Rectangular Hyperbola

A hyperbola whose asymptotes are perpendicular to each other

$$\frac{b}{a} \times \frac{-b}{a} = -1$$

$$b^2 = a^2$$

$$b = a$$

$\therefore$ hyperbola has the equation;

$$\frac{x^2}{a^2} - \frac{y^2}{a^2} = 1$$

$$\boxed{x^2 - y^2 = a^2}$$

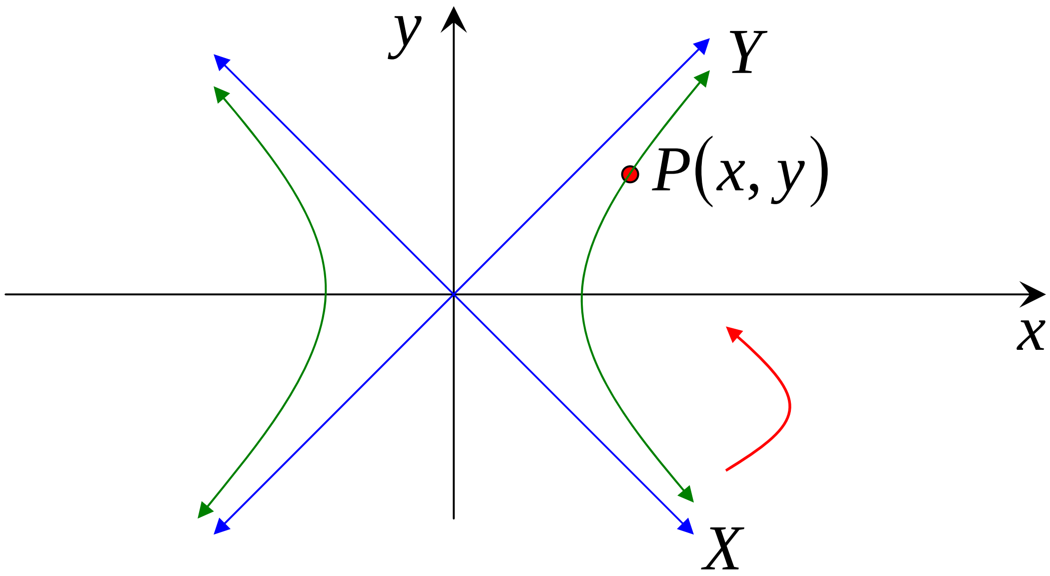
$$a^2\left(e^2 - 1\right) = a^2$$

$$e^2 - 1 = 1$$

$$e^2 = 2$$

$$e = \sqrt{2}$$

$\therefore$ eccentricity is $\sqrt{2}$



In order to make the asymptotes the coordinate axes we need to rotate the curve 45 degrees anticlockwise.

i.e. $P(x, y) = x + iy$ is multiplied by $\text{cis}45^\circ$

$$(x+iy)(\cos 45^\circ + i\sin 45^\circ)$$

$$= (x+iy)\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}}(x+iy)(1+i)$$

$$= \frac{1}{\sqrt{2}}(x+ix+iy-y)$$

$$= \frac{x-y}{\sqrt{2}} + \frac{x+y}{\sqrt{2}}i$$

$$\therefore X = \frac{x-y}{\sqrt{2}} \qquad Y = \frac{x+y}{\sqrt{2}}$$

$$XY = \frac{x^2-y^2}{2}$$

$$XY = \frac{a^2}{2}$$

focus; $(\pm ae, 0)$

$$= (\pm\sqrt{2}a, 0)$$

$$\sqrt{2}a\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)$$

$$= a + ai$$

$\therefore$ focus (a, a)

directrix; $x = \pm\frac{a}{e}$

$$x = \pm\frac{a}{\sqrt{2}}$$

directrices are $\parallel$ to y axis

$\therefore$ when rotated $\parallel$ to $y = -x$

thus in form $x + y + k = 0$

Now distance between directrices is $\frac{2a}{\sqrt{2}}$

$\therefore$ distance from origin to directrix is $\frac{a}{\sqrt{2}}$

$$\frac{|0+0+k|}{\sqrt{2}} = \frac{a}{\sqrt{2}}$$

$$\therefore |k| = a$$

$$k = \pm a$$

$\therefore$ directrices are $x + y = \pm a$

The rectangular hyperbola with x and y axes as aymptotes, has the equation;

$$xy = \frac{1}{2}a^2$$

where;

$\text{foci} : (\pm a, \pm a)$

$\text{directrices} : x + y = \pm a$

$\text{eccentricity} = \sqrt{2}$

Parametric Coordinates of $xy = c^2$

$$x = ct \qquad y = \frac{c}{t}$$

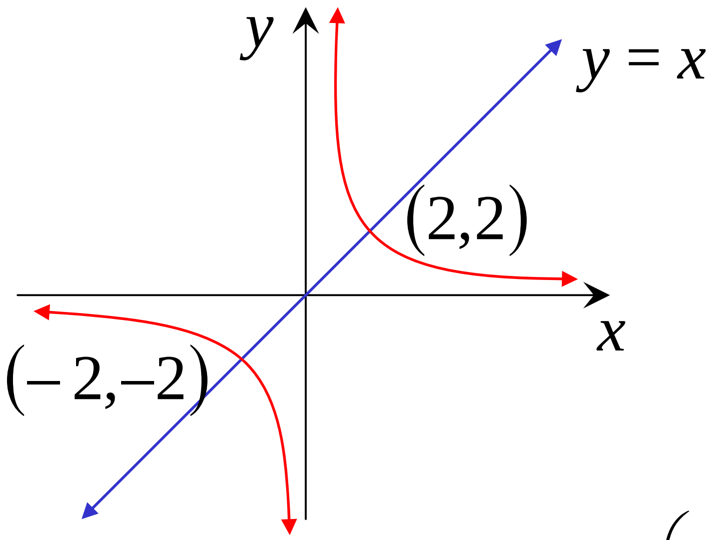
Tangent: $x + t^2 y = 2ct$

Normal: $t^3 x - ty = c(t^4 - 1)$

e.g. (i) (1991)

