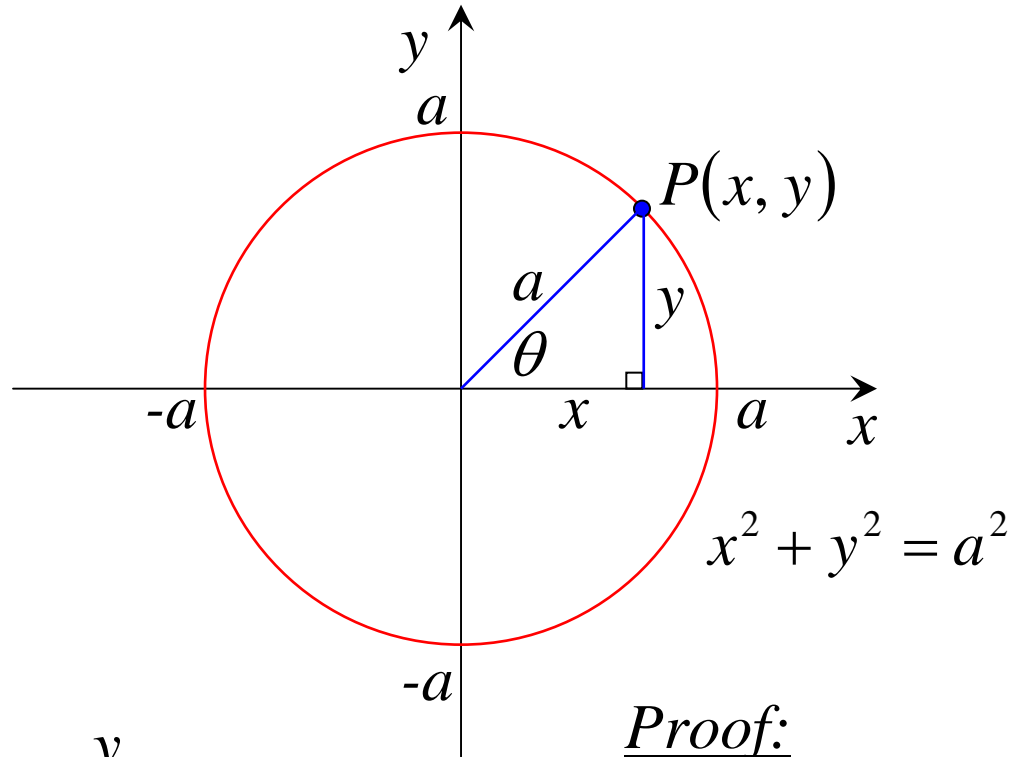


Conics & Parameters

1) Circle



$$\frac{x}{a} = \cos \theta$$

$$\frac{y}{a} = \sin \theta$$

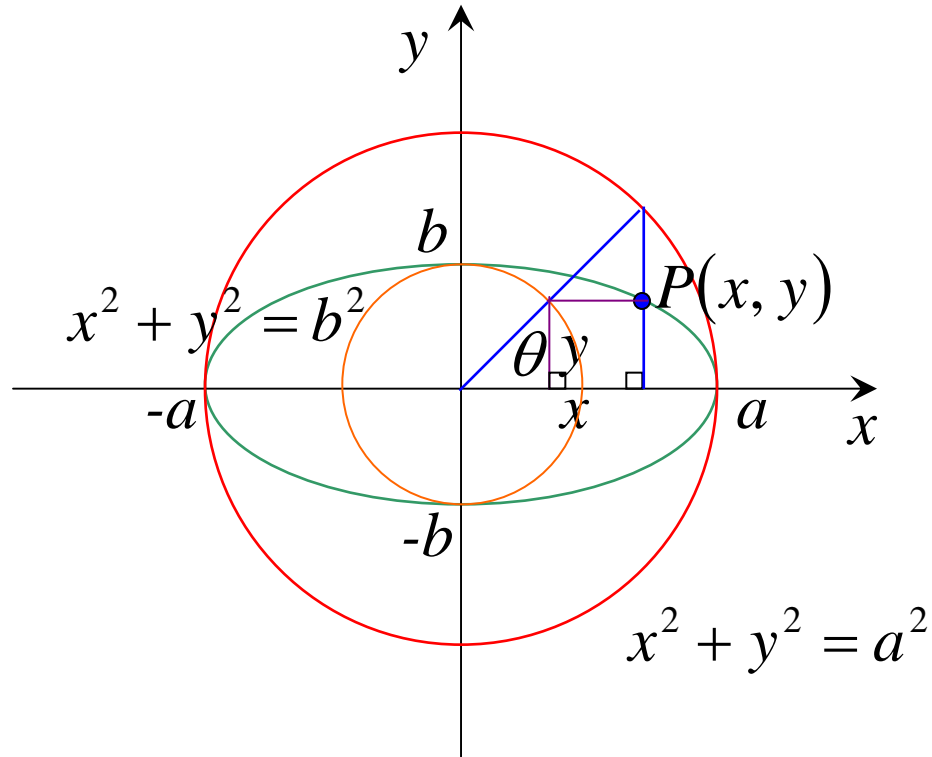
$$x = a \cos \theta$$

$$y = a \sin \theta$$

Proof:

$$\begin{aligned} x^2 + y^2 &= a^2 \cos^2 \theta + a^2 \sin^2 \theta \\ &= a^2 (\cos^2 \theta + \sin^2 \theta) \\ &= a^2 \end{aligned}$$

2) Ellipse



$$\frac{x}{a} = \cos \theta$$

