Name: .... **Maths Class:** ....

SYDNEY TECHNICAL HIGH SCHOOL

YEAR 12 HSC COURSE

Extension 2 Mathematics

TRIAL HIGHER SCHOOL CERTIFICATE

August 2009

TIME ALLOWED: 120 minutes

READING TIME: 5 minutes

Instructions:

- Write your name and class at the top of this page, and on all your answer sheets.
- Hand in your answers attached to the rear of this question sheet.
- All necessary working must be shown. Marks may not be awarded for careless or badly arranged work.
- All questions are of equal value.
- Marks indicated within each question are a guide only and may be varied at the time of marking
- START ALL QUESTIONS ON A NEW PAGE
- Approved calculators may be used.
- A table of *Standard Integrals* is attached. You may detach this page now.

(FOR MARKERS USE ONLY)

1	2	3	4	5	6	7	8	TOTAL
/15	/15	/15	/15	/15	/15	/15	/15	/120

QUESTION 1:

Marks

(a) Find

2 (i) $\int \cos^3 x \, dx$

2 (ii) $\int \frac{dx}{x^2-4x+8}$

2 (iii) $\int_1^5 \frac{dx}{(2x-1)\sqrt{2x-1}}$

4 (b) Prove that $\sec x = \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}$

and hence show that $\int_0^{\frac{\pi}{4}} \sec x \, dx = \ln(\sqrt{2}+1)$

5 (c) (i) Find values of A, B and C so that

$$\frac{5}{(x^2+4)(x+1)} = \frac{Ax+B}{x^2+4} + \frac{C}{x+1}$$

(ii) Hence find $\int \frac{5\, dx}{(x^2+4)(x+1)}$

QUESTION 2: (Start a new page)

Marks

6 (a) If $z = 1 - i$, find

(i) $\bar{z}$ (ii) $|z|$ (iii) $arg\ z$ (iv) $arg\ iz$ (v) z^6 (in simplest form)

2 (b) (i) Sketch the region where the inequalities

$$|z-2| \le |z-2i| \text{ and } |z-1-2i| \le 1$$

hold simultaneously.

3 (ii) P is a point on the boundary of the region in part (i) above, and is represented by the complex number z, where $arg\, z = \frac{\pi}{4}$.

Find the 2 possibilities for z (in the form $a+ib$).

4 (c) A plane curve is defined by the equation

$$x^2 + 2xy + y^5 = 4$$

The curve has a horizontal tangent at the point $P(X, Y)$.

By using implicit differentiation, or otherwise, show that X is the unique solution to

$$X^5 + X^2 + 4 = 0$$

QUESTION 3: (Start a new page)

Marks

2 (a) (i) Without using calculus, sketch the curve $y = (x+1)^2(1-x)$

2 (ii) On a separate diagram from above, but using the same scale on the axes, *and also without calculus*, sketch the curve

$$y^2 = (x+1)^2(1-x)$$

In your answer, pay close attention to the shape of the curve as y approaches zero.

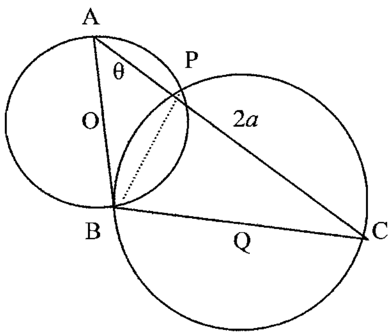
6 (b) Sketch each of the following curves on separate axes for $0 \leq x \leq 2\pi$

(i) $y = sin^2x$ (ii) $y = |sinx|$

(iii) $y = \sqrt{sin^2x}$ (iv) $y = \dfrac{1}{sinx}$

(v) $y = \dfrac{|sinx|}{sinx}$ (vi) $y = e^{sinx}$

(c) The hypotenuse AC of a right-angled triangle ABC has a length of $2a$ units and makes an angle of θ with one of the shorter sides, as shown below.



Circles are drawn using the two shorter sides as diameters, intersecting at points B and P. For this diagram, P is NOT on the side AC.

O and Q are the centres of the circles.

(i) Redraw the diagram in your answer book. (No marks)

2 (ii) Prove that the point P lies on AC (you may initially assume that it doesn't)

3 (iii) Show that the length of PB is $asin2\theta$

QUESTION 4: (Start a new page)

Marks

2 (a) Show that $\int_0^{\frac{\pi}{4}} tan\theta d\theta = \frac{1}{2}\ln 2$

2 (b) (i) Prove that, for any complex numbers z_1 and z_2

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2$$

3 (ii) Hence, using the method of Mathematical Induction, prove that

$$\arg(z_1 z_2 \dots \dots . z_n) = \arg z_1 + \arg z_2 + \dots \dots \dots + \arg z_n$$

(c) A cubic polynomial is given by $P(x) = x^3 + ax + b$

where a and b are constants.

It is given that the polynomial equation $P(x) = 0$ has three roots, α, β, and γ

1 (i) Find the value of $\alpha + \beta + \gamma$

2 (ii) Show that $\alpha^2 + \beta^2 + \gamma^2 = -2a$

3 (iii) If the polynomial has a double root, show that this double root is $\dfrac{-3b}{2a}$

2 (iv) If the polynomial has 3 <u>distinct</u> roots, show that $4a^3 + 27b^2 < 0$

QUESTION 5: (Start a new page)

Marks

5 (a) Given the hyperbola $16x^2 - 9y^2 = 144$, find

(i) the length of the major axis

(ii) the eccentricity

(iii) the co-ordinates of the foci

(iv) the equations of the directrices

(v) the equations of the asymptotes

(b) The parametric co-ordinates of a point P on the curve $y^2 = x^3$ are $x = t^2$ and $y = t^3$

2 (i) Show that the equation of the tangent to this curve at P is

$$t^3 - 3tx + 2y = 0$$

1 (ii) Explain why there can be no more than 3 distinct tangents to $y^2 = x^3$ drawn from any remote point (x_1, y_1), which is not on the curve.

2 (iii) Show that if the tangents to the curve at the points on it having parameters t_1, t_2 and t_3 all pass through the remote point(x_1, y_1), then

$$t_1^2 + t_2^2 + t_3^2 = 6x_1$$

5 (c) The area under the curve $y = x^2$, above the x-axis and between the lines $x = 1$ and $x = 2$, is rotated through 2π radians about the line $x = 2$.

Using the method of cylindrical shells, show that the volume of the solid so formed is $\frac{11\pi}{6}$ cubic units.

