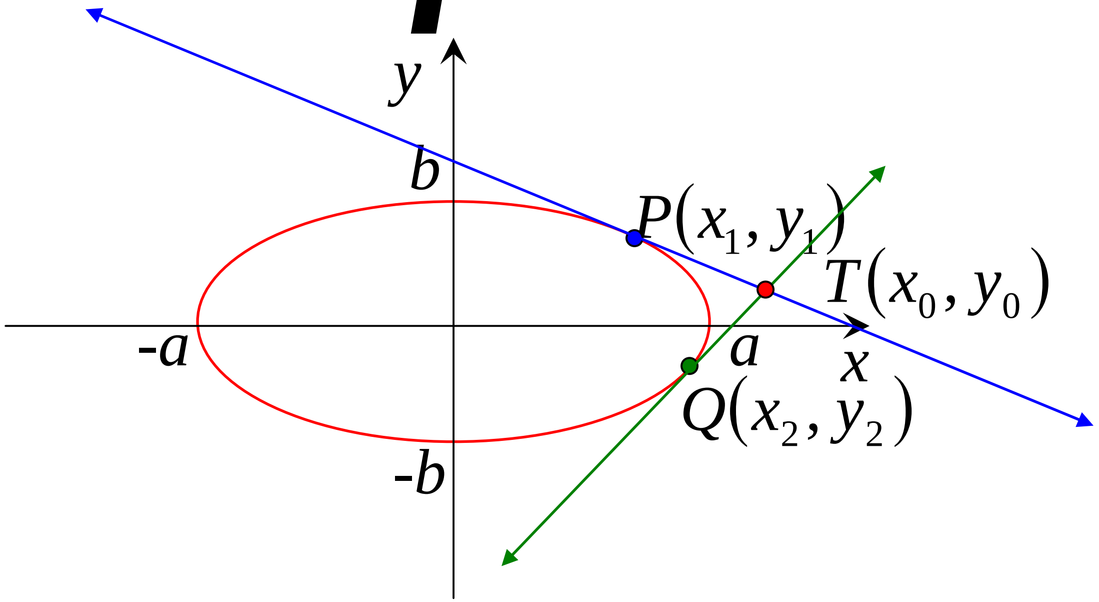
Chord of Contact



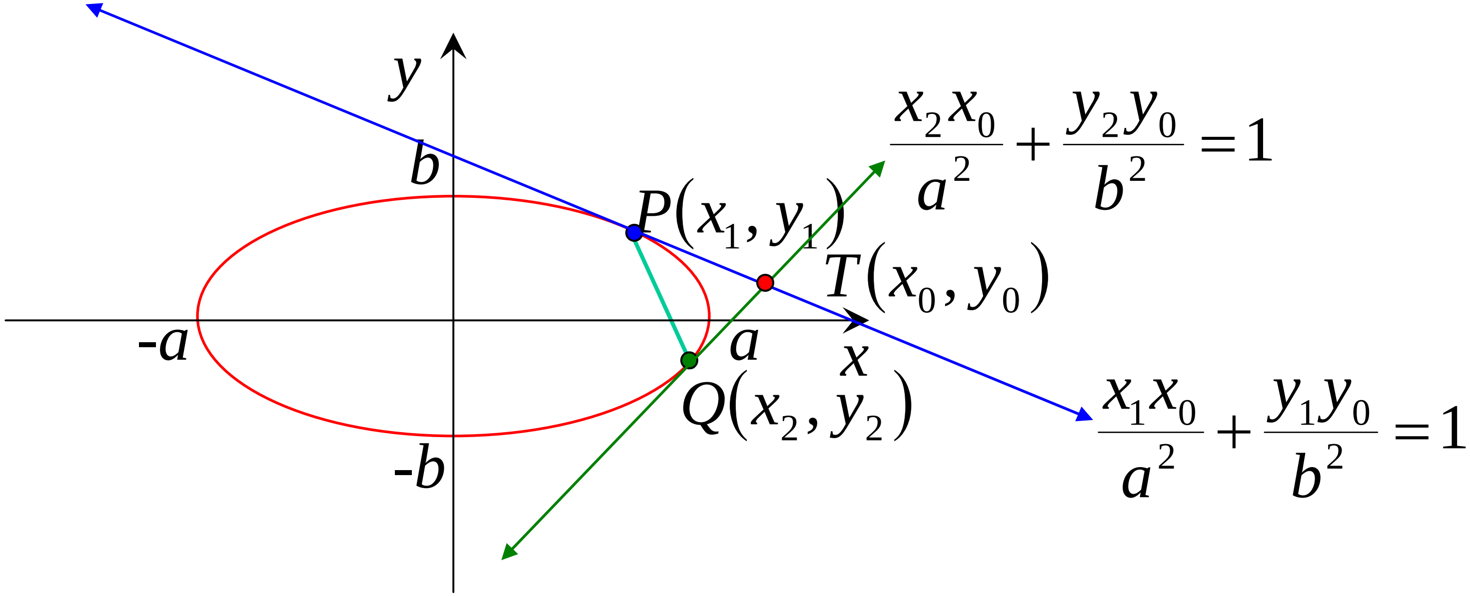
From an external point T, two tangents may be drawn.

