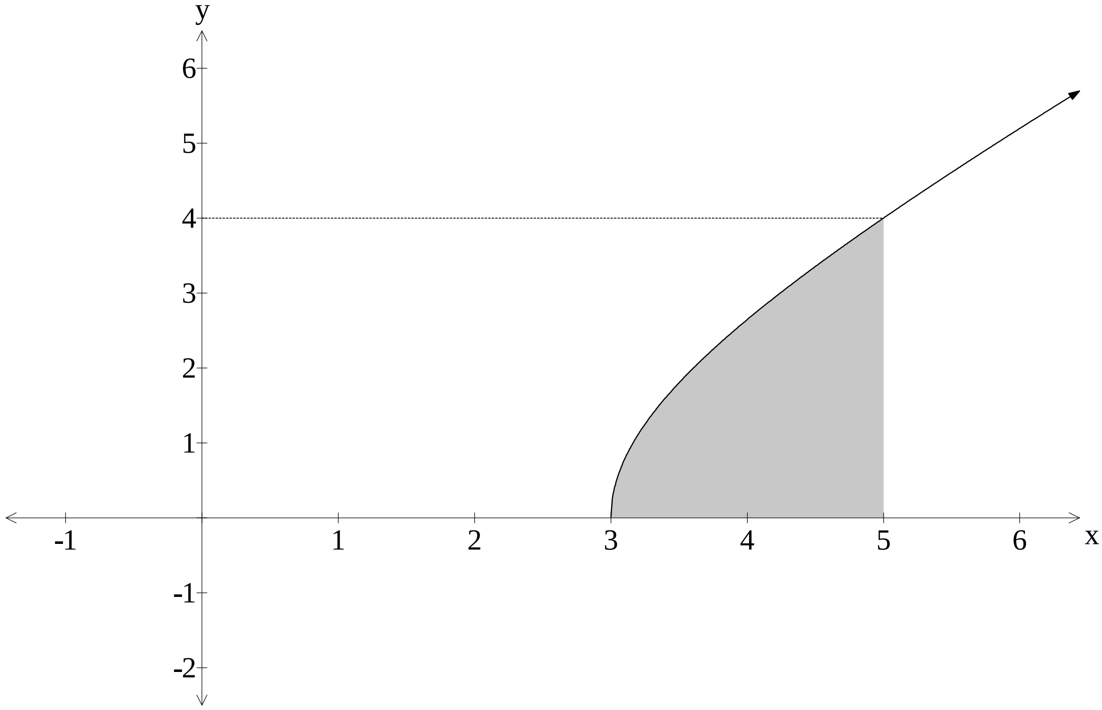
(iii) $y^2 = f(x)$ **2**

(iv) $y = \tan^{-1} f(x)$ **2**

Question 4 continues on page 6

(b) The graph of $f(x)=\sqrt{x^2-9}$ is shown below.

The area between $f(x)$ and the x-axis for $3 \le x \le 5$ is shaded.



(i) Using the method of shells, show that the volume, V, of the solid formed when the shaded area is rotated about the y-axis is given by **2**

$$V = \int_3^5 2\pi x\left(x^2-9\right)^{\frac{1}{2}}\,dx$$

(ii) Hence calculate the volume. **1**

(c) (i) Given $f(x)=f(a-x)$ and using the substitution $u=a-x$, prove that **3**

$$\int_0^a xf(x)\,dx = \frac{a}{2}\int_0^a f(x)\,dx$$

(ii) Hence, or otherwise, prove that $\int_0^{\pi} F(x)\,dx = \frac{\pi^2}{4}$, if $F(x)=\frac{x\sin x}{1+\cos^2 x}$ **2**

Question 5 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Given $I_n = \int_0^1 x^n e^{2x}\, dx$, where n is a positive integer, show that **2**

$$I_n = \frac{1}{2}\left(e^2 - nI_{n-1}\right)$$

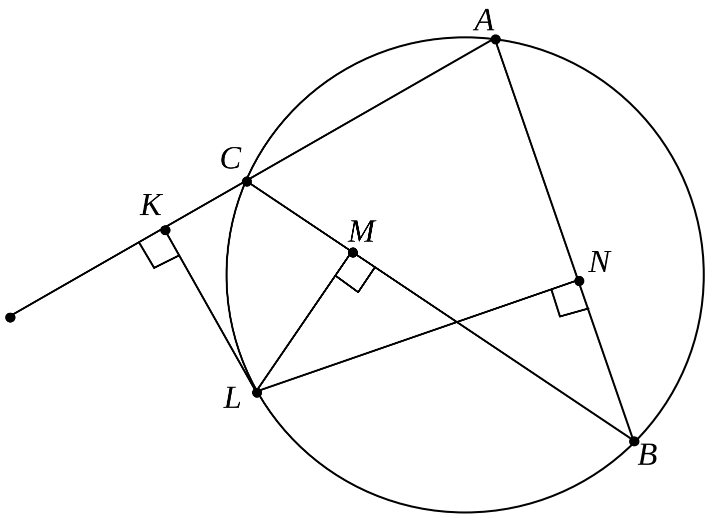
(ii) Hence evaluate $\int_0^1 x^3 e^{2x}\, dx$ **3**

(b) Fifteen new students at NSGHS are distributed evenly among the classes of Miss V, Mr. S and Ms L.
Given that there are three children with red hair among the fifteen and that the students are distributed randomly, find:

(i) the number of ways that all the children with red hair end up in the same class. **2**

(ii) the probability that each class gets one child with red hair. **2**

(c) The diagram below shows triangle ABC inscribed in a circle with L a point on the arc BC.
LK is perpendicular to AC produced and LN is perpendicular to AB.



(i) Copy the diagram into your Answer book

(ii) Explain why $CKLM$ and $MNBL$ are cyclic quadrilaterals. **2**

(iii) Explain why $\angle KCL = \angle ABL$. **1**

(iv) Hence, or otherwise, prove that K, M and N are collinear. **3**

Question 6 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Solve $\sin^{-1} x - \cos^{-1} x = \sin^{-1}(3x - 2)$ **4**

(b) Given that $\sin\left(\frac{1}{2}y\right) = \frac{1}{2}\left(x^2 - 2\right)$ and that $x > 0$ and $y > 0$. **3**

Show by differentiating implicitly that $\dfrac{dy}{dx} = \dfrac{4}{\sqrt{4 - x^2}}$

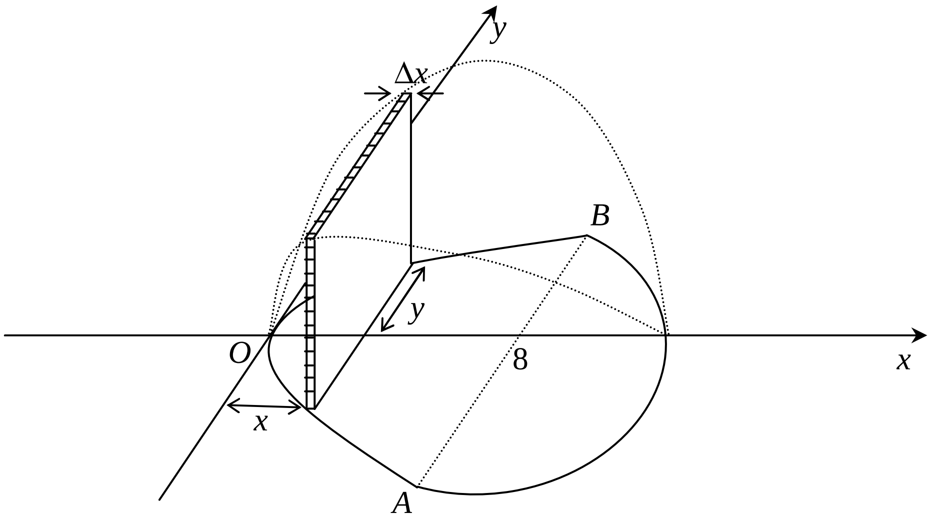
(c) The diagram below shows a solid with its base in the *x*-*y* plane.

Every cross-section perpendicular to the *x*-axis is a square.

One part of the base is the segment *OAB* of the parabola $y^2 = 2x$ cut off by the line $x = 8$.

The other part of the base is a semi-circle with diameter *AB*.

Consider a slice *S*, perpendicular to the *x*-axis, of width Δx.



