CURVE SKETCHING

(A) FEATURES YOU SHOULD NOTICE ABOUT A GRAPH

(1) BASIC CURVES

The following *basic* curve shapes should be recognisable from the equation :

(a) Straight lines : $y = x$ (both pronumerals are to the power of one)

(b) Parabolas : $y = x^2$ (one pronumeral is to the power of one, the other the power of two)
NOTE : general parabola is $y = ax^2 + bx + c$

(c) Cubics : $y = x^3$ (one pronumeral is to the power of one, the other the power of three)
NOTE : general parabola is $y = ax^3 + bx^2 + cx + d$

(d) Polynomials in general

(e) Hyperbolas : $y = \frac{1}{x}$ or $xy = 1$ (one pronumeral is on the bottom of a fraction, the other is not OR pronumerals are multiplied together)

(f) Exponentials : $y = a^x$ (one pronumeral is in the power)

(g) Circles : $x^2 + y^2 = r^2$ (both pronumerals are to the power of two, coefficients are the same)

(h) Ellipses : $ax^2 + by^2 = k$ (both pronumerals are to the power of two, coefficients are NOT the same, *if signs are different then hyperbola*)

(i) Logarithmics : $y = \log_a x$

(j) Trigonometric : $y = \sin x$, $y = \cos x$, $y = \tan x$

(k) Inverse Trigonometric : $y = \sin^{-1} x$, $y = \cos^{-1} x$, $y = \tan^{-1} x$

(2) ODD AND EVEN FUNCTIONS

These curves have symmetry and are thus easier to sketch :

(a) ODD : $f(-x) = -f(x)$ (symmetric about the origin, that is has 180° rotational symmetry)

(b) EVEN : $f(-x) = f(x)$ (symmetric about the y axis)

(3) SYMMETRY IN THE LINE $y = x$

If x and y can be interchanged without changing the function, the curve is reflected in the line $y = x$ e.g. $x^3 + y^3 = 1$ (in other words the curve is its own inverse)

(4) DOMINANCE

As x gets large does a particular term dominate ?

(a) Polynomials : the leading term dominates e.g. $y = x^4 + 3x^3 - 2x + 2$, x^4 dominates.

(b) Exponentials : e^x tends to dominate as it increases so rapidly. e.g. $y = e^x - x^2$

(c) In General : look for the term that increases the most rapidly. i.e. which is the steepest ?

NOTE : It is a good idea to check your limit by substituting large numbers e.g. 1 000 000

(5) ASYMPTOTES

(a) Vertical Asymptotes : the bottom of a fraction cannot equal zero

(b) Horizontal/Oblique Asymptotes : top of a fraction is constant, the fraction can't equal zero

NOTE : If order of the numerator is $\geq$ the order of the denominator, perform a polynomial division, (curves can cross horizontal/oblique asymptotes, good idea to check)

(6) THE SPECIAL LIMIT

Remember the special limit seen in 2 Unit i.e. $\lim_{x \to 0} \frac{\sin x}{x} = 1$, it could come in handy when solving harder graphs.

(B) USING CALCULUS

Calculus is still a tremendous tool that should not be disregarded when curve sketching. However, often it is used as a final tool to determine critical points (points where the derivative is undefined), stationary points, inflections.

(1) CRITICAL POINTS

When $\frac{dy}{dx}$ is undefined the curve has a vertical tangent, these points are called critical points.

(2) STATIONARY POINTS

When $\frac{dy}{dx}=0$ the curve is said to be stationary, these points may be minimum turning points, maximum turning points or points of inflection.

(3) MINIMUM/MAXIMUM TURNING POINTS

(a) When $\frac{dy}{dx}=0$ and $\frac{d^2y}{dx^2}>0$, the point is called a minimum turning point.

(b) When $\frac{dy}{dx}=0$ and $\frac{d^2y}{dx^2}<0$, the point is called a maximum turning point.

NOTE : testing either side of $\frac{dy}{dx}$ for change can be quicker for harder functions

(4) INFLECTION POINTS

(a) When $\frac{d^2y}{dx^2}=0$ and $\frac{d^3y}{dx^3}\neq 0$, the point is called an inflection point.

NOTE : testing either side of $\frac{d^2y}{dx^2}$ for change can be quicker for harder functions

(b) When $\frac{dy}{dx}=0$, $\frac{d^2y}{dx^2}=0$ and $\frac{d^3y}{dx^3}\neq 0$, the point is called a horizontal point of inflection.

(5) INCREASING/DECREASING CURVES

(a) When $\frac{dy}{dx}>0$ the curve has a positive sloped tangent and is thus increasing.

(b) When $\frac{dy}{dx}<0$ the curve has a negative sloped tangent and is thus decreasing.

(6) IMPLICIT DIFFERENTIATION

This technique allows you differentiate complicated functions.