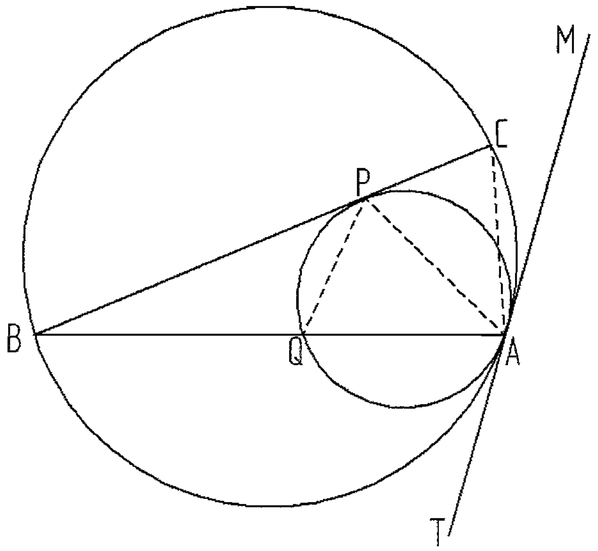
QUESTION 6: (Start a new page)

Marks

(a) Two circles touch internally at a point A and have a common tangent TAM as shown below.

A tangent to the inner circle through a point P (which is not the centre of either circle) meets the outer circle at B and C.

AB cuts the inner circle at Q.



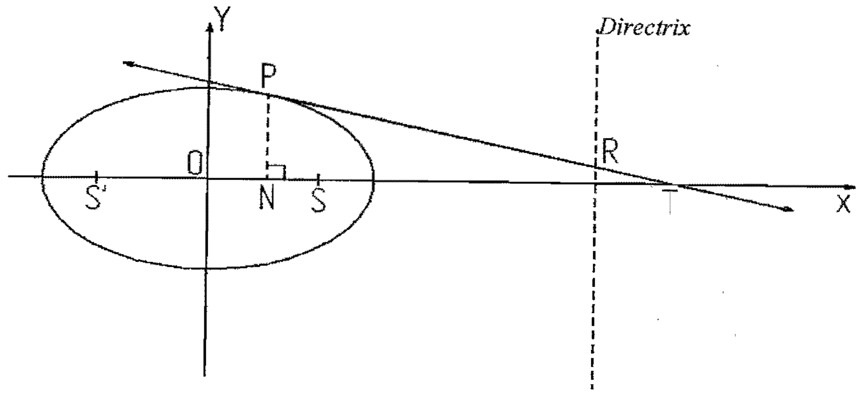
(i) Redraw the diagram neatly onto your answer page *(no marks)*.

5 (ii) Giving all appropriate reasons, prove that AP bisects the angle BAC.

QUESTION 6 continues over the page.....)

QUESTION 6 continued.......)

(b) P$(a\cos\theta, b\sin\theta)$ is any point on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$



The tangent at P cuts the major axis of the ellipse at T and the Directrix at R, while N is the foot of the perpendicular from P to the x-axis.

O is the centre of the ellipse, while S and S' are the foci.

3 (i) Show that the equation of the tangent at P is $\dfrac{x\cos\theta}{a} + \dfrac{y\sin\theta}{b} = 1$
(Show all working)

2 (ii) Show that ON.OT = a^2

5 (iii) Showing all steps carefully, prove that PR subtends a right angle at S.

QUESTION 7: (Start a new page)

Marks

3 (a) Using the substitution $x = a\tan\theta$, or otherwise, find $\int \frac{dx}{(a^2+x^2)^{\frac{3}{2}}}$

(b) You are given the complex polynomial $P(z) = z^5 - 1$

The roots of $P(z) = 0$ are $1, \omega_1, \omega_2, \omega_3, \omega_4$ which are in cyclic order around the unit circle.

3 (i) Prove the following:

(α) $\omega_1 = \overline{\omega_4}$ and $\omega_2 = \overline{\omega_3}$

(β) $\omega_1 + \omega_2 + \omega_3 + \omega_4 = -1$

(γ) $\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$

2 (ii) Using the sum of the products of the roots taken in pairs, or otherwise, show that

$$4\cos\frac{2\pi}{5}\cos\frac{4\pi}{5} + 1 = 0$$

1 (iii) Deduce that $\cos\frac{2\pi}{5}$ and $\cos\frac{4\pi}{5}$ are solutions to $4x^2 + 2x - 1 = 0$

4 (c) (i) If $I_n = \int_0^{\frac{\pi}{4}} \sec^n\theta\, d\theta$,

show that $(n-1)I_n - (n-2)I_{n-2} = (\sqrt{2})^{n-2}$, for $n \geq 2$

2 (ii) Using part (i) above, evaluate $\int_0^{\frac{\pi}{4}} \sec^4\theta\, d\theta$

QUESTION 8: (Start a new page)

Marks

(a) In the right triangular prism shown,

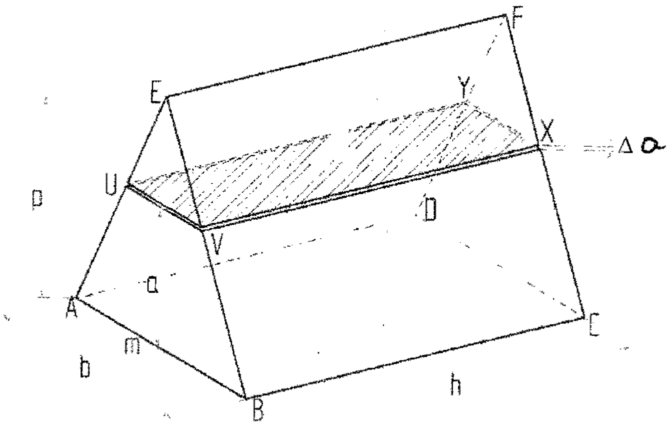
AB=DC= b units

AE=BE=DF=CF

M is the midpoint of AB

EM $= p$ units

BC=AD=EF= h units



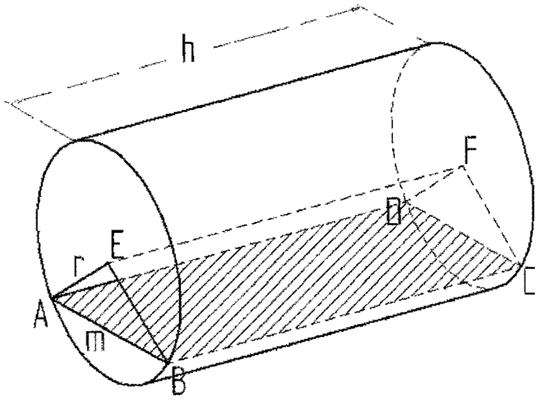
A "slice" UVXY of thickness Δa is taken a units above the base ABCD and parallel to it.