(i) Find the coordinates of *B* and hence find the distance *AB*. **2**

(ii) Show that the volume ΔV of *S* is given by $\Delta V \approx 8x\Delta x$ for $0 \le x \le 8$. **2**

(iii) By first finding an expression for ΔV of *S* when $x > 8$, calculate the volume of the solid. **4**

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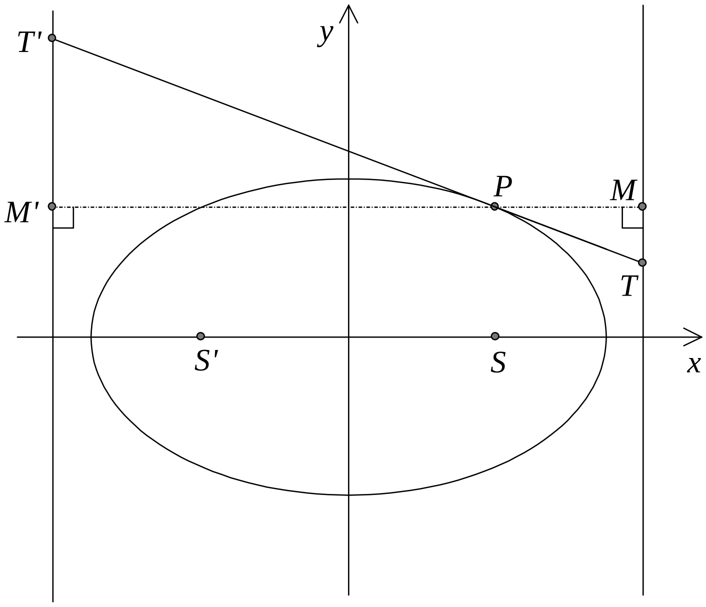
Question 7 (15 marks) Use a SEPARATE writing booklet. **Marks**

The diagram below shows an ellipse $b^2x^2 + a^2y^2 = a^2b^2$, where S and S' are the foci.

The diagram shows a tangent at $P(a\cos\theta, b\sin\theta)$, intersecting the two directrices at T and T'.

M and M' are the foot of the perpendiculars drawn from P to their respective directrices.



(a) Show that $SP + S'P = 2a$. **2**

(b) You may assume that the tangent at P is $xb\cos\theta + ya\sin\theta = ab$. (Do **NOT** prove this)

Let $\alpha = \angle SPT$ and $\beta = \angle S'PT'$

(i) Show that T has coordinates $\left(\dfrac{a}{e}, \dfrac{b(e-\cos\theta)}{ae\sin\theta}\right)$ **1**

(ii) Show that $\angle PST = 90°$ **2**

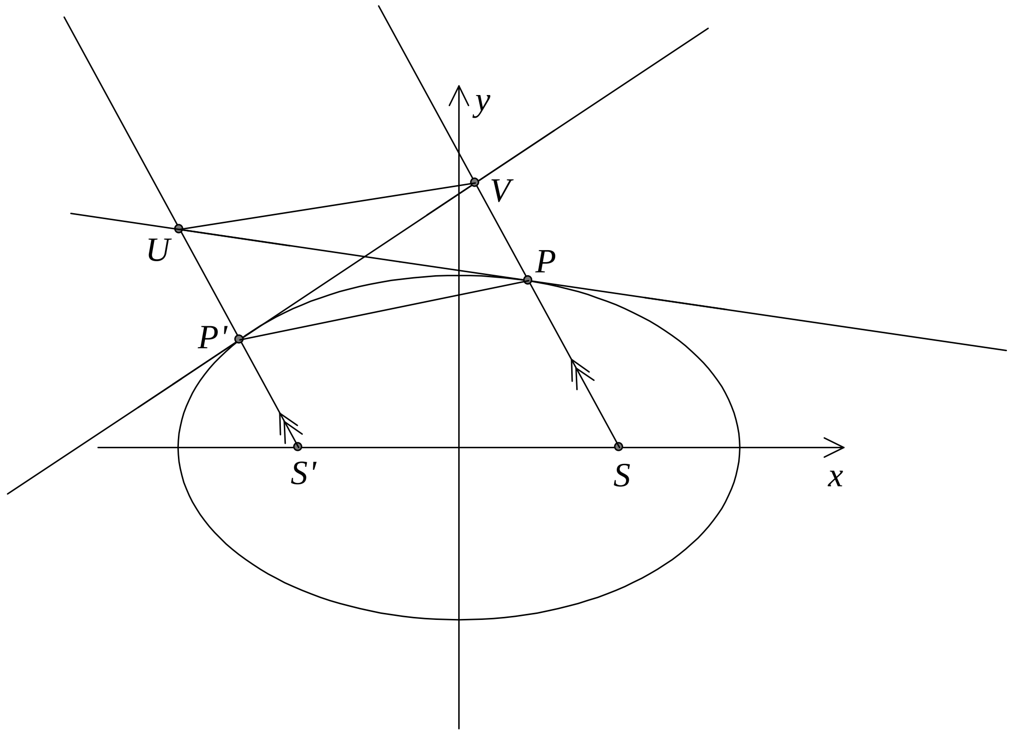
(iii) Show that $\dfrac{PM}{PT} = \dfrac{PM'}{PT'}$ **1**

(iv) Deduce that $\alpha = \beta$. **2**

Question 7 continues on page 11

(c) Consider the diagram below, where $SV \parallel S'U$.

The tangent at P intersects the ray $S'U$ at U and the tangent at P' intersects the ray SV at V.



(i) Copy the diagram into your Answer booklet.

(ii) Using (b) show that $\Delta UPS'$ is isosceles. **3**

(iii) Using (ii) above and also (a), show that $VP = UP'$. **3**

(iv) Deduce that $UV \parallel PP'$ **1**

Question 8 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Show that $\tan^{-1}(n+1)-\tan^{-1}(n-1)=\tan^{-1}\left(\frac{2}{n^2}\right)$, where n is a positive integer. **2**

(ii) Given that $\tan\left(\tan^{-1}x+\tan^{-1}y\right)=\frac{x+y}{1-xy}$, where x and y are real numbers, explain why when $x>1,\ y>1$ that $\tan^{-1}x+\tan^{-1}y=\pi+\tan^{-1}\left(\frac{x+y}{1-xy}\right)$. **1**

(iii) Hence, or otherwise, show that for $n\geq 1$ **3**

$$\sum_{r=1}^{n}\tan^{-1}\left(\frac{2}{r^2}\right)=\frac{3\pi}{4}+\tan^{-1}\left(\frac{2n+1}{1-n-n^2}\right)$$

(iv) Hence write down $\sum_{r=1}^{\infty}\tan^{-1}\left(\frac{2}{r^2}\right)$ **1**

(b) Let $T_n(x)=\frac{{}^nC_0}{x}-\frac{{}^nC_1}{x+1}+\frac{{}^nC_2}{x+2}-\ \ ....\ \ +(-1)^n\frac{{}^nC_n}{x+n}$ for a given integer n and all real x

(i) If $S_k(x)=\frac{k!}{x(x+1)(x+2)....(x+k)}$ where k is an integer, show that **2**

$$S_k(x)-S_k(x+1)=S_{k+1}(x)$$

(ii) Hence prove using mathematical induction that for $n\geq 1$ **4**

$$T_n(x)=\frac{n!}{x(x+1)(x+2)....(x+n)}$$

NOTE: you may use without proof the result ${}^{m+1}C_r={}^mC_r+{}^mC_{r-1}$

(iii) Hence by a suitable substitution prove that **2**

$$\frac{{}^nC_0}{1}-\frac{{}^nC_1}{3}+\frac{{}^nC_2}{5}-\ \ ...\ \ +(-1)^n\frac{{}^nC_n}{2n+1}=\frac{2^n n!}{1\times 3\times 5\times ...\times(2n+1)}$$

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Mathematics Extension 2

Sample Solutions

Question 1

(a) $$\int \cos x \sin^6 x \, dx = \frac{\sin^7 x}{7} + C$$

(b) $$\int_0^1 \frac{2+6x}{\sqrt{4-x^2}}\, dx = 2\int_0^1 \frac{dx}{\sqrt{4-x^2}}\, dx - 3\int_0^1 \frac{-2xdx}{\sqrt{4-x^2}}\, dx$$

$$= 2\left[\sin^{-1}\left(\frac{x}{2}\right)\right]_0^1 - 3\times\left[2\sqrt{4-x^2}\right]_0^1$$

$$= 2\times\frac{\pi}{6} - 6\left[\sqrt{3}-2\right]$$

$$= 12 + \frac{\pi}{3} - 6\sqrt{3}$$

(c) $$\int_0^{\sqrt{3}} x\tan^{-1} x\, dx = \int_0^{\sqrt{3}} \frac{d}{dx}\left(\tfrac{1}{2}x^2\right)\tan^{-1} x\, dx$$

$$= \left[\tfrac{1}{2}x^2 \tan^{-1} x\right]_0^{\sqrt{3}} - \int_0^{\sqrt{3}} \tfrac{1}{2}x^2 \times \frac{1}{1+x^2}\, dx$$

$$= \frac{3}{2}\times\frac{\pi}{3} - \frac{1}{2}\int_0^{\sqrt{3}} \frac{x^2}{1+x^2}\, dx = \frac{\pi}{2} - \frac{1}{2}\int_0^{\sqrt{3}} \frac{(x^2+1)-1}{1+x^2}\, dx$$

$$= \frac{\pi}{2} - \frac{1}{2}\int_0^{\sqrt{3}} \left(1 - \frac{1}{1+x^2}\right) dx$$

$$= \frac{\pi}{2} - \frac{1}{2}\left[x - \tan^{-1} x\right]_0^{\sqrt{3}} = \frac{\pi}{2} - \frac{1}{2}\left(\sqrt{3} - \frac{\pi}{3}\right)$$

$$= \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$$

(d) (i) $$\frac{3}{(x-2)(2x-1)} = \frac{A(2x-1)+B(x-2)}{(x-2)(2x-1)}$$

$\therefore A(2x-1)+B(x-2)=3$

$\therefore 2A+B=0$ [coefficient of x]