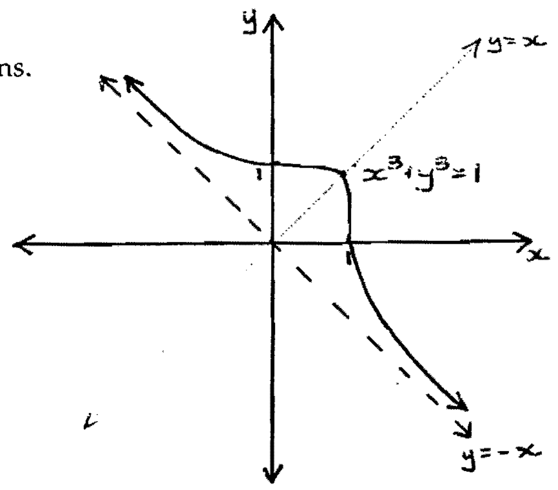
e.g. Sketch $x^3+y^3=1$

NOTE :
- the curve has symmetry in $y=x$
- it passes through (1,0) and (0,1)
- it is asymptotic to the line $y=-x$

$\because y^3=1-x^3$

i.e. $y^3\neq -x^3$

$\therefore y\neq -x$



On differentiating implicitly:

$$3x^2+3y^2\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{-x^2}{y^2}$$

This means that $\frac{dy}{dx}<0$ for all x except at (1,0), which is a critical point and (0,1) which turns out to be a horizontal point of inflection.

(C) TRANSFORMATIONS

Given that the graph of $y = f(x)$ can be sketched, then it is possible to build other sketches through appropriate transformations :

- $y = f(x) \pm a$ OR $(y \mp a) = f(x)$, a is grouped with y,(shift $f(x)$ up or down by a)
- $y = f(x \pm a)$ a is grouped with x,(shift $f(x)$ left or right by a)
- $y = -f(x)$ (reflect $f(x)$ in the x axis)
- $y = f(-x)$ (reflect $f(x)$ in the y axis)
- $y = |f(x)|$ (reflect the part of $f(x)$ where $f(x) < 0$ in the x axis)
- $y = f(|x|)$ (reflect the part of $f(x)$ where $x > 0$ in the y axis)
- $y = kf(x)$ (stretch $f(x)$ vertically, k<1 shallower, k>1 steeper)
- $y = f(kx)$ (stretch $f(x)$ horizontally, k<1 shallower, k>1 steeper)

Exercises

1. Sketch $y = \sin x$ and then sketch all of the above transformations where a = π and k = 2.

2. Sketch the following (start each sketch with the basic curve)

(a) $y = 2e^{3x} - 1$

(b) $y = 1 - 2e^{|x|}$

(c) $y = 3 + 2\sin(2x + \pi)$

3. Sketch $x^2 + 4x + y^2 - 8y = 12$

(D) ADDITION AND SUBTRACTION OF ORDINATES

$y = f(x) + g(x)$ can be graphed by first graphing $y = f(x)$ and $y = g(x)$ separately and then adding their ordinates together.

NOTE : First locate points on $y = f(x) + g(x)$ corresponding to $f(x) = 0$ and $g(x) = 0$, then plot further points by addition and subtraction of ordinates and finally locate the position of stationary points.

$y = f(x) - g(x)$ can be graphed by first graphing $y = f(x)$ and $y = -g(x)$ separately and then adding their ordinates together.

Exercises

Sketch the following curves :

(a) $y = x + \dfrac{1}{x}$

(b) $y = x + \sin x \;,\; 0 \le x \le 4\pi$

(c) $y = x - \log x$

(E) MULTIPLICATION OF FUNCTIONS

The graph of $y = f(x) \cdot g(x)$ can be graphed by first graphing $y = f(x)$ and $y = g(x)$ separately and then examining the sign of this product. Special note needs to be made of points where either $f(x) = 0$ or 1 or $g(x) = 0$ or 1.

NOTE : The regions on the number plane through which the graph must pass should be shaded in as the first step.

Exercises

Sketch the following curves :

(a) $y = (x-1)(x-3)(x+2)$

(b) $y = (x-2)(x+2)^2$

(c) $y = xe^{-x}$

(F) DIVISION OF FUNCTIONS

The graph of $y = \dfrac{f(x)}{g(x)}$ can be graphed by:

STEP 1 First graph $y = f(x)$ and $y = g(x)$ separately.
STEP 2 Mark in the vertical asymptotes.
STEP 3 Shade in the regions in which the curve must be (same as multiplication)
STEP 4 Investigate the behaviour of the function for large values of x (find horizontal/oblique asymptotes, look at dominance)

Exercises

Sketch the following curves :

(a) $y = \dfrac{x(x+1)}{(x-2)}$

(b) $y = \dfrac{x^2}{(x+2)(x-1)}$

(c) $y = \dfrac{\sin x}{x}$

(G) GRAPHS OF RECIPROCAL FUNCTIONS

The graph of $y = \dfrac{1}{f(x)}$ can be sketched by first drawing $y = f(x)$ and noticing :

- when $f(x) = 0$, then $\dfrac{1}{f(x)}$ is undefined, (i.e. a vertical asymptote exists)
- when $f(x) \to \infty$, then $\dfrac{1}{f(x)} \to 0$, (i.e. asymptotes become x intercepts, unless undefined)
- when $f(x)$ is increasing, the reciprocal is decreasing, and visa-versa
- when $f(x)$ is positive, $\dfrac{1}{f(x)}$ is positive, etc.
- the derivative of $\dfrac{1}{f(x)}$ is $\dfrac{-f'(x)}{[f(x)]^2}$, hence stationary points of the original curve are stationary points of its reciprocal.