3 (i) Show that the volume of the rectangular slice is given by

$$\Delta V = \left(\frac{p-a}{p}\right) bh\Delta a$$

2 (ii) Hence, show that the volume of the triangular prism is given by $V = \frac{1}{2}pbh$

4 (iii) The triangular prism above is fitted into a right circular cylinder, of base radius r units and height h units, as shown below, where the points E and F are the centres of the circular bases.



Taking the angle AEB as $\frac{2\pi}{n}$, verify that the volume of the cylinder is $\pi r^2 h$

(In your proof you may use the result $\lim_{x \to 0} \tan x = x$)

QUESTION 8 continues over.......)

QUESTION 8 continued…….)

6 (b) A particle P moves in the *x,y*-plane and its co-ordinates *(x, y)* satisfy the equations

$$\frac{d^2x}{dt^2} = -n^2x \quad \text{and} \quad \frac{d^2y}{dt^2} = -n^2y\,,\ \text{where } n \text{ is a constant}$$

Initially (t=0), it is given that $x = 4,\ y = 0,\ \dfrac{dx}{dt} = 0$ and $\dfrac{dy}{dt} = 3n$

Show that, as t varies, x and y describe the ellipse $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$

END OF EXAMINATION PAPER

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SOLUTIONS AND MARKING

QUESTION 1:

(a) (i) $\int \cos^3 x\, dx = \int \cos x(1-\sin^2 x)\, dx$	2 MARKS 1 for breaking it up
$= \int \cos x\, dx - \int \cos x \sin^2 x\, dx$	
$= \sin x - \frac{1}{3}\sin^3 x + k$	1 for answer.
(ii) $\int \frac{dx}{x^2-4x+8} = \int \frac{dx}{(x-2)^2+4}$	2 MARKS Award 1 for either completing the square or for $\tan^{-1}$.
$= \frac{1}{2}\tan^{-1}\frac{x-2}{2} + k.$	DO NOT PENALISE for no k
(iii) $\int_1^5 \frac{dx}{(2x-1)\sqrt{2x-1}} = \int_1^5 (2x-1)^{-3/2}\, dx$	2 MARKS
$= \left[-(2x-1)^{-1/2}\right]_1^5$ OR $= \left[\frac{-1}{\sqrt{2x-1}}\right]_1^5$	either earns the first mark.
$= -\frac{1}{\sqrt{9}} + \frac{1}{\sqrt{1}}$	
$= -\frac{2}{3}.$	Second mark
(b) $RHS = \frac{\sec x(\tan x + \sec x)}{\sec x + \tan x}$	1 MARK.
$= \sec x = LHS.$	
$\therefore \int_0^{\pi/4} \sec x\, dx = \int_0^{\pi/4} \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}\, dx$	1 MARK for using part (i)
$= \ln(\sec x + \tan x)\Big]_0^{\pi/4}$	1 MARK for recognising this.
$= \ln(\sqrt{2}+1) - \ln 1.$ $= \ln(\sqrt{2}+1)$	1 MARK

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Q1 (d)

$C = 1$

$(Ax+B)(x+1) + C(x^2+4) = 5$

$\Rightarrow A = -1$ and $A + B = 0 \Rightarrow B = 1$

2 marks for A, B, C. no matter how!

$$\int \frac{5\,dx}{(x^2+4)(x+1)} = \int \frac{1-x}{x^2+4}\,dx + \int \frac{dx}{x+1}$$

$$= \int \frac{dx}{x^2+4} - \tfrac{1}{2}\int \frac{2x}{x^2+4}\,dx + \ln(x+1) + k$$

$$= \frac{1}{2}\tan^{-1}\frac{x}{2} - \tfrac{1}{2}\ln(x^2+4) + \ln(x+1) + k$$

1 mark for each part.

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QUESTION 2:

(a) (i) $z = 1 - i$ (ii) $|z| = \sqrt{2}$

$\bar{z} = 1 + i$

(iii) $\arg z = \begin{cases} 7\pi/4 \\ \text{or } -\pi/4 \end{cases}$ (iv) $\arg iz = \pi/4$.

(v) $z = \sqrt{2}\,\text{cis}\left(-\pi/4\right)$

$z^6 = 8\,\text{cis}\left(-3\pi/2\right)$

$= 8\,\text{cis}\left(\pi/2\right)$

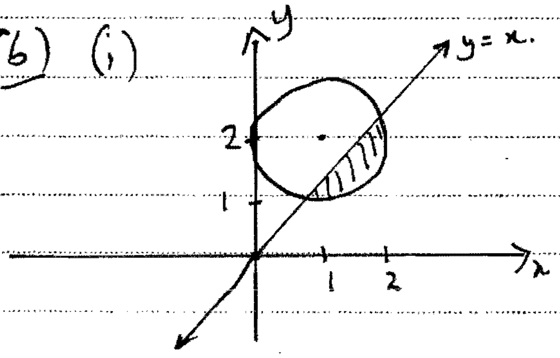
$= 8i$

6 MARKS

1 for each part (i) to (iv)

2 marks for part (v) (1 ONLY for $8\,\text{cis}\,\pi/2$)

(b) (i)



2 MARKS

1 for each of the areas inside circle and below y = x

(ii) P lies on $y = x$ as $\arg z = \pi/4$

∴ Solving simultaneously

$y = x$ and $(x-1)^2 + (y-2)^2 = 1$

gives $2x^2 - 6x + 5 = 1$

$x^2 - 3x + 2 = 0$

$(x-2)(x-1) = 0$

∴ $x = 2$ or, $x = 1$

$y = 2$ $y = 1$

∴ P is $2 + 2i$ or, $1 + i$.

3 MARKS

1 for reasoning this

1 for this step

note: 3 for P no matter how.

1 for this

(c) $2x + 2y + 2x\,\frac{dy}{dx} + 5y^4\frac{dy}{dx} = 0$

∴ $\frac{dy}{dx}\left(2x + 5y^4\right) = -2(x+y)$

$\frac{dy}{dx} = \dfrac{-2(x+y)}{2x + 5y^4}$. ← 1 MARK

4 MARKS

Since there is a horizontal tangent at (X, Y), $\frac{dy}{dx} = 0$ ← 1 for this.