Sub $x=2 \Rightarrow 3A=3$

$\therefore A=1 \Rightarrow B=-2$

(ii) $$\int \frac{3\, dx}{(x-2)(2x-1)} = \int \left(\frac{1}{x-2} + \frac{-2}{2x-1}\right) dx$$

$$= \int \frac{1}{x-2}\, dx - \int \frac{2}{2x-1}\, dx$$

$$= \ln|x-2| - \ln|2x-1| + C$$

$$= \ln\left|\frac{x-2}{2x-1}\right| + C$$

(e) $t = \tan\frac{\theta}{2} \Rightarrow dx = \frac{2dt}{1+t^2}$

$\sin\theta = \frac{2t}{1+t^2}$

$$\int \frac{3}{4-5\sin\theta}\,d\theta = \int \frac{3}{4-5\left(\frac{2t}{1+t^2}\right)} \times \frac{2dt}{1+t^2}$$

$$= \int \frac{6\,dt}{4t^2-10t+4} = \int \frac{3\,dt}{2t^2-5t+2}$$

$$= \int \frac{3\,dt}{(2t-1)(t-2)}$$

$$= \ln\left|\frac{t-2}{2t-1}\right| + C$$

$$= \ln\left|\frac{\tan^{-1}\frac{\theta}{2}-2}{2\tan^{-1}\frac{\theta}{2}-1}\right| + C$$

Question 2

(a) $z=\dfrac{3-6i}{2+i}$

(i) $|z|=\left|\dfrac{3-6i}{2+i}\right|=\dfrac{|3-6i|}{|2+i|}=\dfrac{3\sqrt{5}}{\sqrt{5}}=3$

(ii) $z=\dfrac{3-6i}{2+i}=3\times\dfrac{1-2i}{2+i}$

$\dfrac{1-2i}{2+i}=\dfrac{1-2i}{2+i}\times\dfrac{2-i}{2-i}=\dfrac{-5i}{5}=-i$

$\arg z=\arg(-i)=-\dfrac{\pi}{2}$

(b) $\dfrac{p-5qi}{1+i}=\overline{1-4i}\Rightarrow p-5qi=(1+4i)(1+i)$

$\therefore p-5qi=-3+5i$

$\therefore p=-3,\ -5q=5$ (Equating real and imaginary parts)

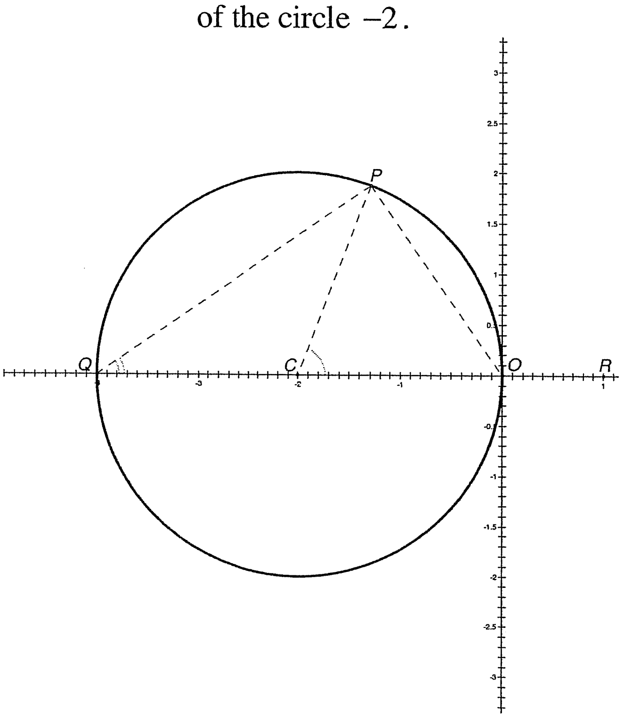
$\therefore p=-3,\ q=-1$

(c) (i) $u=\dfrac{7\sqrt{2}}{2}(1+i)=\dfrac{7\sqrt{2}}{2}\times\sqrt{2}\ \text{cis}\dfrac{\pi}{4}=7\ \text{cis}\dfrac{\pi}{4}$

(ii) $v=\dfrac{uv}{u}=\dfrac{42\text{cis}\frac{\pi}{20}}{7\ \text{cis}\frac{\pi}{4}}=6\ \text{cis}\left(\dfrac{\pi}{20}-\dfrac{\pi}{4}\right)=6\text{cis}\left(-\dfrac{\pi}{5}\right)$

$\therefore r=6,\ \theta=-\dfrac{\pi}{5}$

(d) (i) Let z be represented by the point P. Let Q represent the number -4 and C the centre of the circle -2.



Let $\theta=\arg z\Rightarrow\angle POR=\theta$

$\arg(z+2)=\angle PCO=\pi-2\times\angle POC$

$=\pi-2\times(\pi-\theta)$

$=2\theta-\pi$

(ii) $\arg(z^2+6z+8)=\arg[(z+2)(z+4)]$

$=\arg(z+2)+\arg(z+4)$

Now $\arg(z+4)=\angle PQC=\frac{1}{2}\angle PCO$

(angles at centre and circumference)

$\therefore\arg(z+4)=\frac{1}{2}(2\theta-\pi)=\theta-\frac{\pi}{2}$

$\therefore\arg(z^2+6z+4)=2\theta-\pi+\theta-\frac{\pi}{2}$

$=3\theta-\frac{3\pi}{2}$

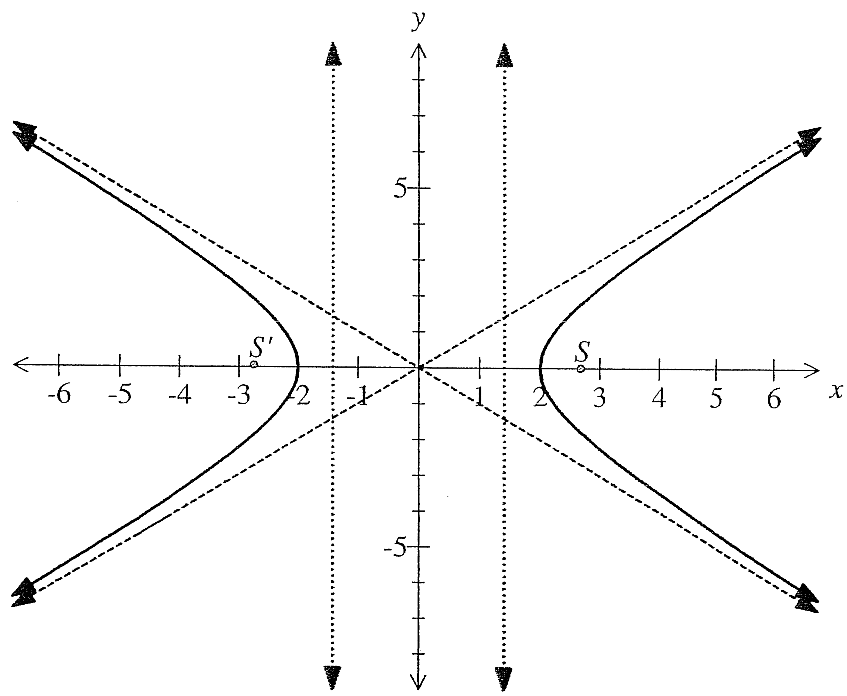
Question 3

(a) (i) $x^2 - y^2 = 4$; $e = \sqrt{2}$

Asymptotes are $y = \pm x$

The directrices are $x = \pm \frac{a}{e} = \pm \frac{2}{\sqrt{2}} = \pm\sqrt{2}$

The foci are at $(\pm ae, 0) = \left(\pm 2\sqrt{2}, 0\right)$ i.e. $S\left(2\sqrt{2}, 0\right)$ and $S'\left(-2\sqrt{2}, 0\right)$



(ii) $x^2 - y^2 = 4 \Rightarrow 2x - 2yy' = 0$

$\therefore y' = \frac{x}{y} \Rightarrow m = \frac{2\sec\theta}{2\tan\theta} = \frac{\sec\theta}{\tan\theta}$

$\therefore y - 2\tan\theta = \frac{\sec\theta}{\tan\theta}(x - 2\sec\theta) \Rightarrow y\tan\theta - 2\tan^2\theta = x\sec\theta - 2\sec^2\theta$

$\therefore x\sec\theta - y\tan\theta = 2\left(\sec^2\theta - \tan^2\theta\right)$

$\therefore x\sec\theta - y\tan\theta = 2$

(iii) $Q: y = 0 \Rightarrow x\sec\theta = 2$

$\therefore Q(2\cos\theta, 0) \Rightarrow M(\cos\theta + \sec\theta, \tan\theta)$

$$\begin{aligned} \text{LHS} &= x^2 - y^2 - 3 \\ &= (\cos\theta + \sec\theta)^2 - (\tan\theta)^2 - 3 \\ &= \cos^2\theta + 2 + \sec^2\theta - \tan^2\theta - 3 \\ &= \cos^2\theta + 2 + 1 - 3 \\ &= \cos^2\theta \end{aligned}$$

$$\begin{aligned} \text{RHS} &= \frac{1}{y^2 + 1} \\ &= \frac{1}{(\tan\theta)^2 + 1} = \frac{1}{\tan^2\theta + 1} \\ &= \frac{1}{\sec^2\theta} = \cos^2\theta \end{aligned}$$