Exercises

Sketch the following curves :

(a) $y = \dfrac{1}{\log x}$

(b) $y = \dfrac{1}{(x-2)(x-1)}$

Exercise 1.1

1 i) Sketch the graph of $y=(x-2)(x^2+6x)$, showing the x-intercepts only. Hence, state the values of x for which the curve is positive or negative.

ii) Sketch the following curves. You are not required to find the co-ordinates of any turning points.

a) $y=\dfrac{1}{(x-2)(x^2+6x)}$ b) $y=\dfrac{x-2}{x^2+6x}$ c) $y=\dfrac{x}{(x-2)(x+6)}$ d) $y=\dfrac{x^2+6x}{x-2}$

2 Sketch the following curves. Also use Calculus to show the co-ordinates of any turning points.

a) $y=\dfrac{16}{x^3-12x}$ b) $y=\dfrac{4x+5}{x^2-1}$ c) $y=\dfrac{(x+1)^2}{(x-1)(x-3)}$ d) $y=\dfrac{(x-1)(x+2)}{x-2}$

3 Sketch the following curves, showing the main features (the x-intercepts, the y-intercept, any asymptotes and the coordinates of any turning points)

a) $y=\dfrac{1}{(x+1)(x+3)}$ b) $y=\dfrac{1}{x^2(x-3)}$ c) $y=\dfrac{x+2}{(x-1)(x+4)}$ d) $y=\dfrac{9-2x}{x(x-4)}$

e) $y=\dfrac{x^2}{2-x}$ f) $y=\dfrac{(x+1)(x-4)}{(x-1)(x+3)}$ g) $y=\dfrac{x^3}{x^2-1}$ h) $y=\dfrac{x^2-1}{x^2+1}$

4 i) Discuss the behaviour of these curves for very small and very large values of x.

ii) Find y', hence, the coordinates of the turning points.

iii) Hence, sketch the following curves.

a) $y=x+\dfrac{1}{x}$ b) $y=x^2+\dfrac{16}{x}$ c) $y=x-\dfrac{4}{x^2}$ d) $y=x-\dfrac{1}{x^3}$

5 i) Discuss the behaviour of the curve at the neighbourhood of $x=1$ and for very large values of x.

ii) Find y', hence, the coordinates of any turning points.

iii) Hence, sketch the following curves.

a) $y=x-1+\dfrac{1}{x-1}$ b) $y=1-x+\dfrac{1}{x-1}$ c) $y=x+1+\dfrac{1}{1-x}$ d) $y=x-1+\dfrac{1}{(x-1)^2}$

6 For the curve $y=\dfrac{x^2+2x+4}{x^2-x-6}$, discuss its behaviour as $x\to\pm\infty$, hence, sketch it. Do not use Calculus.

7 The graph of $f(x)$ is shown in fig. 1.5. Draw separate diagrams for

a) $y=\dfrac{1}{f(x)}$ b) $y=f(-x)$ c) $y=f(|x|)$ d) $y=|f(x)|$

e) $y=f(x-2)$ f) $y=f(2-x)$ g) $y=f'(x)$ h) $|y|=f(x)$

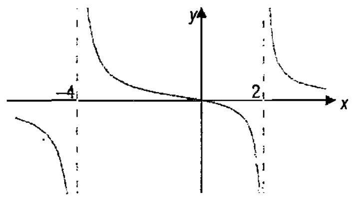


fig. 1.5

8 Repeat question 7 for the following curve

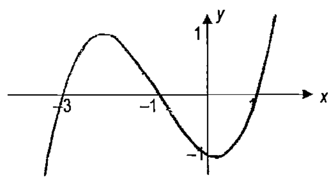


fig. 1.6

9 The graph of $f(x)$ is shown below in fig. 1.7. Draw separate diagrams for

a) $y=xf(x)$ b) $y=x+f(x)$ c) $y=x+1+f(x)$ d) $y=x^2f(x)$

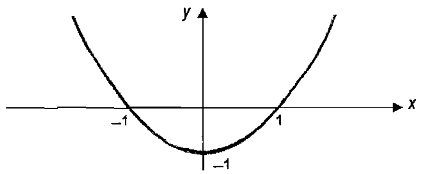


fig. 1.7

10 Repeat question 9 for the following curve

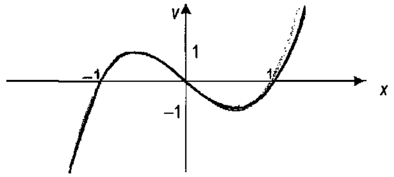


fig. 1.8

11 a) Show that $\dfrac{2x}{1+x^2}\le 1$.

b) Show further that $\dfrac{2x}{1+x^2}$ is an odd function. State the graphical meaning of odd functions.

c) Hence, sketch the graph of $y=\dfrac{2x}{1+x^2}$.