tangent at P has equation $\dfrac{x_1 x}{a^2} + \dfrac{y_1 y}{b^2} = 1$

tangent at Q has equation $\dfrac{x_2 x}{a^2} + \dfrac{y_2 y}{b^2} = 1$

Now T lies on both lines,

$$\therefore \frac{x_1 x_0}{a^2} + \frac{y_1 y_0}{b^2} = 1 \quad \text{and} \quad \frac{x_2 x_0}{a^2} + \frac{y_2 y_0}{b^2} = 1$$



Thus P and Q both must lie on a line with equation;

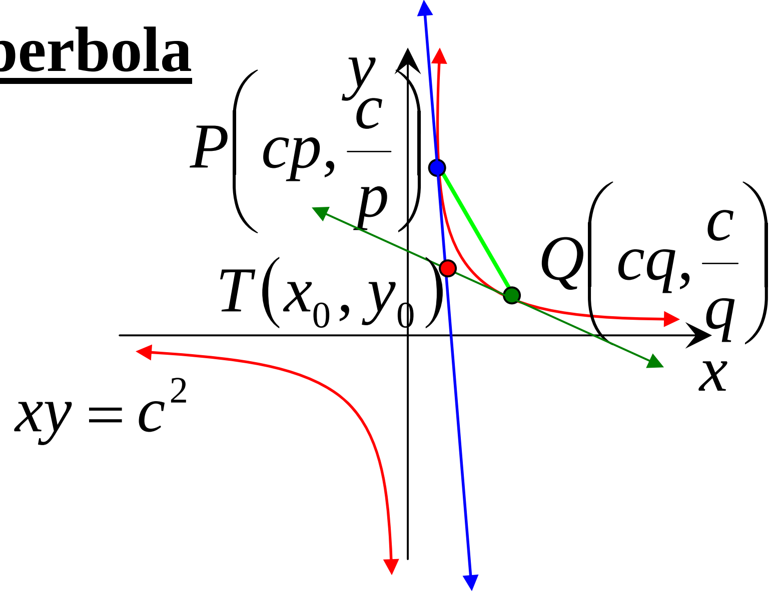
$$\boxed{\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1}$$

which must be the line PQ i.e. chord of contact

Similarly the chord of contact of the hyperbola has the equation;

$$\boxed{\frac{x_0 x}{a^2} - \frac{y_0 y}{b^2} = 1}$$

Rectangular Hyperbola



1) Show that equation PQ is $x + pqy = c(p+q)$……(1)

2) Show that T has coordinates $\left\{\frac{2cpq}{p+q}, \frac{2c}{p+q}\right\}$

$$\therefore x_0 = \frac{2cpq}{p+q} \qquad\qquad y_0 = \frac{2c}{p+q}$$

Substituting into (1); $x + \frac{(p+q)x_0}{2c} y = c(p+q)$

$$x + \frac{2cx_0}{2cy_0} y = \frac{2c^2}{y_0}$$

$$\boxed{xy_0 + x_0 y = 2c^2}$$

Geometric Properties

(1) The chord of contact from a point on the directrix is a focal chord.