$\therefore$ LHS = RHS

So the locus of M is $x^2 - y^2 - 3 = \frac{1}{y^2 + 1}$

Question 3 continued

(b) (i) $u(-1)=u'(-1)=0$ (Multiple Root Theorem)

$u'(x)=7mx^6+6nx^5 \Rightarrow u'(-1)=7m(-1)^6+6n(-1)^5=0$

$\therefore 7m-6n=0$

$\therefore 7m=6n$

(ii) $u(-1)=0 \Rightarrow m(-1)^7+n(-1)^6+1=0$

$\therefore -m+n+1=0 \Rightarrow n-m=-1$ $\quad -(*)$

From (i) $7m=6n$

(*) becomes $7n-7m=-7$ and so $7n-6n=-7 \Rightarrow n=-7$

$\therefore m=-6$ by substituting into (*) or the result in (*)

$m=-6, n=-7$

(c) (i) $x^3+3x+1=0$

Let $y=x^2$

$x^3+3x+1=0 \Rightarrow x(x^2+3)=-1$

$\therefore x^2(x^2+3)^2=1 \Rightarrow y(y+3)^2=1$

$\therefore y^3+6y^2+9y-1=0$

(ii) $\alpha^2, \beta^2, \gamma^2$ are the roots of $y^3+6y^2+9y-1=0$

$\therefore \alpha^2+\beta^2+\gamma^2=-6$ (sum of roots)

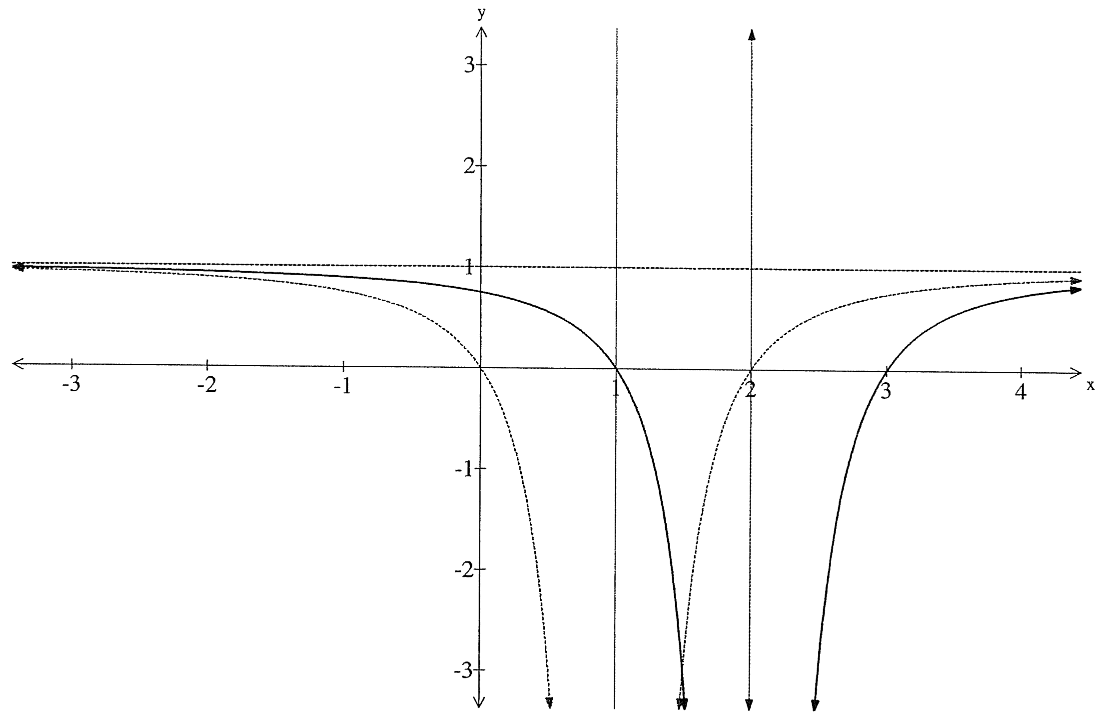
(d) $z^5+z^2-z+c=0$ has real coefficients and so all the roots occur in conjugate pairs.
Diagram B has a root that doesn't have it's conjugate pair showing