12 Sketch the curve $y=\dfrac{x^2-4}{x-a}$, where a is a real number.

(Hint: Consider these cases: $a<-2$, $a=-2$, $-2<a<2$, $a=2$ and $a>2$.)

Checklist

- ☐ Can you find the equations of the asymptotes of rational functions?
- ☐ Can you state the region of the plane where a rational function is above or below the x-axis?
- ☐ Can you sketch the reciprocal curve from the equation of a function?
- ☐ Can you sketch the reciprocal curve from the graph of a curve?
- ☐ Can you sketch $\dfrac{f(x)}{g(x)}$ and $f(x).g(x)$, $f(x)\pm g(x)$ given functions $f(x)$ and $g(x)$?
- ☐ Can you sketch the derivative curve from the graph of a curve?
- ☐ Can you sketch the curves $y=|f(x)|, f(-x), f(|x|), f(x\pm a), f(x)\pm a$ and $|y|=f(x)$ from the graph of $y=f(x)$?

(H) GRAPHS OF THE FORM $y=[f(x)]^n$, where $n>1$ and an integer

The graph of $y=[f(x)]^n$ can be sketched by first drawing $y=f(x)$ and noticing :

- All stationary points must still be stationary points
- All points where the curve cuts the x-axis are also stationary points on the x-axis
- If $|f(x)|>1$ then $|[f(x)]^n|>f(x)$
- If $|f(x)|<1$ then $|[f(x)]^n|<f(x)$
- If n is even then $[f(x)]^n \geq 0$
- If n is odd then the sign of $[f(x)]^n$ is the same as the sign of $f(x)$ for any given value of x

Exercises

(a) If $f(x)=x(x^2-3)$ then sketch $[f(x)]^2$ and $[f(x)]^3$

(b) Sketch $y=\sin^2 x$ and $y=\sin^3 x$

Exercise 1.2

1 Sketch the following pairs of curves, showing any asymptotes and maximum, minimum points

a) $y = 2x - 1$ and $y = (2x-1)^2$

b) $y = \dfrac{2}{x^2+1}$ and $y = \dfrac{4}{(x^2+1)^2}$

c) $y = \dfrac{1}{2x+1}$ and $y = \dfrac{1}{(2x+1)^2}$

d) $y = \dfrac{x}{x^2-1}$ and $y = \dfrac{x^2}{(x^2-1)^2}$

e) $y = \dfrac{2x}{x^2+1}$ and $y = \dfrac{4x^2}{(x^2+1)^2}$

f) $y = \dfrac{1}{2x(x+1)}$ and $y = \dfrac{1}{4x^2(x+1)^2}$

2 Sketch the following pairs of curves, showing important features

a) $y = x^2 - 2$ and $y = (x^2-2)^3$

b) $y = \dfrac{2x}{x^2+1}$ and $y = \dfrac{8x^3}{(x^2+1)^3}$

c) $y = \dfrac{x^2-1}{x}$ and $y = \dfrac{(x^2-1)^3}{x^3}$

d) $y = x^2(x^2-2)$ and $y = \left(x^2(x^2-2)\right)^3$

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3 Refer to question 2, without any further calculation, sketch the following curves

a) $y = (x^2-2)^4$ b) $y = \dfrac{x^4}{(x^2+1)^4}$ c) $y = \dfrac{(x^2-1)^4}{x^4}$ d) $y = \left(x^2(x^2-2)\right)^4$

4 Sketch the following curves, showing important features. You may use Calculus if necessary.

a) $y = x$, $y = x^2$ and $y = x^3$

b) $y = \dfrac{1}{1-x}$, $y = \dfrac{1}{(1-x)^2}$ and $y = \dfrac{1}{(1-x)^3}$

c) $y = \sin x$, $y = \sin^2 x$ and $y = \sin^3 x$

d) $y = \dfrac{1}{x+1}$, $y = \dfrac{1}{x^2+1}$ and $y = \dfrac{1}{x^3+1}$

Hence, or otherwise, determine whether each of the following statements is true or false:

α) $\displaystyle\int_0^{\frac{\pi}{2}} \sin^2 x\,dx < \int_0^{\frac{\pi}{2}} \sin^3 x\,dx$