$$\frac{y}{b} = \sin \theta$$

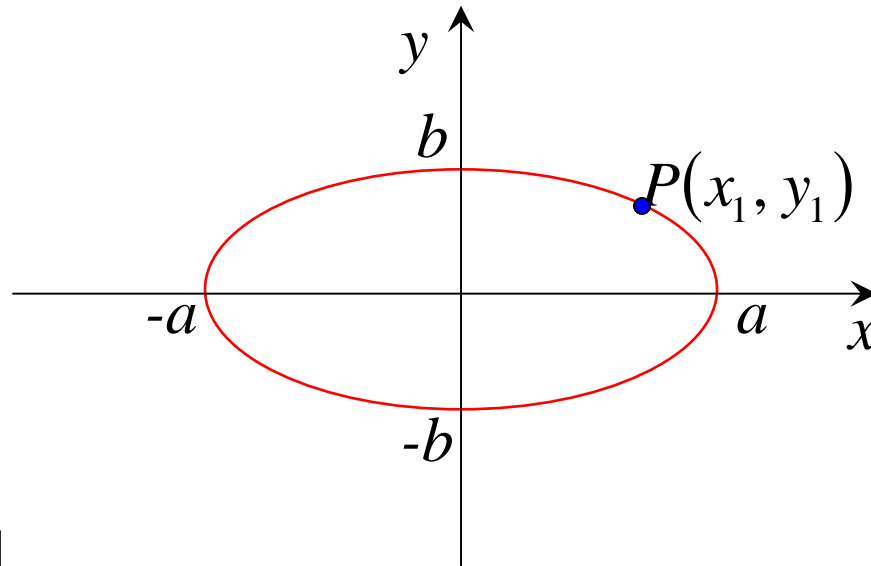
$$x = a \cos \theta$$

$$y = b \sin \theta$$

Proof:

$$\begin{aligned} \frac{x^2}{a^2} + \frac{y^2}{b^2} &= \frac{a^2 \cos^2 \theta}{a^2} + \frac{b^2 \sin^2 \theta}{b^2} \\ &= \cos^2 \theta + \sin^2 \theta \\ &= 1 \end{aligned}$$

Equation of Tangent and Normal



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{2x}{a^2} + \frac{2y}{b^2} \cdot \frac{dy}{dx} = 0$$