Answer: Diagram A

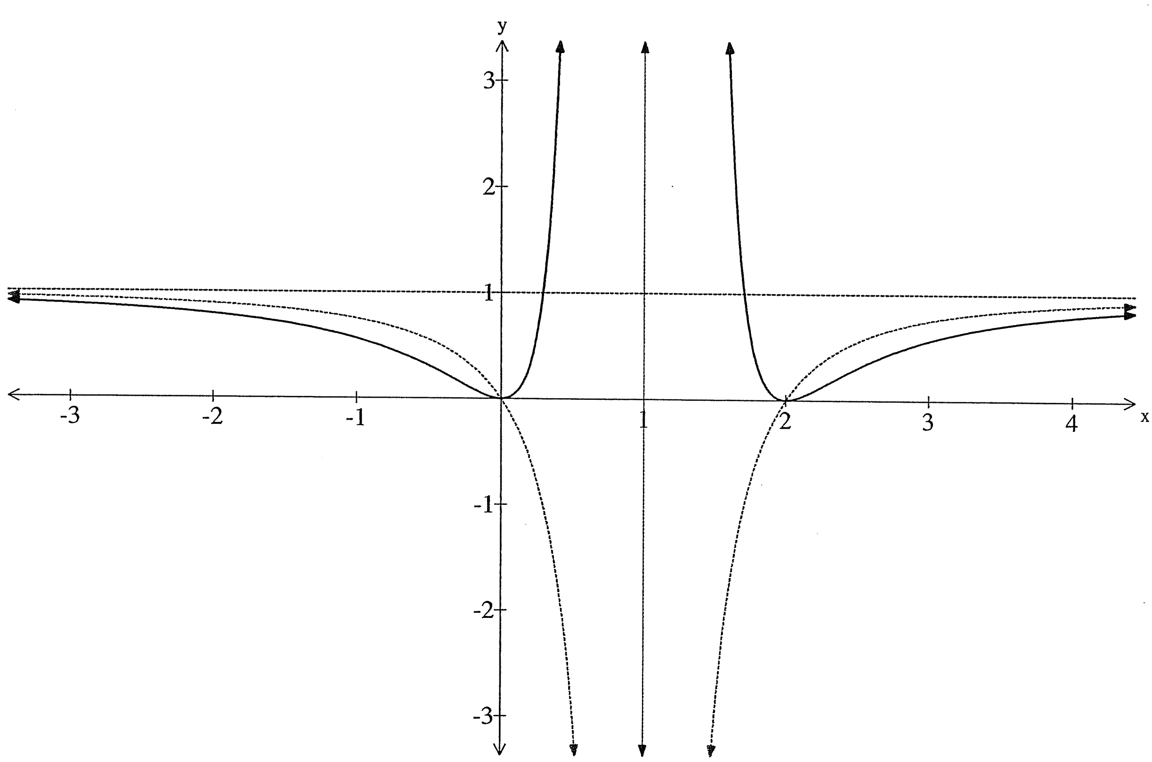
Question 4

(a) Dotted curve existing curve; black curve new transformation

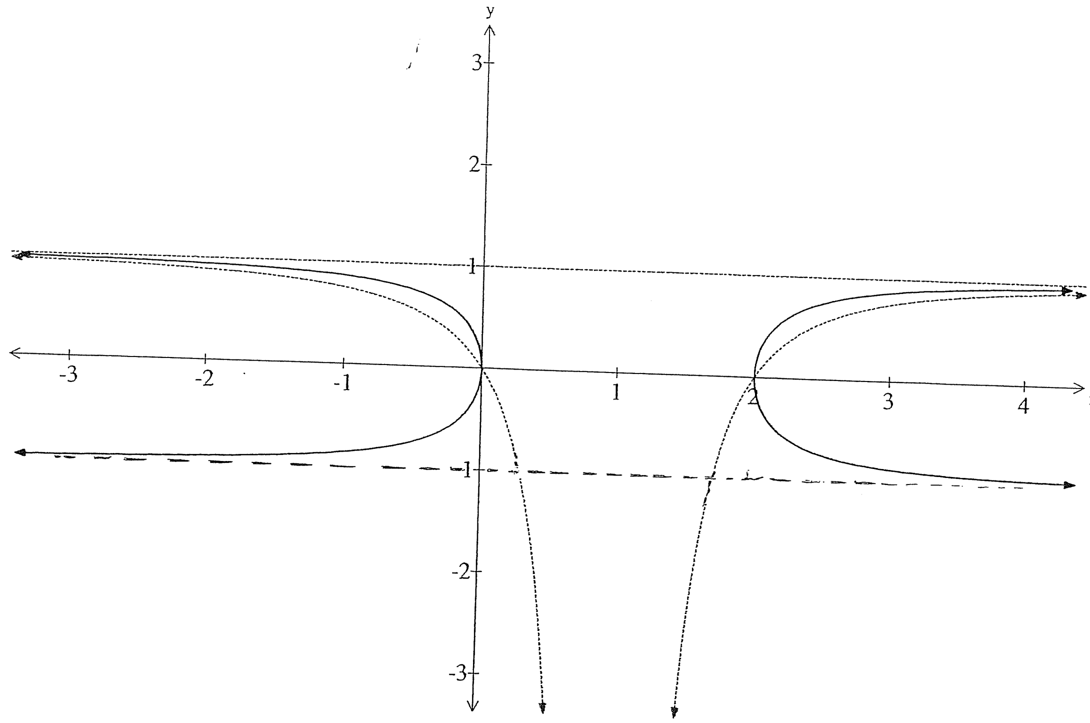
(i) $y = f(x-1)$



(ii) $y = [f(x)]^2$



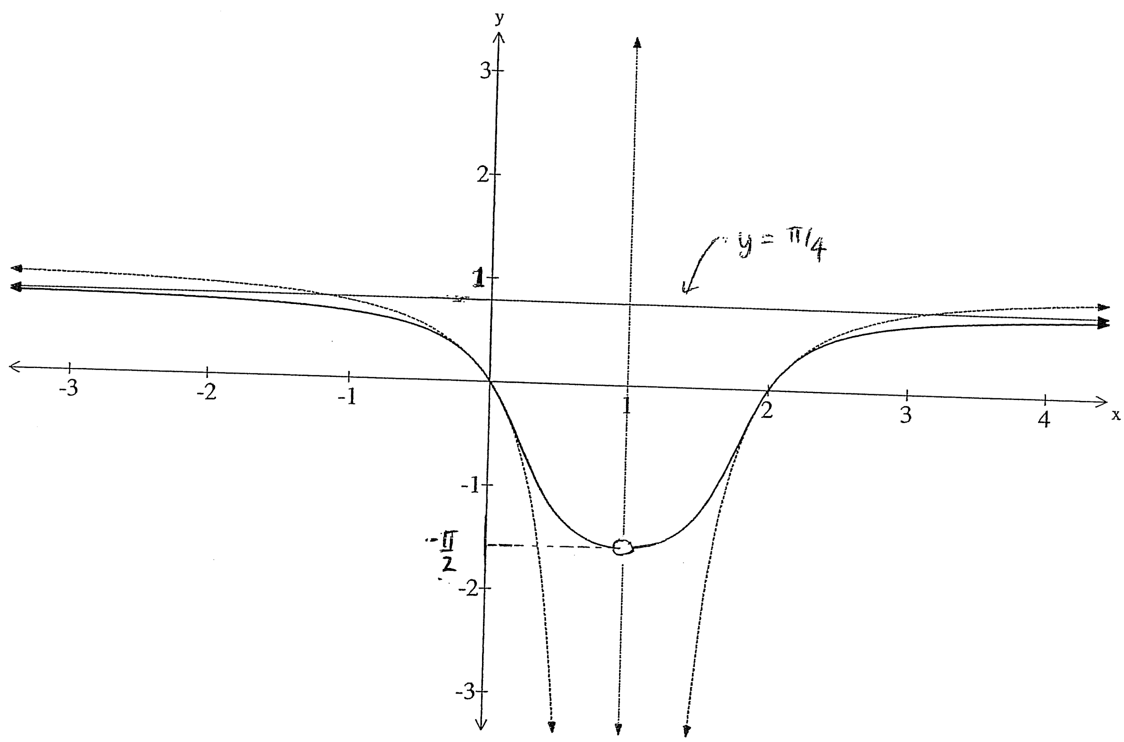
(iii) $y^2 = f(x)$



(iv) $y = \tan^{-1} f(x)$

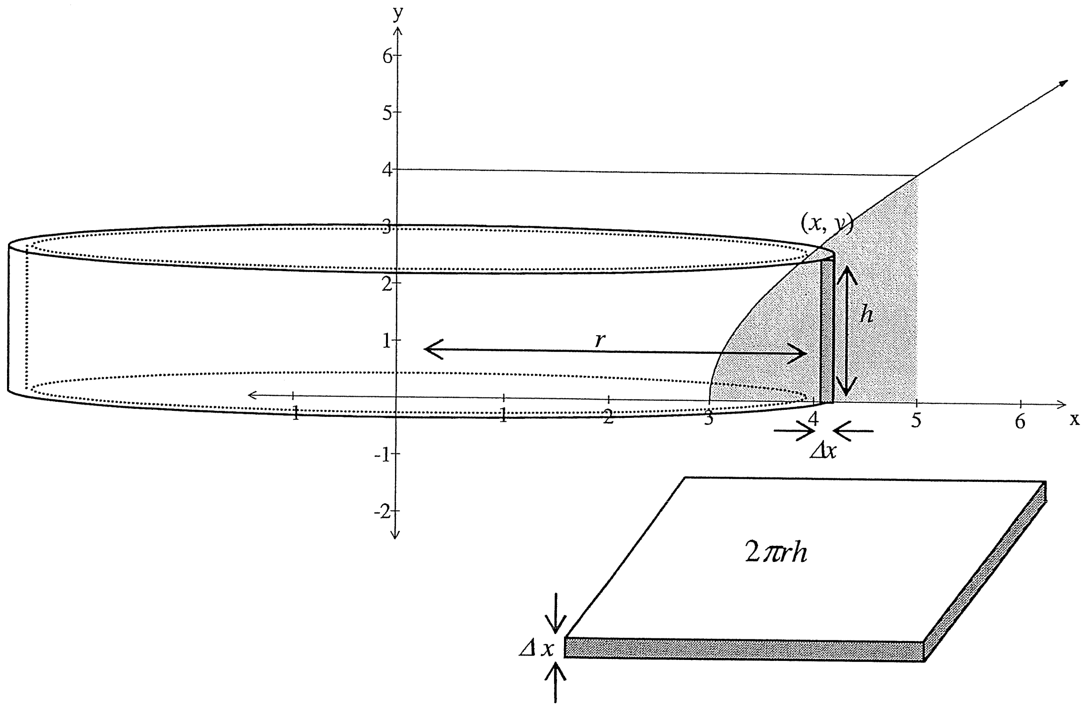
The new horizontal asymptote is $y = \dfrac{\pi}{4}$.

The curve is not defined at $x = 1$, but it isn't a vertical asymptote



Question 4 continued

(b) Cut the shell into what is approximately a rectangular prism of length $2\pi r$ and height h.



(i) $r = x, h = y \Rightarrow \Delta V \approx 2\pi xy$

$\therefore \Delta V \approx 2\pi x\left(x^2-9\right)^{\frac{1}{2}}$

$\therefore V = \lim_{\Delta x \to 0} \sum_{x=3}^{5} \Delta V = \int_3^5 2\pi x\left(x^2-9\right)^{\frac{1}{2}}\,dx$

(ii) $V\pi = \int_3^5 2x\left(x^2-9\right)^{\frac{1}{2}}\,dx$

$= \pi\left[\frac{2}{3}\left(x^2-9\right)^{\frac{3}{2}}\right]_3^5$

$= \frac{2\pi}{3}\left(16^{\frac{3}{2}}-0\right) = \frac{128\pi}{3}$ c.u.

Question 4 continued

(c) (i) $u = a - x \Rightarrow du = -dx$

$x = a - u$

$x = 0 \Rightarrow u = a;\ x = a \Rightarrow u = 0$

$$\int_0^a xf(x)\,dx = \int_a^0 (a-u)f(a-u)(-dx)$$

$$= \int_0^a (a-u)f(a-u)\,du$$

$$= \int_0^a (a-u)f(u)\,du$$

$$= \int_0^a a\,f(u)\,du - \int_0^a u\,f(u)\,du$$

$$= \int_0^a a\,f(x)\,dx - \int_0^a x\,f(x)\,dx$$

$$\therefore 2\int_0^a x\,f(x)\,dx = \int_0^a a\,f(x)\,dx$$

$$\int_0^a x\,f(x)\,dx = \frac{1}{2}\int_0^a a\,f(x)\,dx$$

$$= \frac{a}{2}\int_0^a f(x)\,dx$$

(ii) $$\int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx = \frac{\pi}{2}\int \frac{\sin x}{1+\cos^2 x}\,dx$$

$$= -\frac{\pi}{2}\int \frac{-\sin x}{1+\cos^2 x}\,dx$$

$$= -\frac{\pi}{2}\left[\tan^{-1}(\cos x)\right]_0^\pi$$

$$= -\frac{\pi}{2}\left[\tan^{-1}(-1) - \tan^{-1}(-1)\right]$$

$$= -\frac{\pi}{2}\left[-\frac{\pi}{4} - \frac{\pi}{4}\right]$$

$$= \frac{\pi^2}{4}$$

Question 5

(a) (i)

$$I_n = \int_0^1 x^n e^{2x}\,dx$$

$$= \int_0^1 x^n \frac{d}{dx}\left(\tfrac{1}{2}e^{2x}\right)dx$$

$$= \left[\tfrac{1}{2}e^{2x}x^n\right]_0^1 - \int \frac{e^{2x}}{2}\left(nx^{n-1}\right)dx$$

$$= \tfrac{1}{2}e^2 - \frac{n}{2}\int_0^1 x^{n-1}e^{2x}\,dx$$

$$= \tfrac{1}{2}e^2 - \frac{n}{2}I_{n-1} = \frac{1}{2}\left(e^2 - nI_{n-1}\right)$$