β) $\displaystyle\int_2^3 \frac{1}{1-x}\,dx < \int_2^3 \frac{1}{(1-x)^4}\,dx$

γ) $\displaystyle\int_0^1 x^{2000}\,dx < \int_0^1 x^{2001}\,dx$

δ) $\displaystyle\int_0^1 \frac{1}{1+x^{2000}}\,dx < \int_0^1 \frac{1}{1+x^{2001}}\,dx$

5 The curve $y = \tan^{-1} x$ is shown below. Without using Calculus, sketch the curves of

a) $y = (\tan^{-1} x)^2$

b) $y = (\tan^{-1} x)^3$

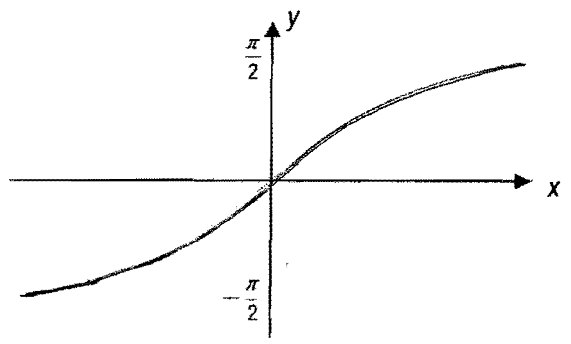


fig. 1.11

6 Sketch the following curves, showing any asymptotes, maximum and minimum points.

a) $y = (\ln x)^2$ b) $y = \ln(x^2)$ c) $y = \ln(\frac{1}{x^2})$ d) $y = x \ln x$

e) $y = x(\ln x)^2$ f) $y = x^2 \ln x$ g) $y = x^2(\ln x)^2$ h) $y = x^2 \ln(x^2)$

i) $y = e^{\frac{1}{x}}$ j) $y = \dfrac{e^x}{x}$ k) $y = \dfrac{e^x}{x^2}$ l) $y = \dfrac{x^2}{e^x}$

(I) GRAPHS OF THE FORM $y = \sqrt{f(x)}$

The graph of $y = \sqrt{f(x)}$ can be sketched by first drawing $y = f(x)$ and noticing :

- $\sqrt{f(x)}$ is only defined if $f(x) \geq 0$
- $\sqrt{f(x)} \geq 0$ for all x in the domain
- $\sqrt{f(x)} < f(x)$ if $f(x) > 1$, and $\sqrt{f(x)} > f(x)$ if $0 < f(x) < 1$
- $\dfrac{dy}{dx} = \dfrac{f'(x)}{2\sqrt{f(x)}}$ implies there are critical points where $f(x) = 0$

Exercises

Sketch :

(a) $y = \sqrt{x^3 - 4x}$

(b) $y = \sqrt{\dfrac{x(x+1)}{(2-x)}}$

1 a) Sketch the cubic curve $y=(x-3)^2(x-1)$, hence, draw the curve $y^2=(x-3)^2(x-1)$.

b) Use Calculus to discuss the gradients at any x-intercepts and to find the coordinates of any turning points.

2 For each of the following curves $y^2=f(x)$,

i) Find the domain by solving $f(x)\geq 0$.

ii) Find y', hence, the gradients at any x-intercepts and the coordinates of any turning points.

iii) Sketch the curve.

a) $y^2=x(x-1)$ b) $y^2=4-x^4$ c) $y^2=(x+5)(x-1)(x-4)$ d) $y^2=(x^2-1)(x^2-4)$

e) $y^2=x(x-1)^2$ f) $y^2=x^2(1-x^2)$ g) $y^2=(x-1)^2(x-3)$ h) $y^2=\dfrac{1}{x^2-4}$

3 Given the curve $y=\dfrac{x^2}{1-x^2}$.

a) Determine whether the curve is odd or even, hence, state its geometrical meaning.
b) Discuss the behaviour of the curve for very large values of x.
c) Find y' and the coordinates of any turning points.
d) Hence, sketch the curve.
e) Without doing any further calculation, sketch the curve $y^2=\dfrac{x^2}{1-x^2}$.

4 Repeat question 3 for the following curves.

i) $y=\dfrac{4x^2}{1+x^2}$ and $y^2=\dfrac{4x^2}{1+x^2}$ ii) $y=\dfrac{4x}{1+x^2}$ and $y^2=\dfrac{4x}{1+x^2}$

iii) $y=\dfrac{1-x^2}{x}$ and $y^2=\dfrac{1-x^2}{x}$ iv) $y=e^x-e^{-x}$ and $y^2=e^x-e^{-x}$

v) $y=\dfrac{1}{e^x+e^{-x}}$ and $y^2=\dfrac{1}{e^x+e^{-x}}$ vi) $y=\dfrac{e^x-e^{-x}}{e^x+e^{-x}}$ and $y^2=\dfrac{e^x-e^{-x}}{e^x+e^{-x}}$