∴ $-2(X + Y) = 0$

∴ $Y = -X$ ← 1 for this

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$\therefore$ Equation becomes

$$x^2 + 2x(-x) + (-x)^5 = 4$$

$$\therefore \quad x^5 + x^2 + 4 = 0$$

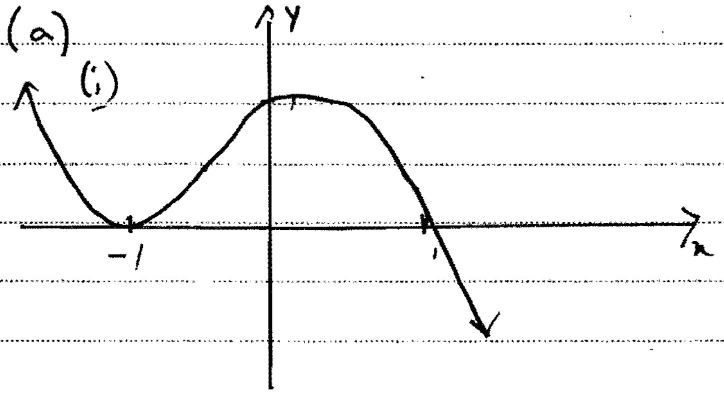
$\leftarrow$ 1 for final statement

Teacher's Name: ____________ **Student's Name/N°:** ____________

QUESTION 3:

(a)

(i)



2 MARKS

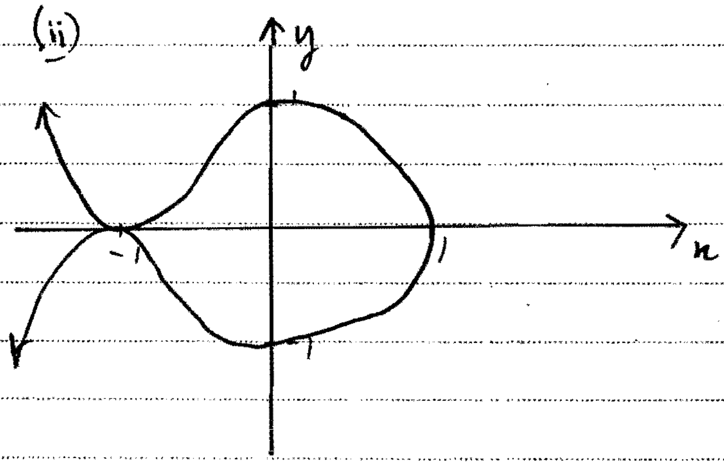
Key features are:
- cutting axis at (1,0)
- bouncing off axis at (-1,0)

} 1 MARK

- Right way up for a negative cubic } 1 MARK

[NO PENALTY for not having (0,1)]

(ii)



2 MARKS

Key features are:

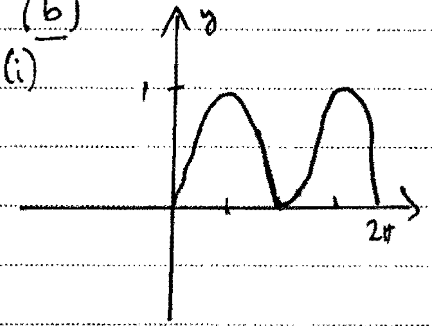
Bounces at (-1,0)
"Rounded" at (1,0) } 1 MARK

1 MARK for having both positive and negative parts.

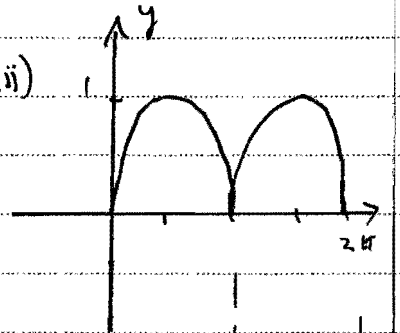
[SUBTRACT 1 MARK if the graph goes outside $x \geq 1$]

(b)

(i)



(ii)

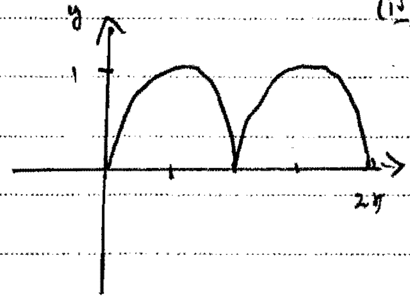


1 EACH (Keypoints are:)

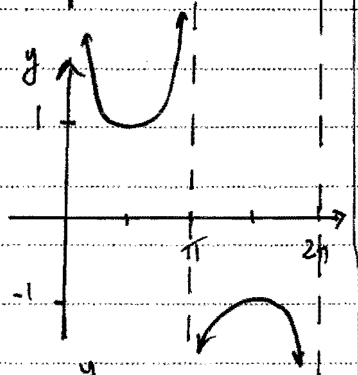
(i) no pointy bits – rounded

(ii) NOT rounded – pointy

(iii)



(iv)

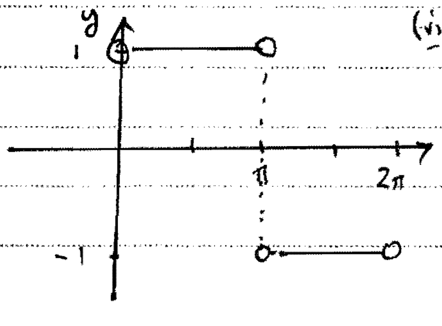


(iii) must be all positive.

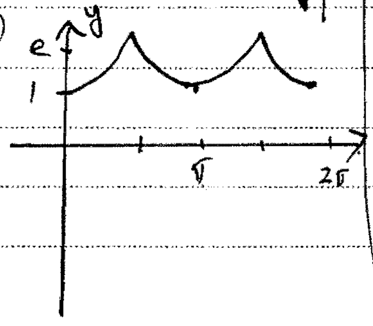
(iv) Asymptotes to be shown or be obvious

(v) Important to show open circles at endpoints

(v)



(vi)



(vi) not necessary to show e. Shape is important

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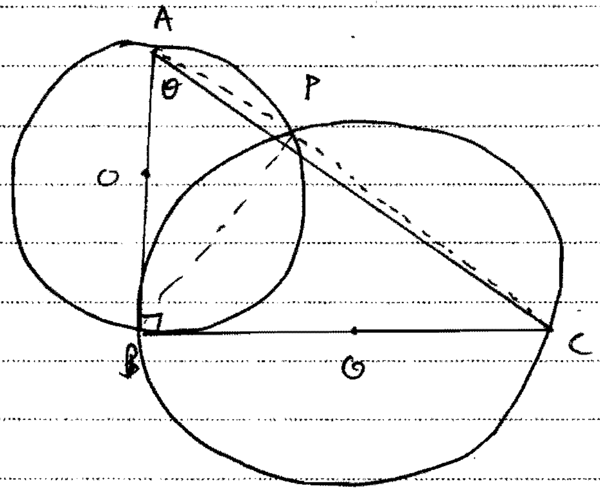
3 CONT.....)

