

Roots and Coefficients

For $P(x) = ax^n + bx^{n-1} + cx^{n-2} + \dots$

$$\sum \alpha = \frac{-b}{a} \quad (\text{sum of roots, 1 at a time i.e. } \alpha + \beta + \gamma + \delta + \dots)$$

$$\sum \alpha\beta = \frac{c}{a} \quad (\text{sum of roots, 2 at a time i.e. } \alpha\beta + \alpha\gamma + \alpha\delta + \dots)$$

$$\sum \alpha\beta\gamma = \frac{-d}{a} \quad (\text{sum of roots, 3 at a time i.e. } \alpha\beta\gamma + \alpha\beta\delta + \dots)$$

$$\sum \alpha\beta\gamma\delta = \frac{e}{a} \quad (\text{sum of roots, 4 at a time})$$

We have already seen that for $ax^2 + bx + c$;

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$$

and

$$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta}$$

This can be generalised to;

$$\sum \alpha^2 = (\sum \alpha)^2 - 2\sum \alpha\beta$$

$$\sum \frac{1}{\alpha} = \frac{\sum \alpha}{\alpha\beta} \quad (\text{order 2})$$

$$= \frac{\sum \alpha\beta}{\alpha\beta\gamma} \quad (\text{order 3})$$

$$= \frac{\sum \alpha\beta\gamma}{\alpha\beta\gamma\delta} \quad (\text{order 4})$$

e.g. (i) $x^3 - 2x^2 + 3x - 4 = 0$, find;

$$\text{a) } \sum \alpha = \underline{2}$$

$$\text{b) } \sum \alpha\beta = \underline{3}$$

$$\text{c) } \sum \alpha\beta\gamma = \underline{4}$$

$$\begin{aligned} \text{d) } \sum \alpha^2 &= (\sum \alpha)^2 - 2\sum \alpha\beta \\ &= 2^2 - 2(3) \\ &= \underline{-2} \end{aligned}$$

$$\text{e) } \sum \alpha^3 \qquad \alpha^3 - 2\alpha^2 + 3\alpha - 4 = 0$$

$$\beta^3 - 2\beta^2 + 3\beta - 4 = 0$$

$$\gamma^3 - 2\gamma^2 + 3\gamma - 4 = 0$$

$$\sum \alpha^3 - 2\sum \alpha^2 + 3\sum \alpha - 12 = 0$$

$$\sum \alpha^3 - 2\sum \alpha^2 + 3\sum \alpha - 12 = 0$$

$$\sum \alpha^3 = 2\sum \alpha^2 - 3\sum \alpha + 12$$

$$= 2(-2) - 3(2) + 12$$

$$= \underline{2}$$

$$\text{f) } \sum \alpha^4$$

$$\alpha^4 - 2\alpha^3 + 3\alpha^2 - 4\alpha = 0$$

$$\beta^4 - 2\beta^3 + 3\beta^2 - 4\beta = 0$$

$$\gamma^4 - 2\gamma^3 + 3\gamma^2 - 4\gamma = 0$$

$$\sum \alpha^4 - 2\sum \alpha^3 + 3\sum \alpha^2 - 4\sum \alpha = 0$$

$$\sum \alpha^4 = 2\sum \alpha^3 - 3\sum \alpha^2 + 4\sum \alpha$$

$$= 2(2) - 3(-2) + 4(2)$$

$$= \underline{18}$$

$$\begin{aligned} \text{g)} & (\alpha + \beta)(\alpha + \gamma)(\beta + \gamma) \\ &= (\sum \alpha - \gamma)(\sum \alpha - \beta)(\sum \alpha - \alpha) \\ &= (2 - \gamma)(2 - \beta)(2 - \alpha) \\ &= (4 - 2\gamma - 2\beta + \beta\gamma)(2 - \alpha) \\ &= 8 - 4\gamma - 4\beta + 2\beta\gamma - 4\alpha + 2\alpha\gamma + 2\alpha\beta - \alpha\beta\gamma \\ &= 8 - 4\sum \alpha + 2\sum \alpha\beta - \sum \alpha\beta\gamma \\ &= 8 - 4(2) + 2(3) - 4 \\ &= \underline{2} \end{aligned}$$

(ii) (1996)

Consider the polynomial equation $x^4 + ax^3 + bx^2 + cx + d = 0$ where a, b, c and d are all integers.

Suppose the equation has a root of the form ki , where k is real and $k \neq 0$

a) State why the conjugate $-ki$ is also a root.

Roots appear in conjugate pairs when the coefficients are real.

b) Show that $c = k^2 a$

$$\begin{aligned}x^4 + ax^3 + bx^2 + cx + d &= (x + ki)(x - ki)(x^2 + px + q) \\ &= (x^2 + k^2)(x^2 + px + q) \\ &= x^4 + px^3 + qx^2 + k^2x^2 + k^2px + k^2q \\ &= x^4 + px^3 + (q + k^2)x^2 + k^2px + k^2q\end{aligned}$$

$$\therefore p = a \dots (1) \quad q + k^2 = b \dots (2) \quad k^2 p = c \dots (3) \quad k^2 q = d \dots (4)$$

Substitute (1) into (3) $k^2 a = c$

c) Show that $c^2 + a^2d = abc$

Using (2); $q = b - k^2$

Sub into (4); $k^2(b - k^2) = d$

$$bk^2 - k^4 = d$$

$$\frac{bc}{a} - \frac{c^2}{a^2} = d$$

$$\left(\because k^2 = \frac{c}{a}\right)$$

$$abc - c^2 = a^2d$$

$$\underline{c^2 + a^2d = abc}$$

d) If 2 is also a root of the equation, and $b = 0$, show that c is even.

Let α be the 4th root

As the other three roots are integer multiples, and the product of the roots is an integer;

$\therefore \alpha$ is an integer

$$(-ki)(ki) + 2(-ki) + 2(ki) + \alpha(-ki) + \alpha(ki) + 2\alpha = b \quad (\sum \alpha\beta)$$

$$k^2 + 2\alpha = 0$$

$$k^2 = -2\alpha$$

From c) ... $c = k^2 a$

$$c = -2\alpha a$$

$c = 2M$ α and a are both integers, $\therefore M$ is an integer

Thus c is an even number, as it is divisible by 2

Exercise 5D; 1 to 9