<u>*ellipse*</u>

As T is on the directrix it has coordinates $\left(\frac{a}{e}, y_0\right)$ i.e. $x_0 = \frac{a}{e}$

$\therefore$ chord of contact will have the equation;

$$\frac{x\left(\frac{a}{e}\right)}{a^2} + \frac{yy_0}{b^2} = 1$$

$$\frac{x}{ae} + \frac{yy_0}{b^2} = 1$$

Substitute in focus $(ae, 0)$

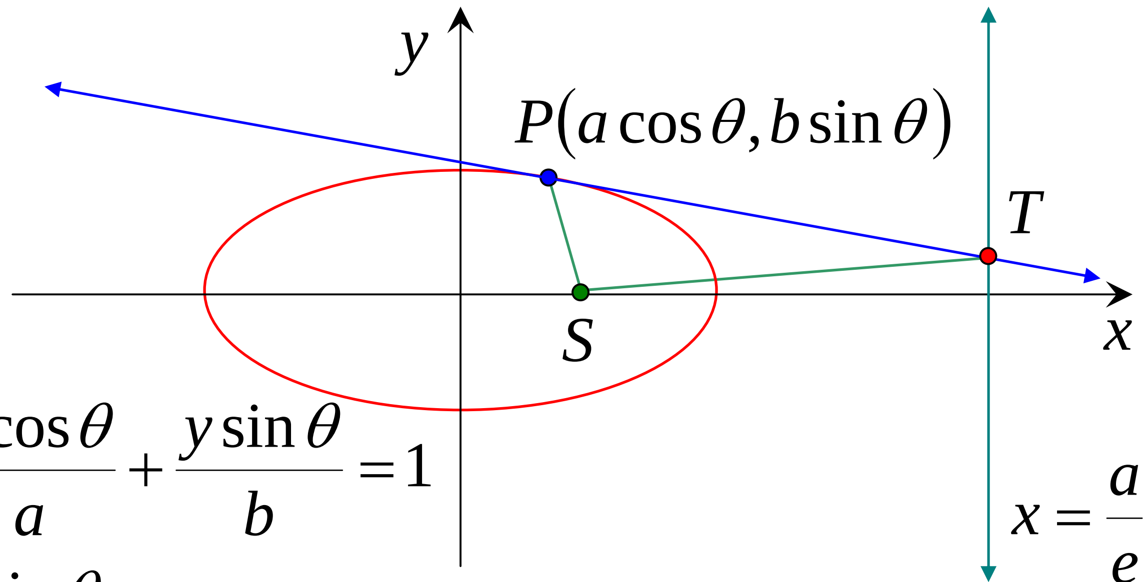
$$\frac{ae}{ae} + 0 = 1 + 0$$

$$= 1$$

$\therefore$ focus lies on chord of contact

<u>i.e. it is a focal chord</u>

(2) That part of the tangent between the point of contact and the directrix subtends a right angle at the corresponding focus.



<u>Prove:</u> $\angle PST = 90^\circ$

equation of tangent is $\dfrac{x\cos\theta}{a} + \dfrac{y\sin\theta}{b} = 1$

when $x = \dfrac{a}{e}$, $\dfrac{\frac{a}{e}\cos\theta}{a} + \dfrac{y\sin\theta}{b} = 1$