(c)



(ii) By joining AP and PC,
In smaller circle, AB is a diameter
$\therefore \angle APB = 90^\circ$ (angle in a semi-circle)
Similarly in the larger circle $\angle CPB = 90^\circ$
$\therefore \angle APC = 180^\circ$
$\therefore$ P lies on AC.

(ii) 2 MARKS
They have to convince you.
Do NOT accept things like "obvious".

(iii) In $\triangle ABC$, $BC/AC = \sin\theta$

3 MARKS

$\therefore BC = 2a\sin\theta$

1 MARK

In $\triangle BPC$, $\angle PCB = (90-\theta)^\circ$ (angle sum of $\triangle BPC$)

$\therefore PB/BC = \sin(90-\theta)$

$\therefore PB = BC\cos\theta$

1 MARK

$= 2a\sin\theta\cos\theta$

$= a\sin 2\theta$

1 MARK

OR

In $\triangle ABC$, $AB/AC = \cos\theta$

$\therefore AB = AC\cos\theta$

$= 2a\cos\theta$

1 MARK

In $\triangle APB$, $PB/AB = \sin\theta$

$\therefore PB = AB\sin\theta$

$= 2a\cos\theta\sin\theta$

1 MARK

$= a\sin 2\theta$

1 MARK

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QUESTION 4:

(a) $\int_0^{\pi/4} \tan\theta \, d\theta = \int_0^{\pi/4} \frac{\sin\theta}{\cos\theta} \, d\theta$ 2 MARKS

$= -\ln\cos\theta \Big]_0^{\pi/4}$ 1 for this line

$= -\ln \frac{1}{\sqrt{2}} + \ln 1$

$= \frac{1}{2}\ln 2$ 1 for this

(b) (i) Let $z_1 = r_1 \operatorname{cis}\theta_1$ and $z_2 = r_2 \operatorname{cis}\theta_2$

$\therefore z_1 z_2 = r_1 r_2 \operatorname{cis}\theta_1 \operatorname{cis}\theta_2$

$= r_1 r_2 [\cos\theta_1 + i\sin\theta_1][\cos\theta_2 + i\sin\theta_2]$

$= r_1 r_2 \left(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2) \right)$ 2 MARKS

$= r_1 r_2 \left(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2) \right)$

$= r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2)$

$\therefore \arg z_1 z_2 = \arg_1 + \arg z_2$

(ii) For $n=2$ the formula is true (above) 3 MARKS

Assume the formula is true for $n=k$

ie $\arg(z_1 z_2 \cdots z_k) = \arg z_1 + \arg z_2 + \cdots + \arg z_k$

For $n=k+1$

$\arg(z_1 z_2 \cdots z_k z_{k+1}) = \arg[(z_1 \cdots z_k) z_{k+1}]$ ← 1 MARK

$= \arg(z_1 \cdots z_k) + \arg z_{k+1}$

(from part (i))

$= \underbrace{\arg z_1 + \arg z_2 + \cdots + \arg z_k}_{\text{from assumption}} + \arg z_{k+1}$ ← 1 MARK

$\therefore$ If the formula is true for $n=k$, it is true for $n=k+1$

But it is true for $n=2$ 1 MARK

$\therefore$ " " " " $n=3$ and so on.

ie true $\forall n$.

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Q 4 (cont...)

(c) (i) $\alpha + \beta + \gamma = 0$ — 1 MARK

(ii) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \alpha\gamma)$ — 1 MARK

$= -2a$ — 1 MARK

(iii) If there is a double root, it solves $P'(x) = 0$ — 3 MARKS

ie $3x^2 + a = 0$

$\therefore \quad x = \pm\sqrt{-a/3}$ — 1 MARK

$\therefore P\left(\sqrt{-a/3}\right) = 0 \Rightarrow \left(-\frac{a}{3}\right)^{3/2} + a\left(\sqrt{-a/3}\right) + b = 0$

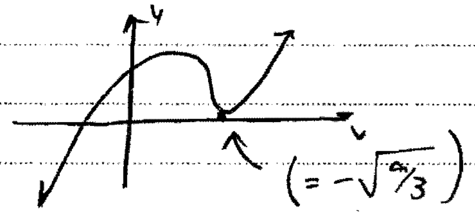
$\therefore \left(-\frac{a}{3}\right)^{1/2}\left[\left(-\frac{a}{3}\right) + a\right] + b = 0$ — 2 MARKS

$\therefore \left(-\frac{a}{3}\right)^{1/2} = -b/\tfrac{2a}{3}$

$= -\frac{3b}{2a}$

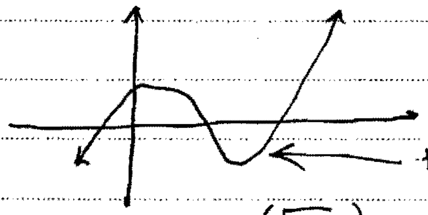
$\therefore$ double root is $-\frac{3b}{2a}$

(iv) If this polynomial has a double root its graph looks like: — 2 MARKS



For 3 distinct roots, this becomes

This is the key to it = 1 MARK



For this area $P(x) < 0$

ie $P\left(\sqrt{-a/3}\right) < 0 \Rightarrow \left(-\frac{a}{3}\right)^{1/2} < -\frac{3b}{2a}$

$\therefore -\frac{a}{3} > \frac{9b^2}{4a^2}$ — 1 MARK for algebra

$\therefore -4a^3 > 27b^2$. — (WATCH negative "fudges")

$27b^2 + 4a^3 < 0$

Teacher's Name: ______________________ **Student's Name/Nº:** ______________________

QUESTION 5:

(a) $x^2/9 - y^2/16 = 1$

(i) length of axis $= 6$ (5 MARKS)