5 The sketch of $y=f(x)$ is shown in fig. 1.14. Draw separate diagrams for

a) $y=\sqrt{f(x)}$ b) $y=f^2(x)$ c) $y=\dfrac{1}{f(x)}$ d) $y^2=\dfrac{1}{f(x)}$

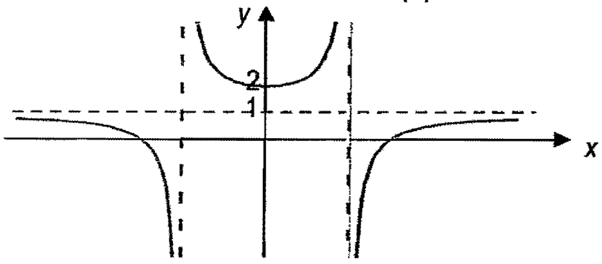


fig. 1.14

6 Repeat question 5 for the following curve

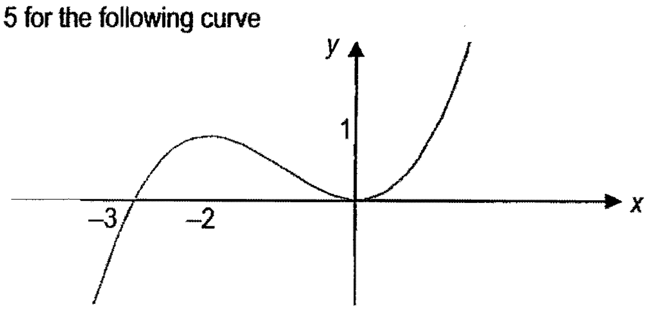


fig. 1.15

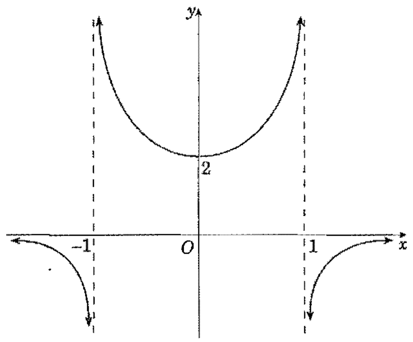
7 Sketch the following curves

a) A traffic officer on point duty $x^2y^2=x^2+y^2$. b) A dumb-bell $y^2=x^4-x^6$.

(J) Other Graphs

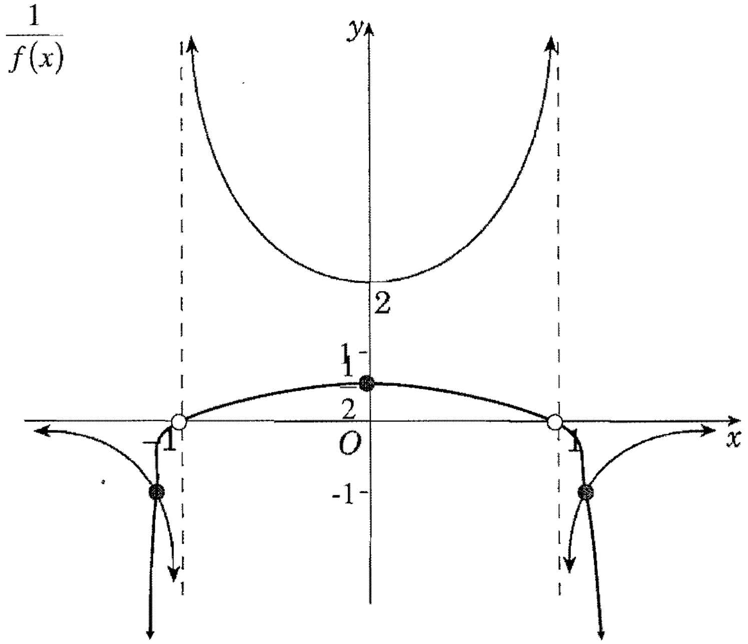
e.g. (*i*) (2003)

The diagram shows the graph of $y = f(x)$

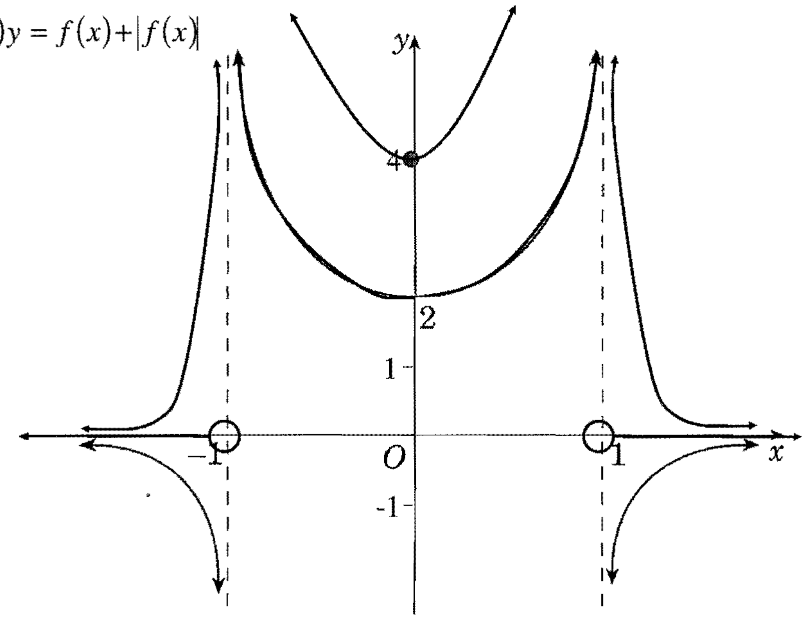


Draw separate sketches of the graphs of the following;