(ii)

$$\int_0^1 x^3 e^{2x}\,dx = I_3$$

$$= \tfrac{1}{2}\left(e^2 - 3I_2\right) = \tfrac{1}{2}e^2 - \tfrac{3}{2}I_2$$

$$= \tfrac{1}{2}e^2 - \tfrac{3}{2}\left[\tfrac{1}{2}\left(e^2 - 2I_1\right)\right]$$

$$= \tfrac{1}{2}e^2 - \tfrac{3}{4}e^2 + \tfrac{3}{2}I_1$$

$$= \tfrac{1}{2}e^2 - \tfrac{3}{4}e^2 + \tfrac{3}{2}\left[\tfrac{1}{2}\left(e^2 - I_0\right)\right]$$

$$= \tfrac{1}{2}e^2 - \tfrac{3}{4}e^2 + \tfrac{3}{4}e^2 - \tfrac{3}{4}I_0$$

$$= \tfrac{1}{2}e^2 - \tfrac{3}{4}I_0$$

$$= \tfrac{1}{2}e^2 - \tfrac{3}{4}\times\tfrac{1}{2}\left(e^2 - 1\right)$$

$$= \tfrac{1}{8}e^2 + \tfrac{3}{8}$$

$$I_0 = \int_0^1 e^{2x}\,dx$$

$$= \frac{1}{2}\left[e^{2x}\right]_0^1$$

$$= \frac{1}{2}\left(e^2 - 1\right)$$

Question 5 continued

(b) (i) Since the classes are "distinguishable" then there are $\binom{3}{1}$ ways of picking the class that has all the red heads. Then the remaining 2 students for that class need to be picked from the remaining 12 students in $\binom{12}{2}$ ways. Then $\binom{10}{5}$ ways to place 5 of the remaining girls in one of the other class, this leaves the last 5 students to be allocated to the remaining class.

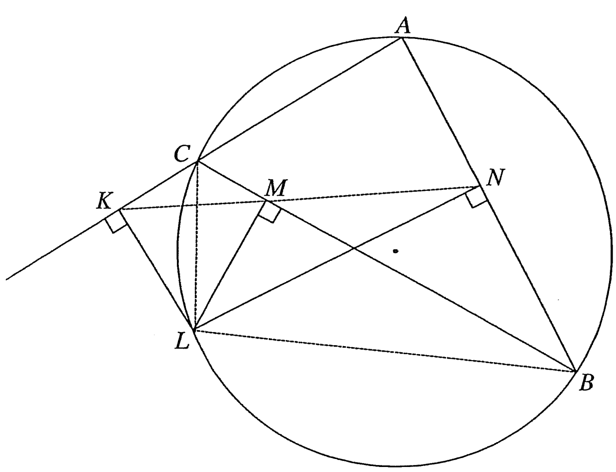
i.e. $\binom{3}{1}\times\binom{12}{2}\times\binom{10}{5} = 49\,896$ ways.

(ii) Miss V can be allocated one of the redheads in $\binom{3}{1}$ ways and her remaining students in $\binom{12}{4}$ ways. Mr S can be allocated his redhead in $\binom{2}{1}$ ways and the remaining students in $\binom{8}{4}$ ways. The remaining students all go to Ms L's class. i.e.

$\binom{3}{1}\times\binom{12}{4}\times\binom{2}{1}\times\binom{8}{4} = 207\,900$ ways.

Without restriction all the students can be allocated in $\binom{15}{5}\times\binom{10}{5}$ ways i.e. in 756 756 ways. So the probability of this happening is $\frac{207\,900}{756\,756} = \frac{25}{91}$.

(c) (i) Construct KM, MN, LB and CL.



(ii) In $CKLM$, $\angle CKL = \angle CML = 90°$.

Opposite angles are supplementary and so the quadrilateral is cyclic.

In $MNBL$, $\angle LMB = \angle LNB = 90°$.

By the converse of angles in the same segment, the quadrilateral is cyclic.

(iii) $\angle KCL$ is the exterior angle of quadrilateral $ACLB$ and so $\angle KCL = \angle ABL$ by the exterior angle theorem for cyclic quadrilaterals.

(iv) As $CKLM$ is a cyclic quadrilateral, $\angle KCL = \angle KML$ (angles in same segment)
As $MNBL$ is a cyclic quadrilateral, $\angle LMN = 180° - \angle ABL$ (opposite angles supp.)
From (iii) $\angle KCL = \angle ABL$ and so $\angle LMN = 180° - \angle KML$
$\therefore \angle KMN = \angle KML + \angle LMN = 180°$
So K, M, and N are collinear.

Question 6

(a) $\sin\left(\sin^{-1}x - \cos^{-1}x\right) = \sin\left[\sin^{-1}(3x-2)\right]$

$\therefore \sin\left(2\sin^{-1}x - \frac{\pi}{2}\right) = 3x-2$ $\qquad \left[\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}\right]$

$\therefore -\sin\left(\frac{\pi}{2} - 2\sin^{-1}x\right) = 3x-2$ $\qquad \left[\sin(-x) = -\sin x\right]$

$\therefore -\cos\left(2\sin^{-1}x\right) = 3x-2$ $\qquad \left[\sin\left(\frac{\pi}{2}-x\right) = \cos x\right]$

$\therefore -\left[1-2\sin^2\left(\sin^{-1}x\right)\right] = 3x-2$ $\qquad \left[\text{Let } \alpha = \sin^{-1}x;\ \cos 2\alpha = 1-2\sin^2\alpha\right]$

$\therefore 2x^2 - 1 = 3x-2$ $\qquad \left[\sin^2\left(\sin^{-1}x\right) = \left[\sin\left(\sin^{-1}x\right)\right]^2\right]$

$\therefore 2x^2 - 3x + 1 = 0$

$\therefore (2x-1)(x-1) = 0$

$\therefore x = \frac{1}{2}, 1$

Now test the solutions in the original equation i.e. $\sin^{-1}x - \cos^{-1}x = \sin^{-1}(3x-2)$

$x = \frac{1}{2}$: LHS $= \sin^{-1}\frac{1}{2} - \cos^{-1}\frac{1}{2} = \frac{\pi}{6} - \frac{\pi}{3} = -\frac{\pi}{6}$

RHS $= \sin^{-1}\left(3\times\frac{1}{2} - 2\right) = \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$

$x = 1$: LHS $= \sin^{-1}1 - \cos^{-1}1 = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

RHS $= \sin^{-1}(3\times 1 - 2) = \sin^{-1}(1) = \frac{\pi}{2}$

$\therefore x = \frac{1}{2}, 1$

(b) $\sin\left(\frac{1}{2}y\right) = \frac{1}{2}\left(x^2-2\right)$

$\therefore \frac{1}{2}y'\cos\left(\frac{1}{2}y\right) = x$

$$y' = \frac{2x}{\cos\left(\frac{1}{2}y\right)} = \frac{2x}{\sqrt{1-\sin^2\left(\frac{1}{2}y\right)}}$$

$$= \frac{2x}{\sqrt{1-\left[\frac{1}{2}\left(x^2-2\right)\right]^2}} = \frac{4x}{\sqrt{4-\left(x^2-2\right)^2}}$$

$$= \frac{4x}{\sqrt{4-\left(x^2-2\right)^2}}$$

$$= \frac{4x}{\sqrt{x^2\left(4-x^2\right)}} = \frac{4x}{x\sqrt{4-x^2}} = \frac{4}{\sqrt{4-x^2}}$$

ALTERNATIVELY

$$\tfrac{1}{2}y = \sin^{-1}\left(\frac{x^2}{2} - 1\right)$$

$$\therefore \frac{d}{dx}\left(\tfrac{1}{2}y\right) = \frac{d}{dx}\sin^{-1}\left(\frac{x^2}{2} - 1\right)$$

$$\therefore \tfrac{1}{2}y' = \frac{1}{\sqrt{1-\left(\frac{x^2}{2}-1\right)^2}} \times x = \frac{x}{\sqrt{1-\left(\frac{x^4}{4} - x^2 + 1\right)}}$$

$$\therefore y' = \frac{4x}{\sqrt{4x^2 - x^4}} = \frac{4x}{x\sqrt{4x - x^2}} = \frac{4}{\sqrt{4x - x^2}}$$