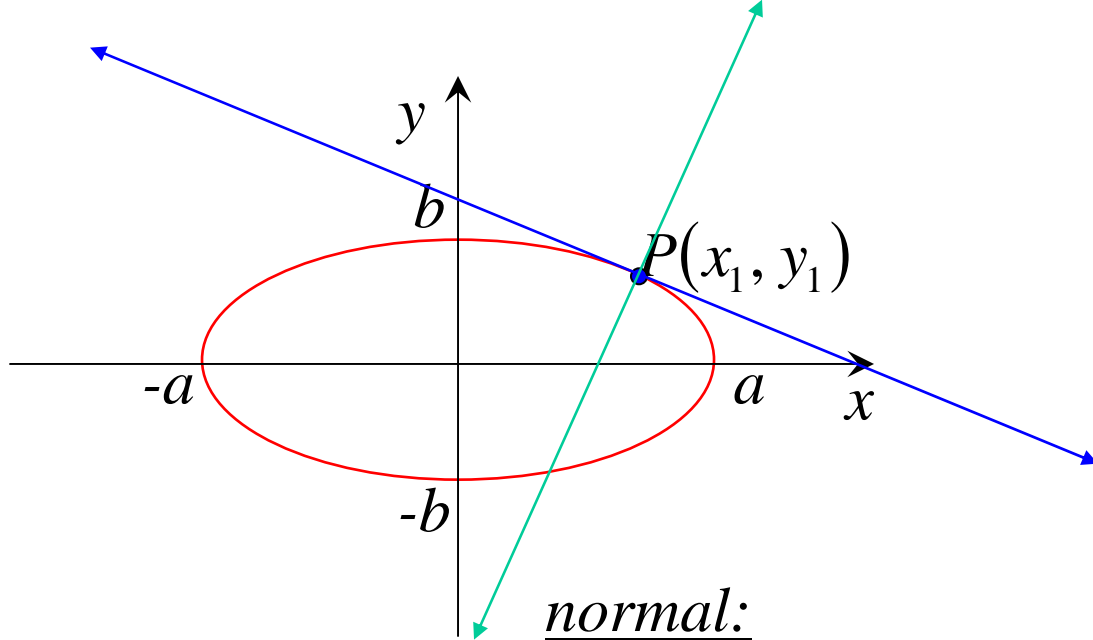
$$\left[\frac{df}{dx} = \frac{df}{dy} \cdot \frac{dy}{dx} \right]$$

$$\frac{2y}{b^2} \cdot \frac{dy}{dx} = -\frac{2x}{a^2}$$

$$\frac{dy}{dx} = -\frac{b^2 x}{a^2 y}$$

at $P(x_1, y_1)$

$$\frac{dy}{dx} = -\frac{b^2 x_1}{a^2 y_1}$$



tangent:

$$y - y_1 = -\frac{b^2 x_1}{a^2 y_1} (x - x_1)$$

$$a^2 y_1 y - a^2 y_1^2 = -b^2 x_1 x + b^2 x_1^2$$

$$b^2 x_1 x + a^2 y_1 y = b^2 x_1^2 + a^2 y_1^2$$

$$\frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}$$

$$\frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} = 1$$

normal:

$$y - y_1 = \frac{a^2 y_1}{b^2 x_1} (x - x_1)$$

$$b^2 x_1 y - b^2 x_1 y_1 = a^2 y_1 x - a^2 x_1 y_1$$

$$a^2 y_1 x - b^2 x_1 y = a^2 x_1 y_1 - b^2 x_1 y_1$$

$$\frac{a^2 x}{x_1} - \frac{b^2 y}{y_1} = a^2 - b^2$$

Using Parametric Coordinates;

$$\begin{aligned} \text{at } P(a \cos \theta, b \sin \theta) \quad \frac{dy}{dx} &= -\frac{ab^2 \cos \theta}{a^2 b \sin \theta} \\ &= -\frac{b \cos \theta}{a \sin \theta} \end{aligned}$$

tangent:

$$\begin{aligned} y - b \sin \theta &= -\frac{b \cos \theta}{a \sin \theta} (x - a \cos \theta) \\ ay \sin \theta - ab \sin^2 \theta &= -bx \cos \theta + ab \cos^2 \theta \end{aligned}$$

$$bx \cos \theta + ay \sin \theta = ab \sin^2 \theta + ab \cos^2 \theta$$

$$\frac{x \cos \theta}{a} + \frac{y \sin \theta}{b} = \sin^2 \theta + \cos^2 \theta$$

$$\underline{\frac{x \cos \theta}{a} + \frac{y \sin \theta}{b} = 1}$$

normal:

$$\begin{aligned} y - b \sin \theta &= \frac{a \sin \theta}{b \cos \theta} (x - a \cos \theta) \\ by \cos \theta - b^2 \sin \theta \cos \theta & \end{aligned}$$

$$= ax \sin \theta - a^2 \sin \theta \cos \theta$$

$$\begin{aligned} ax \sin \theta - by \cos \theta & \\ &= (a^2 - b^2) \sin \theta \cos \theta \end{aligned}$$