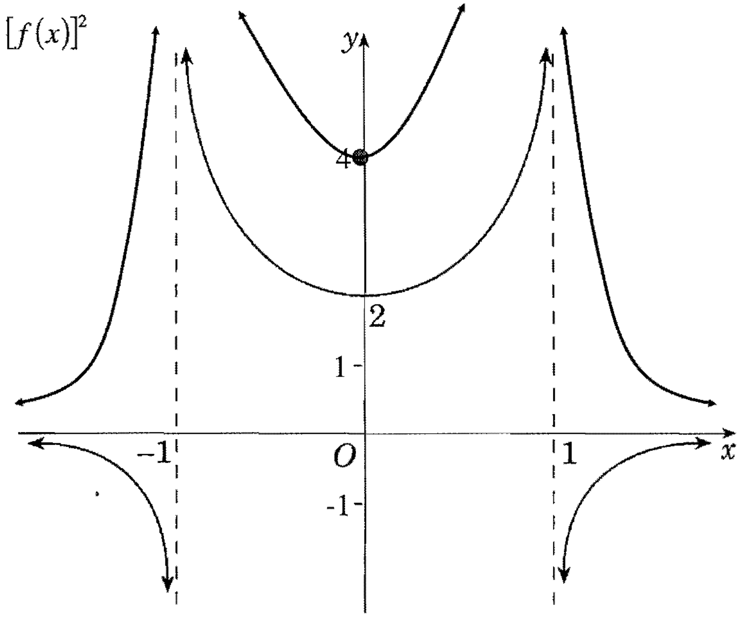
$(i)\ y = \dfrac{1}{f(x)}$



$(ii)\ y = f(x) + |f(x)|$



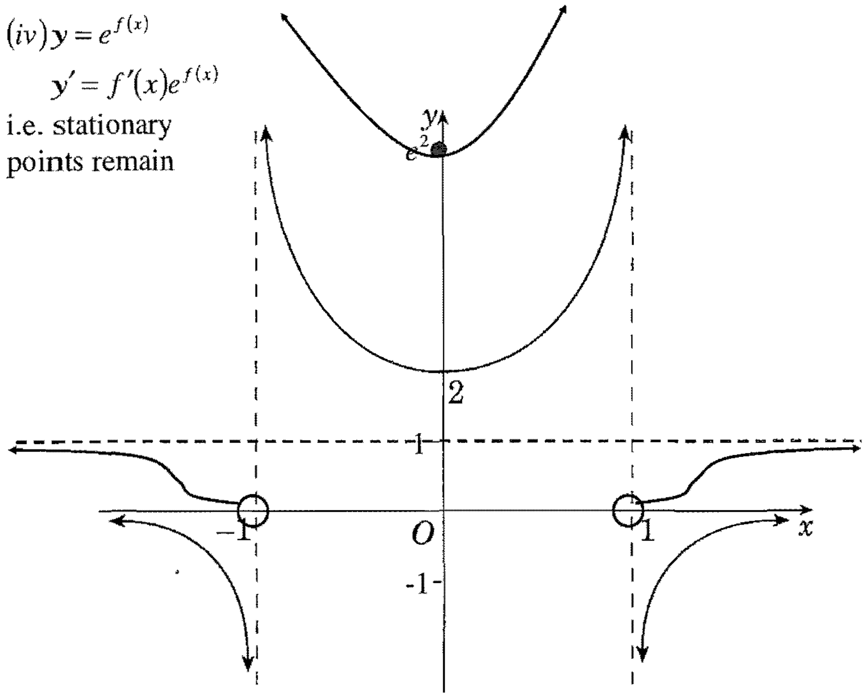
$(iii)\ y = [f(x)]^2$



$(iv)\ y = e^{f(x)}$

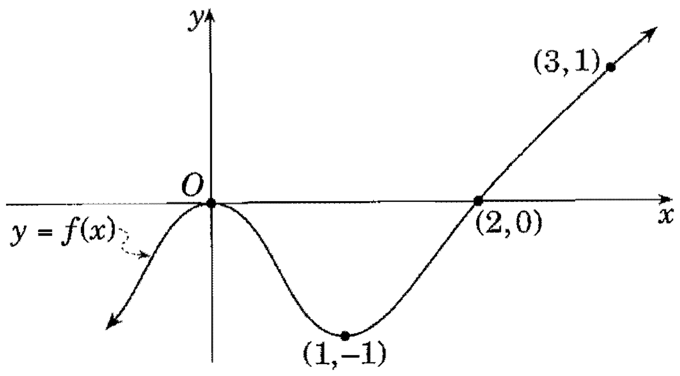
$y' = f'(x)e^{f(x)}$

i.e. stationary points remain



