Factorising Complex Expressions

If a polynomial's coefficients are all real then the roots will appear in complex conjugate pairs.

Every polynomial of degree *n* can be;

- factorised as a mixture of quadratic and linear factors over the real field
- factorised to *n* linear factors over the complex field

NOTE: odd ordered polynomials must have a real root

$$e.g.\ (i)\ x^2+2x+2 = (x+1)^2+1$$

$$= (x+1+i)(x+1-i)$$

$(ii)\ z^4 + z^2 - 12 = 0$

$$(z^2 - 3)(z^2 + 4) = 0$$

$$(z + \sqrt{3})(z - \sqrt{3})(z^2 + 4) = 0 \quad \text{(factorised over Real numbers)}$$

$$(z + \sqrt{3})(z - \sqrt{3})(z + 2i)(z - 2i) = 0 \quad \text{(factorised over Complex numbers)}$$

$$\underline{z = \pm\sqrt{3} \quad \text{or} \quad z = \pm 2i}$$

(iii) Factorise $2x^3 - 3x^2 + 8x + 5$

as it is a cubic it must have a real factor

$$P\left(-\frac{1}{2}\right) = 2\left(-\frac{1}{2}\right)^3 - 3\left(-\frac{1}{2}\right)^2 + 8\left(-\frac{1}{2}\right) + 5$$

$$= -\frac{1}{4} - \frac{3}{4} - 4 + 5$$

$$= 0$$

$$\therefore 2x^3 - 3x^2 + 8x + 5 = (2x+1)(x^2 - 2x + 5)$$

$$= (2x+1)\left[(x-1)^2 + 4\right]$$

$$\underline{= (2x+1)(x-1-2i)(x-1+2i)}$$

$$(iv)\ z^5+z^4+z^3+z^2+z+1=0$$

Now $z^6-1=(z-1)\left(z^5+z^4+z^3+z^2+z+1\right)$

And $z^6-1=0$ has solutions;

$$z=cis\left[\frac{2\pi k}{6}\right]\ k=0,\pm1,\pm2,3$$

$$z^5+z^4+z^3+z^2+1=0$$

$$\frac{(z-1)\left(z^5+z^4+z^3+z^2+1\right)}{(z-1)}=0$$

$$z^6-1=0, z\neq 1$$

$$z=cis\frac{\pi}{3},cis-\frac{\pi}{3},cis\frac{2\pi}{3},cis-\frac{2\pi}{3},cis\pi$$

$$z=\frac{1}{2}+\frac{\sqrt{3}}{2}i,\frac{1}{2}-\frac{\sqrt{3}}{2}i,-\frac{1}{2}+\frac{\sqrt{3}}{2}i,-\frac{1}{2}-\frac{\sqrt{3}}{2}i,-1$$

(v) 1996 HSC

Let $\omega = \cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9}$

a) Show that ω^k is a solution of $z^9 - 1 = 0$, where k is an integer

$$z^9 = 1$$

$$z = cis\left[\frac{2\pi k}{9}\right] \qquad k = 0,1,2,3,4,5,6,7,8$$

$$z = \left\{cis\frac{2\pi}{9}\right\}^k$$

$$\underline{z = \omega^k}$$

b) Prove that $\omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 + \omega^7 + \omega^8 = -1$

$$z^9 - 1 = 0$$

$$\therefore 1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 + \omega^7 + \omega^8 = 0 \qquad (\text{sum of roots})$$

$$\underline{\omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 + \omega^7 + \omega^8 = -1}$$

c) Hence show that $\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}=\frac{1}{8}$

$$\sin A+\sin B=2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

(2 sine half sum cos half diff)

$$\cos A+\cos B=2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

(2 cos half sum cos half diff)

$$\cos A-\cos B=-2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

(minus 2 sine half sum sine half diff)

$$\omega+\omega^2+\omega^3+\omega^4+\omega^5+\omega^6+\omega^7+\omega^8=-1$$

roots appear in conjugate pairs

$$2\cos\frac{2\pi}{9}+2\cos\frac{4\pi}{9}+2\cos\frac{6\pi}{9}+2\cos\frac{8\pi}{9}=-1$$

$$\cos\frac{2\pi}{9}+\cos\frac{4\pi}{9}+\cos\frac{6\pi}{9}+\cos\frac{8\pi}{9}=-\frac{1}{2}$$

$$2\cos\frac{3\pi}{9}\cos\frac{\pi}{9}+2\cos\frac{7\pi}{9}\cos\frac{\pi}{9}=-\frac{1}{2}$$

$$2\cos\frac{3\pi}{9}\cos\frac{\pi}{9}+2\cos\frac{7\pi}{9}\cos\frac{\pi}{9}=-\frac{1}{2}$$

$$\cos\frac{\pi}{9}\left(\cos\frac{3\pi}{9}+\cos\frac{7\pi}{9}\right)=-\frac{1}{4}$$

$$\cos\frac{\pi}{9}\left(2\cos\frac{5\pi}{9}\cos\frac{2\pi}{9}\right)=-\frac{1}{4}$$

$$\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{5\pi}{9}=-\frac{1}{8}$$

But $\cos\frac{5\pi}{9}=-\cos\frac{4\pi}{9}$

$$-\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}=-\frac{1}{8}$$

$$\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}=\frac{1}{8}$$

OR

$$z^9 - 1$$

$$= (z-1)(z-\omega)\left(z-\omega^8\right)\left(z-\omega^2\right)\left(z-\omega^7\right)\left(z-\omega^3\right)\left(z-\omega^6\right)\left(z-\omega^4\right)\left(z-\omega^5\right)$$

$$= (z-1)\left(z^2 - 2\cos\frac{2\pi}{9}z + 1\right)\left(z^2 - 2\cos\frac{4\pi}{9}z + 1\right)$$

$$\left(z^2 - 2\cos\frac{6\pi}{9}z + 1\right)\left(z^2 - 2\cos\frac{8\pi}{9}z + 1\right)$$

$$= (z-1)\left(z^2 - 2\cos\frac{2\pi}{9}z + 1\right)\left(z^2 - 2\cos\frac{4\pi}{9}z + 1\right)$$

$$\left(z^2 + z + 1\right)\left(z^2 - 2\cos\frac{8\pi}{9}z + 1\right)$$

Let $z = i$

$$i^9 - 1 = (i-1)\left(-2\cos\frac{2\pi}{9}i\right)\left(-2\cos\frac{4\pi}{9}i\right)(i)\left(-2\cos\frac{8\pi}{9}i\right)$$

$$i^9 - 1 = (i-1)\left(-2\cos\frac{2\pi}{9}i\right)\left(-2\cos\frac{4\pi}{9}i\right)(i)\left(-2\cos\frac{8\pi}{9}i\right)$$

$$i - 1 = -i^4(i-1)\left(2\cos\frac{2\pi}{9}\right)\left(2\cos\frac{4\pi}{9}\right)\left(2\cos\frac{8\pi}{9}\right)$$

$$-1 = 8\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}\cos\frac{8\pi}{9}$$

$$-1 = 8\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}\left(-\cos\frac{\pi}{9}\right)$$

$$\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9} = \frac{1}{8}$$

Exercise 4J; 1 to 4, 7ac