$$\frac{\cos\theta}{e} + \frac{y\sin\theta}{b} = 1$$

$$\frac{y\sin\theta}{b} = \frac{e-\cos\theta}{e}$$

$$y = \frac{b(e-\cos\theta)}{e\sin\theta}$$

$$\therefore T\left(\frac{a}{e}, \frac{b(e-\cos\theta)}{e\sin\theta}\right)$$

$$m_{PS}=\frac{b\sin\theta-0}{a\cos\theta-ae}$$

$$=\frac{b\sin\theta}{a(\cos\theta-e)}$$

$$m_{TS}=\frac{\dfrac{b(e-\cos\theta)}{e\sin\theta}-0}{\dfrac{a}{e}-ae}$$

$$=\frac{b(e-\cos\theta)}{e\sin\theta}\times\frac{e}{a-ae^2}$$

$$=\frac{b(e-\cos\theta)}{a(1-e^2)\sin\theta}$$

$$=\frac{b(e-\cos\theta)}{\dfrac{a^2(1-e^2)}{a}\sin\theta}$$

$$=\frac{a(e-\cos\theta)}{b\sin\theta}$$

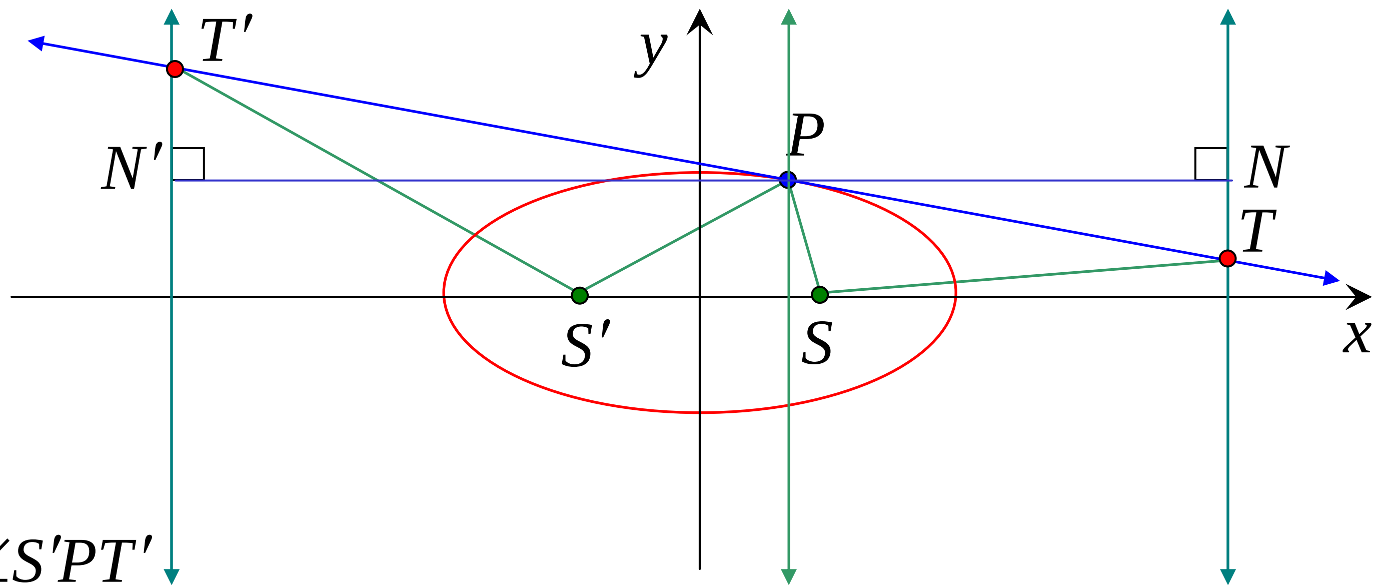
$$m_{PS}\times m_{TS}=\frac{b\sin\theta}{a(\cos\theta-e)}\times\frac{a(e-\cos\theta)}{b\sin\theta}$$

$$=-1$$

$$\therefore\angle PST=90^\circ$$

(3) Reflection Property

Tangent to an ellipse at a point P on it is equally inclined to the focal chords through P.



Prove: $\angle SPT = \angle S'PT'$

Construct a line || y axis passing through P

$$\frac{PT}{PT'} = \frac{PN}{PN'}$$ (ratio of intercepts of || lines)

$$\therefore \frac{PT}{PN} = \frac{PT'}{PN'}$$

$$ePN = PS \quad \text{and} \quad ePN' = PS'$$

$$\therefore \frac{PT}{\frac{PS}{e}} = \frac{PT'}{\frac{PS'}{e}}$$

$$\frac{PT}{PS} = \frac{PT'}{PS'}$$

$$\angle PST = \angle PS'T' = 90^\circ \quad \text{(proven in property (2))}$$

$$\therefore \sec\angle SPT = \sec\angle S'PT'$$

$$\angle SPT = \angle S'PT'$$

e.g. Find the cartesian equation of $|z+2|+|z-2|=8$

The sum of the focal lengths of an ellipse is constant

$$2a=8 \qquad ae=2 \qquad b^2=a^2\left(1-e^2\right)$$

$$a=4 \qquad 4e=2 \qquad b^2=16\left(1-\frac{1}{4}\right)$$

$$e=\frac{1}{2} \qquad =12$$

$\therefore$ locus is the ellipse $\dfrac{x^2}{16}+\dfrac{y^2}{12}=1$

Exercise 6E; 1, 2, 4, 7, 8, 10