The hyperbola H is $xy = 4$

a) Sketch H showing where H intersects the axis of symmetry.



$$xy = 4$$
$$x^2 = 4$$
$$x = \pm 2$$

b) Show that the tangent at $P\left(2t, \frac{2}{t}\right)$ is $x + t^2 y = 4t$

$$y = \frac{4}{x}$$

$$\frac{dy}{dx} = -\frac{4}{x^2}$$

when $x = 2t, \frac{dy}{dx} = -\frac{4}{(2t)^2}$

$$= \frac{-1}{t^2}$$

$$y - \frac{2}{t} = -\frac{1}{t^2}(x - 2t)$$

$$t^2 y - 2t = -x + 2t$$

$$\underline{x + t^2 y = 4t}$$

c) $s \neq 0, s^2 \neq t^2$, show that the tangents at P and $Q\left(2s, \frac{2}{s}\right)$

intersect at $M\left(\frac{4st}{s+t}, \frac{4}{s+t}\right)$

$$P: x + t^2 y = 4t$$

$$Q: x + s^2 y = 4s$$

$$\left(t^2 - s^2\right)y = 4t - 4s$$

$$(t+s)(t-s)y = 4(t-s)$$

$$y = \frac{4}{s+t}$$

$$x + \frac{4t^2}{s+t} = 4t$$

$$x = \frac{4st + 4t^2 - 4t^2}{s+t}$$

$$= \frac{4st}{s+t}$$

$$\therefore M \text{ is } \left(\frac{4st}{s+t}, \frac{4}{s+t}\right)$$

d) Suppose that $s = \frac{-1}{t}$, show that the locus of M is a straight line through the origin, but not including the origin.

$$x = \frac{4st}{s+t} \qquad y = \frac{4}{s+t}$$

$$s = \frac{-1}{t}$$

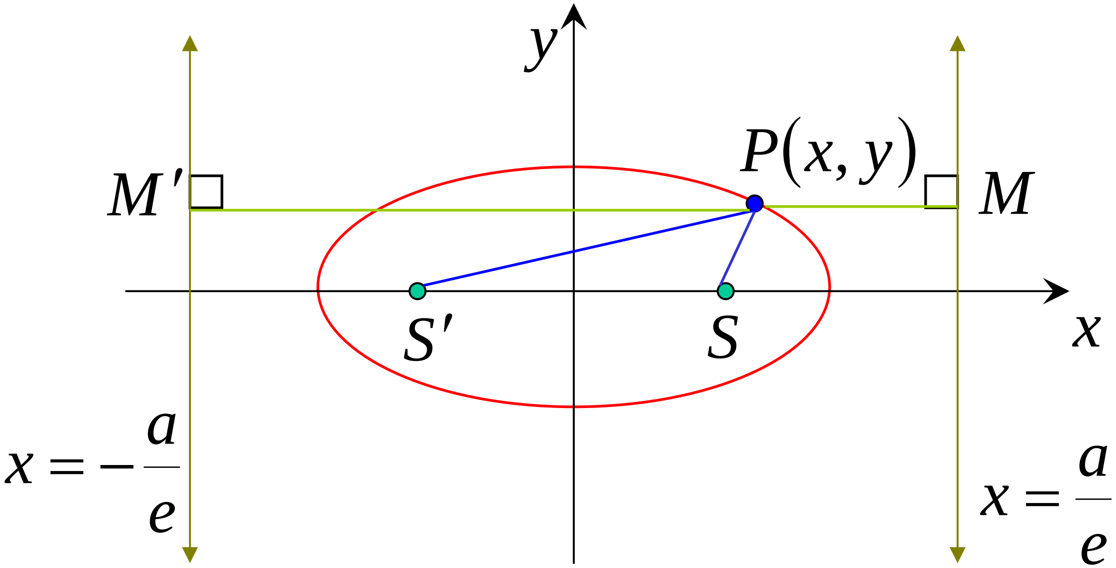
$$st = -1$$

$$x = \frac{-4}{s+t} \qquad \begin{aligned} y &= \frac{4}{s+t} \\ &= -x \end{aligned}$$

$$\therefore y = -x \qquad \frac{4}{s+t} \neq 0, \text{ thus } M \neq (0,0)$$

$\therefore$ locus of M is $y = -x$, excluding $(0,0)$

(ii) Show that $PS + PS' = 2a$



By definition of an ellipse;

$$\begin{aligned} PS + PS' &= ePM + ePM' \\ &= e(PM + PM') \\ &= e\left(\frac{2a}{e}\right) \\ &= \underline{2a} \end{aligned}$$

Exercise 6D; 3, 4, 7, 10, 11a, 12, 14, 19, 21, 26, 29, 31, 43, 47