Forming Polynomials With The Roots Of Another

If $\alpha, \beta, \gamma, \ldots$ are the roots of a polynomial, to form an equation with roots;

(1) $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}, \ldots$ let $y = \frac{1}{x}$ and substitute $x = \frac{1}{y}$

(2) $k\alpha, k\beta, k\gamma, \ldots$ let $y = kx$ and substitute $x = \frac{y}{k}$

(3) $\alpha + \text{c}, \beta + c, \gamma + c, \ldots$ let $y = x + c$ and substitute $x = y - c$

(4) $\alpha^2, \beta^2, \gamma^2, \ldots$ let $y = x^2$ and substitute $x = y^{\frac{1}{2}}$

e.g. If α, β, γ are the roots of $x^3 + x + 2 = 0$, form an equation whose roots are;

a) $\dfrac{1}{\alpha}, \dfrac{1}{\beta}, \dfrac{1}{\gamma}$

let $y = \dfrac{1}{x}$

$x = \dfrac{1}{y}$

$$\left(\frac{1}{y}\right)^3 + \frac{1}{y} + 2 = 0$$

$$1 + y^2 + 2y^3 = 0$$

b) $\alpha+1, \beta+1, \gamma+1$

let $y = x+1$

$x = y-1$

$$(y-1)^3 + (y-1) + 2 = 0$$

$$y^3 - 3y^2 + 3y - 1 + y - 1 + 2 = 0$$

$$y^3 - 3y^2 + 4y = 0$$

c) $\alpha^2, \beta^2, \gamma^2$

let $y = x^2$

$x = y^{\frac{1}{2}}$

$$\left(y^{\frac{1}{2}}\right)^3 + y^{\frac{1}{2}} + 2 = 0$$

$$y^{\frac{3}{2}} + y^{\frac{1}{2}} + 2 = 0$$

$$y^{\frac{1}{2}}(y+1) = -2$$

$$y(y+1)^2 = 4$$

$$y^3 + 2y^2 + y = 4$$

$$y^3 + 2y^2 + y - 4 = 0$$

d) $\frac{1}{\alpha^2}, \frac{1}{\beta^2}, \frac{1}{\gamma^2}$

let $y = \frac{1}{x^2}$

$x = y^{-\frac{1}{2}}$

$$y^{-\frac{3}{2}} + y^{-\frac{1}{2}} + 2 = 0$$

$$y^{-\frac{3}{2}}(y+1) = -2$$

$$(y+1) = -2y^{\frac{3}{2}}$$

$$(y+1)^2 = 4y^3$$

$$y^2 + 2y + 1 = 4y^3$$

$$4y^3 - y^2 - 2y - 1 = 0$$

e) Find $\alpha^2 + \beta^2 + \gamma^2$

$$\alpha^2 + \beta^2 + \gamma^2$$
$$= \left(\sum \alpha\right)^2 - 2\sum \alpha\beta$$
$$= (0)^2 - 2(1)$$
$$= -2$$

OR using equation found in c)

$$\alpha^2 + \beta^2 + \gamma^2$$
$$= \frac{-b}{a}$$
$$= \frac{-2}{1}$$
$$= -2$$

Exercise 5D; 10 to 16

Exercise 5E; 2cd, 4b, 7, 9, 10, 18, 27, 30, 34, 35