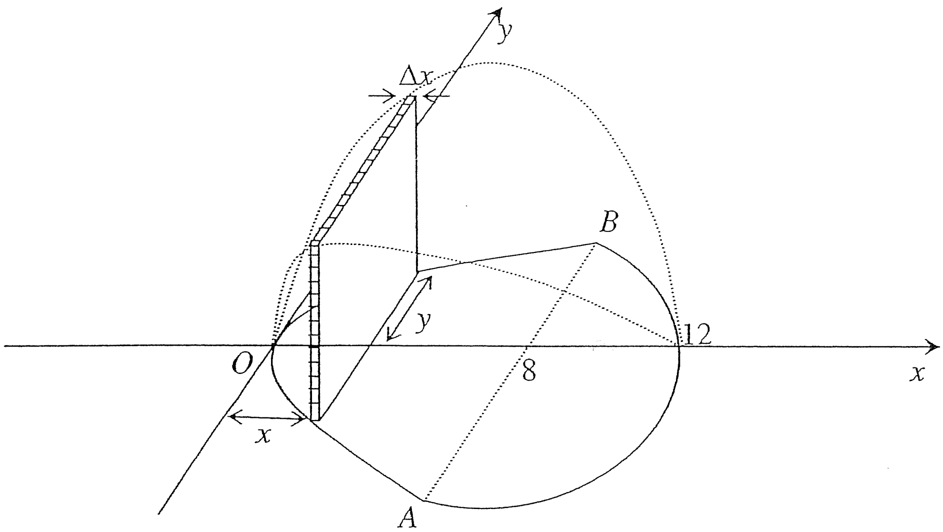
Question 6 continued

(c) (i) $x = 8 \Rightarrow y^2 = 2 \times 8 \Rightarrow y = \pm 4$

$\therefore B(8, 4)$

$\therefore AB = 2 \times 4 = 8$

So the semi-circle has radius 4 and so the extreme x-value is $x = 12$



(ii) The square has side length $2y \Rightarrow \Delta V \approx (2y)^2 \, \Delta x$

$\therefore \Delta V \approx 4y^2 \, \Delta x = 4(2x)\Delta x = 8x \, \Delta x$

(iii) For $8 < x < 12$, the base is $(x-8)^2 + y^2 = 16$

$\therefore \Delta V \approx 4y^2 \, \Delta x = 4\left[16 - (x-8)^2\right]\Delta x = \left[64 - 4(x-8)^2\right]\Delta x$

$$V = \int_0^8 8x \, dx + \int_8^{12} \left[64 - 4(x-8)^2\right] dx$$

$$= \left[4x^2\right]_0^8 + \left[64x - \frac{4}{3}(x-8)^3\right]_8^{12}$$

$$= 256 + \left[\left(768 - \frac{4}{3} \times 64\right) - (512 - 0)\right]$$

$$= 426\tfrac{2}{3}$$

ALTERNATIVELY

With the semi-circular section

$$V_{\text{semi-circular}} = 4\int_8^{12} \left[16 - (x-8)^2\right] dx$$

$$= 4\int_0^4 \left[16 - x^2\right] dx$$

$$= 4\left[16x - \frac{x^3}{3}\right]_0^4$$

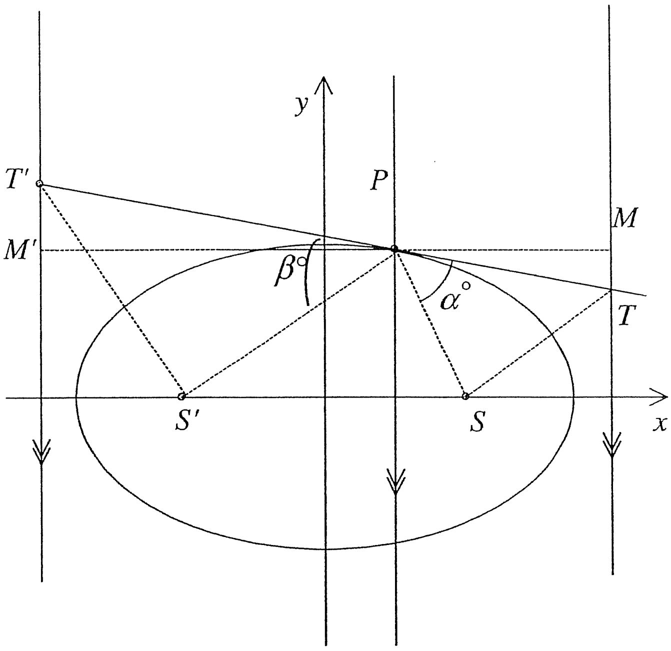
$$= 4\left(64 - \frac{64}{3}\right)$$

$$= 4\left(\tfrac{2}{3} \times 64\right) = 170\tfrac{2}{3}$$

Question 7

(a) For a conic $SP = ePM$

$$SP + S'P = ePM + ePM'$$
$$= e(PM + PM')$$
$$= e\left(2\times\frac{a}{e}\right)$$
$$= 2a$$



(b) (i) $T:\ x = \frac{a}{e} \Rightarrow \left(\frac{a}{e}\right)b\cos\theta + ya\sin\theta = ab$

$$\therefore \frac{b\cos\theta}{e} + y\sin\theta = b$$

$$\therefore y\sin\theta = b - \frac{b\cos\theta}{e} = \frac{b(e-\cos\theta)}{e}$$

$$\therefore y = \frac{b(e-\cos\theta)}{e\sin\theta} \Rightarrow T\left(\frac{a}{e}, \frac{b(e-\cos\theta)}{e\sin\theta}\right)$$

(ii) $m_{SP} = \dfrac{b\sin\theta - 0}{a\cos\theta - ae} = \dfrac{b\sin\theta}{a(\cos\theta - e)}$

$$m_{ST} = \frac{\frac{b(e-\cos\theta)}{e\sin\theta} - 0}{\frac{a}{e} - ae} \times \frac{e\sin\theta}{e\sin\theta} = \frac{b(e-\cos\theta)}{a\sin\theta(1-e^2)}$$

$$= \frac{b(e-\cos\theta)}{a\sin\theta\times\frac{b^2}{a^2}} \qquad \left[e^2 = 1 - \frac{b^2}{a^2}\right]$$

$$= \frac{a(e-\cos\theta)}{b\sin\theta} = -\frac{a(\cos\theta - e)}{b\sin\theta}$$

$\therefore m_{ST}\times m_{SP} = -1 \Rightarrow \angle PST = 90°$

Similarly $\angle PS'T' = 90°$

Question 7 continued

(iii) $PM : PM' = PT : PT'$ (parallel lines preserve ratio)

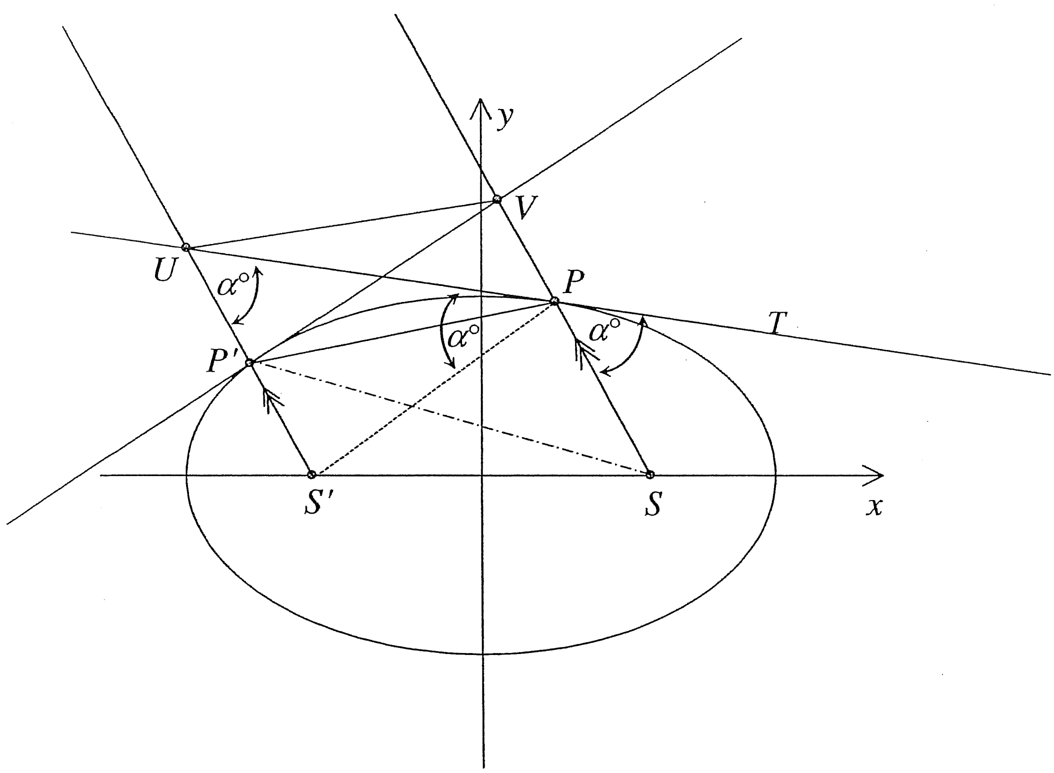
$$\therefore \frac{PM}{PM'} = \frac{PT}{PT'} \Rightarrow \frac{PM}{PT} = \frac{PM'}{PT'}$$

(iv) Using (ii) $\cos\alpha = \frac{SP}{PT} = \frac{ePM}{PT} = e\frac{PM}{PT}$ and similarly $\cos\beta = e\frac{PM'}{PT'}$

$\therefore \cos\alpha = \cos\beta$

$\therefore \alpha = \beta$ $[\because 0 \le \alpha, \beta \le 90°]$

(c) (i)



(ii) From (b) $\angle UPS' = \angle SPT = \alpha$
$\angle S'UP = \alpha$ (Corresponding angles are equal on parallel lines, $SV \parallel S'U$)
$\therefore \Delta UPS'$ is isosceles.