$$\underline{\frac{ax}{\cos \theta} - \frac{by}{\sin \theta} = a^2 - b^2}$$

For ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

tangent at (x_1, y_1)

$$\frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} = 1$$

normal at (x_1, y_1)

$$\frac{a^2 x}{x_1} - \frac{b^2 y}{y_1} = a^2 - b^2$$

tangent at $(a \cos \theta, b \sin \theta)$

$$\frac{x \cos \theta}{a} + \frac{y \sin \theta}{b} = 1$$

normal at $(a \cos \theta, b \sin \theta)$

$$\frac{ax}{\cos \theta} - \frac{by}{\sin \theta} = a^2 - b^2$$

e.g. (i) Find the equation of the tangent to the ellipse $\frac{x^2}{16} + y^2 = 1$

at the point $\left(2, \frac{-\sqrt{3}}{2}\right)$

$$\frac{x}{8} + 2y \frac{dy}{dx} = 0 \quad \text{at} \left(2, \frac{-\sqrt{3}}{2}\right), \frac{dy}{dx} = \frac{-2}{16 \left(\frac{-\sqrt{3}}{2}\right)}$$
$$\frac{dy}{dx} = \frac{-x}{16y}$$

$$\frac{dy}{dx} = \frac{1}{4\sqrt{3}}$$

$$y + \frac{\sqrt{3}}{2} = \frac{1}{4\sqrt{3}}(x - 2)$$

$$4\sqrt{3}y + 6 = x - 2$$

$$\underline{x - 4\sqrt{3}y - 8 = 0}$$

(ii) Find the equation of the normal to the ellipse $x = 2\cos\theta$, $y = \sin\theta$

at the point $\theta = \frac{\pi}{6}$

$$y = \sin f(x)$$

$$\frac{dy}{dx} = f'(x)\cos f(x)$$

$$y = \cos f(x)$$

$$\frac{dy}{dx} = -f'(x)\sin f(x)$$

$$x = 2\cos\theta$$

$$y = \sin\theta$$

$$\frac{dx}{d\theta} = -2\sin\theta$$

$$\frac{dy}{d\theta} = \cos\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

$$= \frac{\cos\theta}{-2\sin\theta}$$

$$\text{at } \theta = \frac{\pi}{6}, \frac{dy}{dx} = \frac{\sqrt{3}}{2} \\ = \frac{-\sqrt{3}}{2}$$

$$y - \frac{1}{2} = \frac{2}{\sqrt{3}}(x - \sqrt{3})$$

$$2\sqrt{3}y - \sqrt{3} = 4x - 4\sqrt{3}$$

$$4x - 2\sqrt{3}y - 3\sqrt{3} = 0$$

(iii) Show that if $y = mx + k$ is a tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ then } a^2 m^2 + b^2 = k^2$$

$$\begin{array}{lll} x = a \cos \theta & y = b \sin \theta & \frac{dy}{dx} = \frac{-b \cos \theta}{a \sin \theta} \\ \frac{dx}{d\theta} = -a \sin \theta & \frac{dy}{d\theta} = b \cos \theta & \end{array}$$

If $y = mx + k$ is a tangent then $\frac{dy}{dx} = m$

$$\text{i.e. } m = \frac{-b \cos \theta}{a \sin \theta}$$

$$am \sin \theta = -b \cos \theta$$

$$am \sin \theta + b \cos \theta = 0 \dots\dots(1)$$

If tangent meets ellipse at $(a \cos \theta, b \sin \theta)$ then;

$$b \sin \theta = am \cos \theta + k$$

$$am \cos \theta - b \sin \theta = -k \dots\dots(2)$$

$$(1)^2 : a^2 m^2 \sin^2 \theta + 2abm \sin \theta \cos \theta + b^2 \cos^2 \theta = 0 \quad (+)$$

$$(2)^2 : a^2 m^2 \cos^2 \theta - 2abm \sin \theta \cos \theta + b^2 \sin^2 \theta = k^2$$

$$a^2 m^2 (\sin^2 \theta + \cos^2 \theta) + b^2 (\sin^2 \theta + \cos^2 \theta) = k^2$$

$$\underline{a^2 m^2 + b^2 = k^2}$$

Exercise 6C; 1, 3, 11, 15, 18a