(ii) $16 = 9(e^2 - 1)$ 1 each

$\therefore e^2 = 1 + 16/9$

$e = 5/3$

(iii) foci at $(\pm 5, 0)$

(iv) Directrices at $x = \pm 9/5$

(v) Asymptotes are $y = \pm 4/3 x$

(b) (i) P is (t^2, t^3) 2 MARKS

$2y \frac{dy}{dx} = 3x^2$

$\therefore \frac{dy}{dx} = \frac{3x^2}{2y}$

At P, $m_T = 3t/2$ 1 for slope

Equation of tangent is:

$y - t^3 = 3t/2\,(x - t^2)$

$\therefore t^3 - 3tx + 2y = 0$ 1 for equation

(ii) The parameters of any point P (not on the curve) which has a tangent to the curve at (x_1, y_1) solve $t^3 - 3tx + 2y = 0$ and this has at most 3 solutions for t ie. there are no more than 3 tangents

1 for seeing this connection

(iii) If there are 3 points, then their parameters t_1, t_2, t_3 are solutions to $t^3 - 3tx + 2y = 0$ where $x = x_1$ and $y = y_1$ 2 MARKS

1 for connecting the roots and equation

Sum of these $= t_1 + t_2 + t_3 = 0$

Product in pairs $= t_1t_2 + t_1t_3 + t_2t_3 = -3x_1$ 1 for all this

Since $t_1^2 + t_2^2 + t_3^2 = (t_1 + t_2 + t_3) - 2(t_1t_2 + t_1t_3 + t_2t_3)$

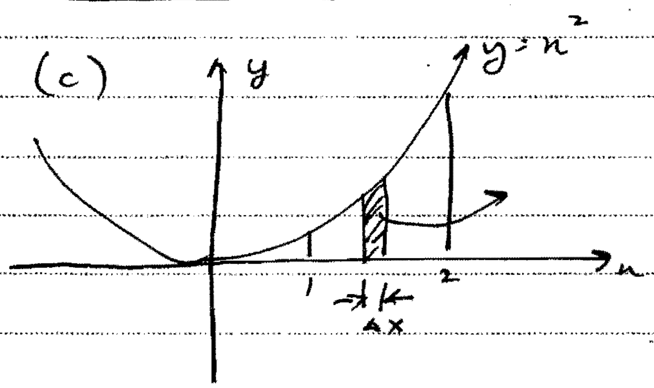
$= 0 + 6x_1$

$= 6x_1$

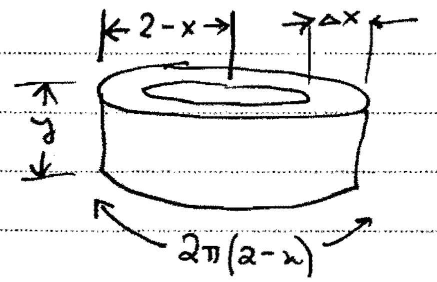
Teacher's Name: Student's Name/N°:

Q 5 CONT...)

(c)



5 MARKS



1 for {radius / circumference} of shell as $2\pi(2-x)$

1 for height of y (or x^2)

$$\Delta V = 2\pi(2-x)\,y\,\Delta x$$
$$= 2\pi(2-x)\,x^2\,\Delta x.$$

$$\text{VOL} = \lim_{\Delta x \to 0} \sum_1^2 2\pi(2-x)x^2\,\Delta x$$

$$= 2\pi \int_1^2 (2-x)x^2\,dx. \qquad \leftarrow 1 \text{ for expression of volume}$$

$$= 2\pi\left[\tfrac{2}{3}x^3\right]_1^2 - 2\pi\left[\tfrac{1}{4}x^4\right]_1^2$$

$$= 2\pi\left(\tfrac{16}{3} - \tfrac{2}{3}\right) - 2\pi\left(4 - \tfrac{1}{4}\right)$$

$$= \frac{28\pi}{3} - \frac{30\pi}{4}$$

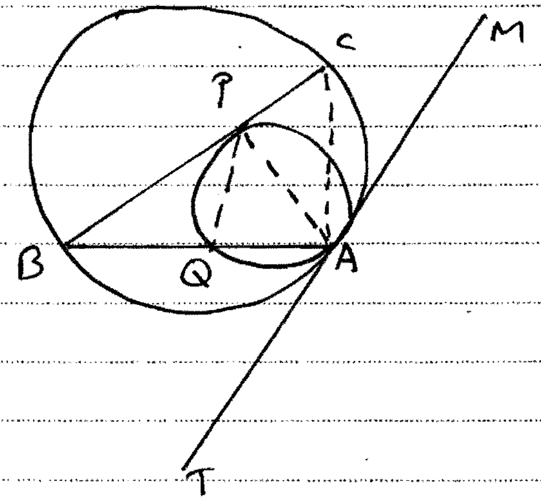
$$= \tfrac{11\pi}{6} \text{ cu units}$$

2 for working and answer

Teacher's Name: ______ Student's Name/N°: ______

QUESTION 6:

(a) (i)



(i) NO MARKS

(ii) Taking the tangent TM and the BIG circle — 5 MARKS

let $\angle MAC = \alpha^\circ$ — 1 MARK

$\therefore \angle CBA = \alpha^\circ$ (angle in the alt segment in big circle)

Using the tangent BC and the SMALL circle, let $\angle BPQ = \beta^\circ$ — 1 MARK

$\therefore \angle PAQ = \beta^\circ$ (angle in the alternate segment)

For $\triangle BPQ$, $\angle PQA = (\alpha+\beta)^\circ$ [external angle of $\triangle BPQ$] — 1 MARK

This is the angle in the alternate segment for the small circle and the chord PA with tangent TM.

$\therefore \angle PAM = (\alpha+\beta)^\circ$ — 1 MARK

But since $\angle MAC = \alpha$

$\therefore \angle PAC = \beta^\circ$

$= \angle PAQ$ — 1 MARK.

$\therefore$ PA bisects $\angle BAC$

(subtract 1 if there are no reasons throughout)

Teacher's Name: **Student's Name/N°:**

Q6 cont...)