(iii) Similarly $\Delta SP'V$ is isosceles
$VP = VS - SP$
$= SP' - SP$ $[\Delta SP'V$ isosceles$]$
$= SP' - (2a - S'P)$ [From (a)]
$= S'P - (2a - SP')$
$= US' - (2a - SP')$ $[\Delta UPS'$ isosceles$]$
$= US' - S'P'$ [From (a) but with $S'P' + SP' = 2a$]
$= UP'$

(iv) $VP = UP'; VP \parallel UP' \Rightarrow UVPP'$ is a parallelogram
$\therefore UV \parallel PP'$ (opposite sides of a parallelogram are parallel)

Question 8

(a) (i) $$\tan\Big[\underbrace{\tan^{-1}(n+1)}_{\alpha}-\underbrace{\tan^{-1}(n-1)}_{\beta}\Big]=\tan(\alpha-\beta)$$

$$=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

$$=\frac{(n+1)-(n-1)}{1+(n+1)(n-1)}$$

$$=\frac{2}{1+n^2-1}=\frac{2}{n^2}$$

$$\therefore \tan^{-1}(n+1)-\tan^{-1}(n-1)=\tan^{-1}\left(\frac{2}{n^2}\right)$$

(ii) $$x>1\Rightarrow\frac{\pi}{4}<\tan^{-1}x<\frac{\pi}{2}$$

$$\therefore x>1,\ y>1\Rightarrow\frac{\pi}{4}+\frac{\pi}{4}<\tan^{-1}x+\tan^{-1}y<\frac{\pi}{2}+\frac{\pi}{2}$$

$\therefore \frac{\pi}{2}<\tan^{-1}x+\tan^{-1}y<\pi$ i.e. $\tan^{-1}x+\tan^{-1}y$ lies in the second quadrant.

BUT $x>1,\ y>1$ $\frac{x+y}{1-xy}<0$ and so $-\frac{\pi}{2}<\tan^{-1}\left(\frac{x+y}{1-xy}\right)<0$

$$\therefore -\frac{\pi}{2}+\pi<\pi+\tan^{-1}\left(\frac{x+y}{1-xy}\right)<0+\pi\Rightarrow\frac{\pi}{2}<\pi+\tan^{-1}\left(\frac{x+y}{1-xy}\right)<\pi$$

So $\tan^{-1}x+\tan^{-1}y=\pi+\tan^{-1}\left(\frac{x+y}{1-xy}\right)$

(iii) $$\sum_{r=1}^{n}\tan^{-1}\left(\frac{2}{r^2}\right)=\sum_{r=1}^{n}\left[\tan^{-1}(r+1)-\tan^{-1}(r-1)\right]$$

$$=\Big[\underbrace{\tan^{-1}(2)-\tan^{-1}(0)}_{r=1}\Big]+\Big[\underbrace{\tan^{-1}(3)-\tan^{-1}(1)}_{r=2}\Big]+\Big[\underbrace{\tan^{-1}(4)-\tan^{-1}(2)}_{r=3}\Big]+\ldots$$

$$+\ldots+\Big[\underbrace{\tan^{-1}(n-1)-\tan^{-1}(n-3)}_{r=n-2}\Big]+\Big[\underbrace{\tan^{-1}(n)-\tan^{-1}(n-2)}_{r=n-1}\Big]$$

$$+\Big[\underbrace{\tan^{-1}(n+1)-\tan^{-1}(n-1)}_{r=n}\Big]$$

$$=\tan^{-1}(n+1)+\tan^{-1}(n)-\tan^{-1}(1)$$

$$=\pi+\tan^{-1}\left(\frac{2n+1}{1-n-n^2}\right)-\frac{\pi}{4}$$

$$=\frac{3\pi}{4}+\tan^{-1}\left(\frac{2n+1}{1-n-n^2}\right)$$

Question 8 continued

(iv) $$\sum_{r=1}^{\infty}\tan^{-1}\left(\frac{2}{r^2}\right)=\lim_{n\to\infty}\sum_{r=1}^{n}\tan^{-1}\left(\frac{2}{r^2}\right)$$

$$=\lim_{n\to\infty}\left[\frac{3\pi}{4}+\tan^{-1}\left(\frac{2n+1}{1-n-n^2}\right)\right]$$

$$=\frac{3\pi}{4}+\lim_{n\to\infty}\left[\tan^{-1}\left(\frac{2n+1}{1-n-n^2}\right)\right]$$

$$=\frac{3\pi}{4} \qquad \left[\because \lim_{n\to\infty}\frac{2n+1}{1-n-n^2}=0\right]$$

(b) (i) $$\text{LHS}=S_k(x)-S_k(x+1)$$

$$=\frac{k!}{x(x+1)(x+2)....(x+k)}-\frac{k!}{(x+1)(x+2)(x+3)....(x+k+1)}$$

$$=\frac{k!(x+k+1)-xk!}{x(x+1)(x+2)(x+3)....(x+k+1)}$$

$$=\frac{k![(x+k+1)-x]}{x(x+1)(x+2)(x+3)....(x+k+1)}$$

$$=\frac{k!(k+1)}{x(x+1)(x+2)(x+3)....(x+k+1)}$$

$$=\frac{(k+1)!}{x(x+1)(x+2)(x+3)....(x+k+1)}$$

$$=S_{k+1}(x)$$

(ii) Test $n=1$

$$\text{LHS}=T_1(x)=\frac{{}^1C_0}{x}-\frac{{}^1C_1}{x+1}=\frac{1}{x}-\frac{1}{x+1}=\frac{1}{x(x+1)}$$

$$\text{RHS}=\frac{1!}{x(x+1)}=\frac{1}{x(x+1)}$$

So true for $n=1$

Assume true for some integer $n=k$ i.e. $T_k(x)=\dfrac{k!}{x(x+1)(x+2)....(x+k)}$

Need to prove it is true for $n=k+1$ i.e. $T_{k+1}(x)=\dfrac{(k+1)!}{x(x+1)(x+2)....(x+k+1)}$

Question 8 continued

$$\text{LHS} = T_{k+1}(x)$$

$$= \frac{^{k+1}C_0}{x} - \frac{^{k+1}C_1}{x+1} + \frac{^{k+1}C_2}{x+2} - \quad .... \quad + (-1)^k \frac{^{k+1}C_k}{x+k} + (-1)^{k+1} \frac{^{k+1}C_{k+1}}{x+k+1}$$

$$= \frac{^kC_0}{x} - \frac{\left(^kC_1 + {}^kC_0\right)}{x+1} + \frac{\left(^kC_2 + {}^kC_1\right)}{x+2} - \quad .... \quad + (-1)^k \frac{\left(^kC_k + {}^kC_{k-1}\right)}{x+k} + (-1)^{k+1} \frac{^kC_k}{(x+1)+k}$$

$$= \frac{^kC_0}{x} - \frac{^kC_1}{x+1} + \frac{^kC_2}{x+2} - \quad .... \quad + (-1)^k \frac{^kC_k}{x+k}$$
$$- \left[\frac{^kC_0}{x+1} - \frac{^kC_1}{x+2} + ... + (-1)^{k-1} \frac{^kC_{k-1}}{x+k} + (-1)^k \frac{^kC_k}{x+1+k} \right]$$

$$= T_k(x) - T_k(x+1)$$

$$= S_k(x) - S_k(x+1) \qquad \text{[By assumption]}$$

$$= S_{k+1}(x) \qquad \text{[From (i)]}$$

So the formula is true for $n = k+1$ when the formula is true for $n = k$.

So by the principle of mathematical induction the formula is true for $n \geq 1, n \in \mathbb{Z}$

(iii) Sub $x = \frac{1}{2}$ into both sides of

$$\frac{^nC_0}{x} - \frac{^nC_1}{x+1} + \frac{^nC_2}{x+2} - \quad .... \quad + (-1)^n \frac{^nC_n}{x+n} = \frac{k!}{x(x+1)(x+2)....(x+k)}$$

$$\therefore \frac{^nC_0}{\frac{1}{2}} - \frac{^nC_1}{\frac{1}{2}+1} + \frac{^nC_2}{\frac{1}{2}+2} - \quad .... \quad + (-1)^n \frac{^nC_n}{\frac{1}{2}+n} = \frac{n!}{\frac{1}{2}\left(\frac{1}{2}+1\right)\left(\frac{1}{2}+2\right)....\left(\frac{1}{2}+n\right)}$$

$$\therefore 2\left(\frac{^nC_0}{1} - \frac{^nC_1}{3} + \frac{^nC_2}{5} - \quad .... \quad + (-1)^n \frac{^nC_n}{2n+1}\right) = \frac{n!}{\underbrace{\frac{1}{2}\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)....\left(\frac{2n+1}{2}\right)}_{n+1 \text{ terms}}}$$

$$\therefore 2\left(\frac{^nC_0}{1} - \frac{^nC_1}{3} + \frac{^nC_2}{5} - \quad .... \quad + (-1)^n \frac{^nC_n}{2n+1}\right) = \frac{2^{n+1}n!}{1\times3\times5\times...\times(2n+1)}$$

$$\frac{^nC_0}{1} - \frac{^nC_1}{3} + \frac{^nC_2}{5} - \quad .... \quad + (-1)^n \frac{^nC_n}{2n+1} = \frac{2^n n!}{1\times3\times5\times...\times(2n+1)}$$

End of solutions