(b)(i) $\frac{d}{dx}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)=0$

$$\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$$

$$\therefore \frac{dy}{dx}=-\frac{xb^2}{ya^2}$$

At P $\quad m_T=\frac{-a\cos\theta\, b^2}{a^2 b\sin\theta}$ — 3 MARKS

$$=-\frac{b\cos\theta}{a\sin\theta}$$ ← 1 mark to here (can be quoted)

At P, tangent is:

$$y-b\sin\theta=-\frac{b\cos\theta}{a\sin\theta}(x-a\cos\theta)$$

$$\therefore ay\sin\theta-ab\sin^2\theta=-bx\cos\theta+ab\cos^2\theta$$

$$\therefore ay\sin\theta+bx\cos\theta=ab(\sin^2\theta+\cos^2\theta)$$ 1 mark to here

$$\therefore \frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$$ 1 for division by ab

(ii) N is the point $(a\cos\theta, 0)$ — 2 MARKS

T is where the tangent cuts $y=0$ — 1 for N

ie T is $\left(\frac{a}{\cos\theta}, 0\right)$ — 1 for T

$$\therefore ON.OT=a\cos\theta\cdot\frac{a}{\cos\theta}$$

$$=a^2$$

(iii) R is where the tangent meets $x=\frac{a}{e}$ — 5 MARKS

i.e. $\frac{\cos\theta}{e}+\frac{y\sin\theta}{b}=1$

$$\therefore y=\frac{b}{\sin\theta}\left(1-\frac{\cos\theta}{e}\right)$$ 1 for R

Slope $PS=m_1=\frac{b\sin\theta}{a\cos\theta-ae}$

slope $RS=m_2=\frac{\frac{b}{\sin\theta}\left(1-\frac{\cos\theta}{e}\right)}{\frac{a}{e}-ae}$

1 mark for both

Teacher's Name: Student's Name/N°:

Q 6 cont...)

$$\therefore m_1 m_2 = \frac{b\sin\theta}{a(\cos\theta - e)} \times \frac{\frac{b}{\sin\theta}\left(1 - \frac{\cos\theta}{e}\right)}{a\left(\frac{1}{e} - e\right)}$$

$$= \frac{\frac{b^2}{e}(e - \cos\theta)}{\frac{a^2}{e}(1-e^2)(\cos\theta - e)}$$

$$= \frac{b^2}{-a^2(1-e^2)}$$

2 for this working

and since $b^2 = a^2(1-e^2)$ — 1 for recognising this.

$m_1 m_2 = -1 \Rightarrow PS \perp RS$

$\therefore$ PR subtends a right angle at S

no special marks for this.

Teacher's Name: Student's Name/N°:

QUESTION 7: 3 MARKS

(a) $x = a\tan\theta$

$\frac{dx}{d\theta} = a\sec^2\theta$

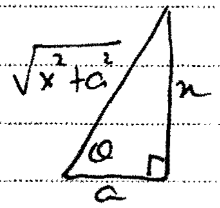
$\therefore dx = a\sec^2\theta\, d\theta$ 1 for this or equivalent

$$\therefore \int \frac{dx}{(a^2+x^2)^{3/2}} = \int \frac{a\sec^2\theta\, d\theta}{(a^2+a^2\tan^2\theta)^{3/2}}$$

$$= \int \frac{a\sec^2\theta\, d\theta}{a^3(1+\tan^2\theta)}$$

$$= \int \frac{d\theta}{a^2\sec\theta}$$

$$= \frac{1}{a^2}\sin\theta + k$$ 1 to get here.



Since $x = a\tan\theta$

$\therefore \sin\theta = \frac{x}{\sqrt{x^2+a^2}}$ 1 for $\sin\theta$

$$\therefore \int \frac{dx}{(a^2+x^2)^{3/2}} = \frac{x}{a^2\sqrt{x^2+a^2}} + k$$ [NO PENALTY for no k]

(b) Let the roots of $z^5=1$ be the solutions to $\text{cis}\,5\theta = 1$

$\therefore 5\theta = 0, 2\pi, 4\pi, 6\pi, 8\pi$

$\therefore \theta = 0, \frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}, \frac{8\pi}{5}$

$\therefore z = 1, \text{cis}\frac{2\pi}{5}, \text{cis}\frac{4\pi}{5}, \text{cis}\frac{6\pi}{5}, \text{cis}\frac{8\pi}{5}$

(i) 3 MARKS

(α) $\omega_1 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$

$\omega_4 = \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5}$

$= \cos\frac{2\pi}{5} - i\sin\frac{2\pi}{5}$ 1 MARK.

$= \overline{\omega_1}$

Similarly $\omega_3 = \overline{\omega_2}$ It is OK here to say "similarly".

Teacher's Name: Student's Name/N°:

Q 7 CONT...)

(β) Sum of roots of $z^5 - 1 = 0$ — 1 MARK

$\therefore 1 + \omega_1 + \omega_2 + \omega_3 + \omega_4 = 0$

$\therefore \omega_1 + \omega_2 + \omega_3 + \omega_4 = -1$

(γ) $\omega_1 + \omega_2 + \omega_3 + \omega_4 = -1$

$\therefore \omega_1 + \omega_2 + \bar{\omega}_2 + \bar{\omega}_1 = -1$

$\therefore 2\cos 2\pi/5 + 2\cos 4\pi/5 = -1$ — 1 MARK

$\therefore \cos 2\pi/5 + \cos 4\pi/5 = -1/2$

(ii) Sum of roots in pairs is

$\underbrace{\omega_1 + \omega_2 + \omega_3 + \omega_4} + \omega_1\omega_2 + \omega_2\omega_3 + \omega_1\omega_4 + \omega_2\omega_3 + \omega_2\omega_4 + \omega_3\omega_4 = 0$

$\therefore -1 + \omega_1\omega_2 + \omega_1\bar{\omega}_2 + \omega_1\bar{\omega}_1 + \omega_2\bar{\omega}_2 + \omega_2\bar{\omega}_1 + \bar{\omega}_1\bar{\omega}_2 = 0$

$\therefore -1 + \omega_1(\omega_2 + \bar{\omega}_2) + |\omega_1|^2 + |\omega_2|^2 + \bar{\omega}_1(\omega_2 + \bar{\omega}_2) = 0$

$\therefore -1 + (\omega_2 + \bar{\omega}_2)(\omega_1 + \bar{\omega}_1) + 2 = 0$ — 1 for using $|\omega|^2 = 1$

$\therefore 2\cos 4\pi/5 \cdot 2\cos 2\pi/5 + 1 = 0$ — 1 for using conjugates.

$\therefore 4\cos 2\pi/5 \cos 4\pi/5 + 1 = 0$

(iii) If the roots of the quadratic are $\cos 2\pi/5$ and $\cos 4\pi/5$ then

Sum $= \cos 2\pi/5 + \cos 4\pi/5 = -1/2$

PRODUCT $= \cos 2\pi/5 \cos 4\pi/5 = -1/4$ — 1 for seeing this

$\therefore$ QUADRATIC is

$x^2 + 1/2\, x - 1/4 = 0$

$\therefore 4x^2 + 2x - 1 = 0$

Teacher's Name: Student's Name/N°:

Q7 cont...)

(c)(i) $I_n = \int \sec^n\theta \, d\theta$ — 4 marks

$= \int \sec^2\theta \sec^{n-2}\theta \, d\theta$

$= \tan\theta \sec^{n-2}\theta - \int \tan\theta \frac{d}{d\theta} \sec^{n-2}\theta \, d\theta$ — 1 mark

Now $\frac{d}{d\theta} \sec^{n-2}\theta = (n-2)\sec^{n-3}\theta(-1)\cos^{-2}\theta(-\sin\theta)$

$= (n-2)\sec^{n-2}\theta \tan\theta$ — 1 for the differentiation of $\sec\theta$

$\therefore \int \tan\theta \frac{d}{d\theta} \sec^{n-2}\theta \, d\theta$

$= (n-2)\int \tan^2\theta \sec^{n-2} d\theta$

$= (n-2)\int (\sec^2\theta - 1)\sec^{n-2}\theta \, d\theta$ — 1 for changing $\tan^2\theta$ to $(\sec^2\theta - 1)$

$\therefore I_n = \tan\theta \sec^{n-2}\theta - (n-2)\int \sec^n\theta \, d\theta + (n-2)\int \sec^{n-2}\theta \, d\theta$

$= \tan\theta \sec^{n-2}\theta - (n-2)I_n + (n-2)I_{n-2}$

Using limits.

$\therefore (n-1)I_n = \tan\theta \sec^{n-2}\theta \Big]_0^{\pi/4} + (n-2)I_{n-2}$ — 1 for evaluating the limits

$\therefore (n-1)I_n - (n-2)I_{n-2} = (\sqrt{2})^{n-2}$

(ii) $\therefore \int_0^{\pi/4} \sec^4\theta \, d\theta$ means $n = 4$. — 2 marks

$\therefore 3I_4 - 2I_{n-2} = (\sqrt{2})^2$ — 1 for this

$\therefore 3I_4 = 2 + 2\int_0^{\pi/4} \sec^2\theta \, d\theta$

$= 2 + 2[\tan\theta]_0^{\pi/4}$ — 1 for this

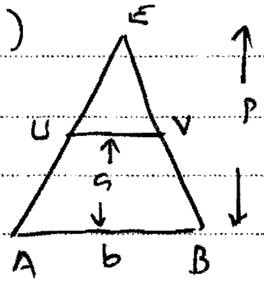
$= 4.$

$\therefore \int_0^{\pi/4} \sec^4\theta \, d\theta = 4/3$ — No particular mark

Teacher's Name: **Student's Name/N°:**

<u>QUESTION 8:</u>

(a)(i)



By similarity

$$\frac{UV}{b} = \frac{p-a}{p}$$

$$\therefore UV = \frac{b(p-a)}{p}$$

3 MARKS

2 for getting the width UV.

$$\therefore (\text{slice}) \Delta V = \frac{b(p-a)}{p} . h . \Delta a$$

1 MARK

(ii) $$\text{Vol} = \lim_{\Delta a \to 0} \sum_{0}^{p} \frac{b(p-a)}{p} h \Delta a$$

2 MARKS

$$= \int_0^p \left(hb - \frac{bah}{p} \right) da$$

← 1 for this

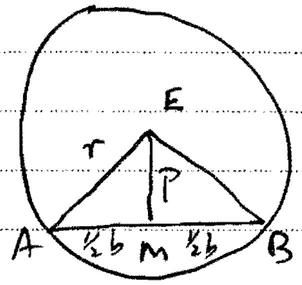
$$= hba \Big]_0^p - \frac{ba^2 h}{2p} \Big]_0^p$$

$$= hbp - \frac{bhp}{2}$$

$$= \tfrac{1}{2} bhp .$$

1 for completion

(iii)



If $\angle AEB = \frac{2\pi}{n}$, there are n of the triangular prisms in the cylinder

As $n \to \infty$, $\angle AEB \to 0$ and $p \to r$

5 MARKS

1 for seeing this

So volume of cylinder is

$\lim_{n \to \infty} nV$ where V is from (ii) above

<u>NOW</u> In $\triangle EMB$, $\tan \frac{\pi}{n} = \frac{\frac{1}{2} b}{p}$

As $n \to \infty$, $\frac{\pi}{n} \to 0$, so $\tan\left(\frac{\pi}{n}\right) \to \frac{\pi}{n}$ (given)

1 for using this fact

$\therefore \frac{1}{2} \frac{b}{p} \to \frac{\pi}{n}$

ie. $b \to \frac{2\pi p}{n}$

1 for getting b.

PTO

Teacher's Name: Student's Name/N°:

Q8 (c) CONT...)

Since $\text{vol}_{cylinder} = \lim_{n\to\infty} nV$

$= \lim_{n\to 0} n\left(\frac{1}{2}pbh\right)$

$= n\left(\frac{\pi p^2 h}{n}\right)$ 1 for simplification

$= \pi p^2 h$

$= \pi r^2 h$ $\left[\text{since } p \to r\right]$ 1 for $p \to r$

(c) 6 MARKS

$\frac{d^2x}{dt^2} = -n^2 x$ and $\frac{d^2y}{dt^2} = -n^2 y$

$\Rightarrow x = a\cos(nt+\alpha)$ $y = b\cos(nt+\beta)$

$\therefore \dot{x} = -an\sin(nt+\alpha)$ $\dot{y} = -nb\sin(nt+\beta)$ 1 for both of these

At $t=0$, $\frac{dx}{dt} = 0$

At $t=0$, $\dot{y} = 3n$.

$\therefore 3n = -bn\sin\beta)$

$\therefore \alpha = 0$

$\therefore \sin\beta = -3/b$ 1 for α

ie $\dot{x} = -an\sin(nt)$

ALSO at $t=0$, $y=0$

ALSO at $t=0$ $x=4$

$\therefore 0 = b\cos\beta$ 1 for a

$4 = a$

$\therefore \beta = \pi/2$ 1 for β

$\therefore x = 4\cos nt$

Since $\sin\beta = -3/b$

$b = -3$ 1 for b

$\therefore y = 3\sin(nt+\pi/2)$

$= 3\sin(nt)$

$\therefore x^2/16 + y^2/9 = \frac{16\cos^2 nt}{16} + \frac{9\sin^2 nt}{9}$ 1 for finishing

$= 1$