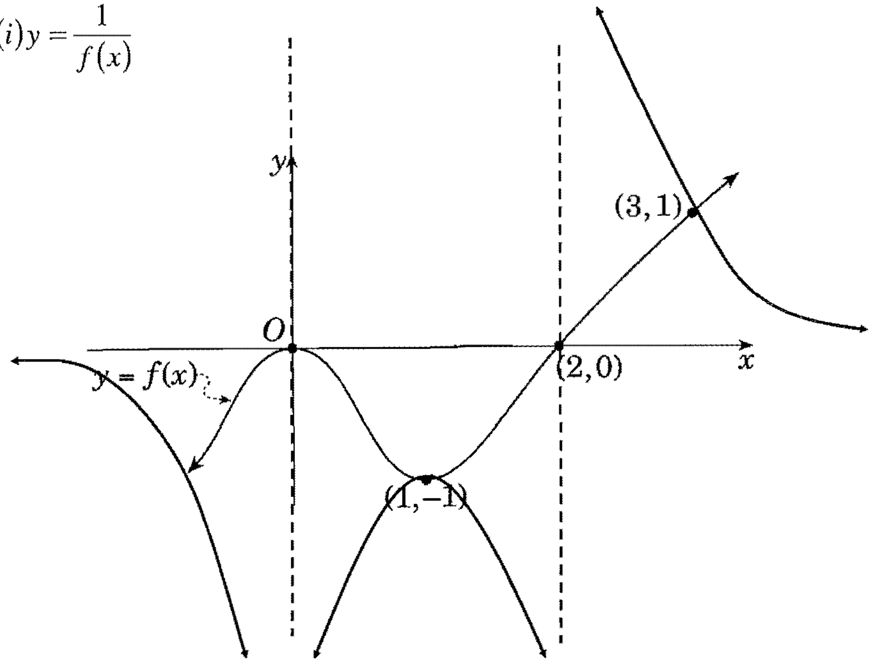
e.g. (*ii*) (2002)

The diagram shows the graph of $y = f(x)$

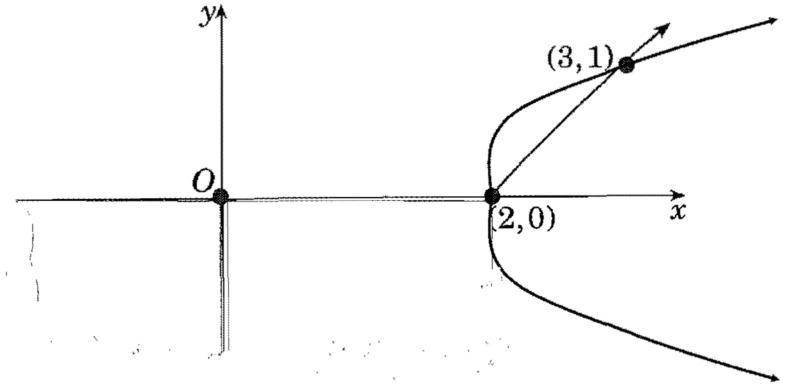


Draw separate sketches of the graphs of the following;

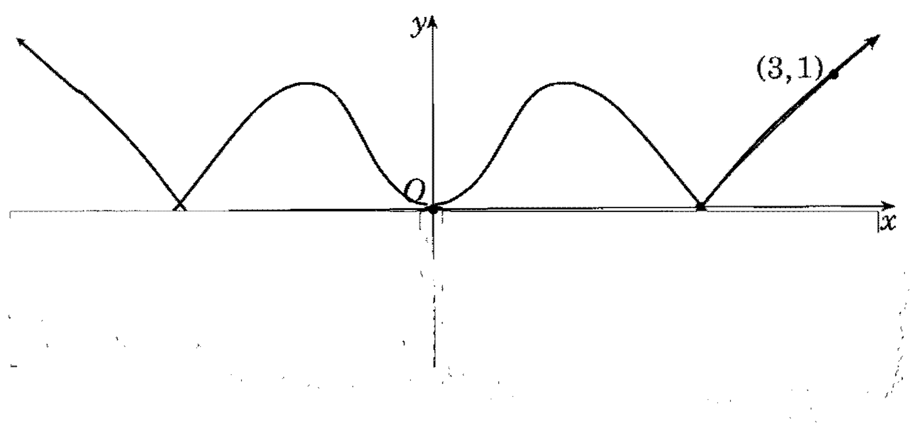
$(i)\ y = \dfrac{1}{f(x)}$



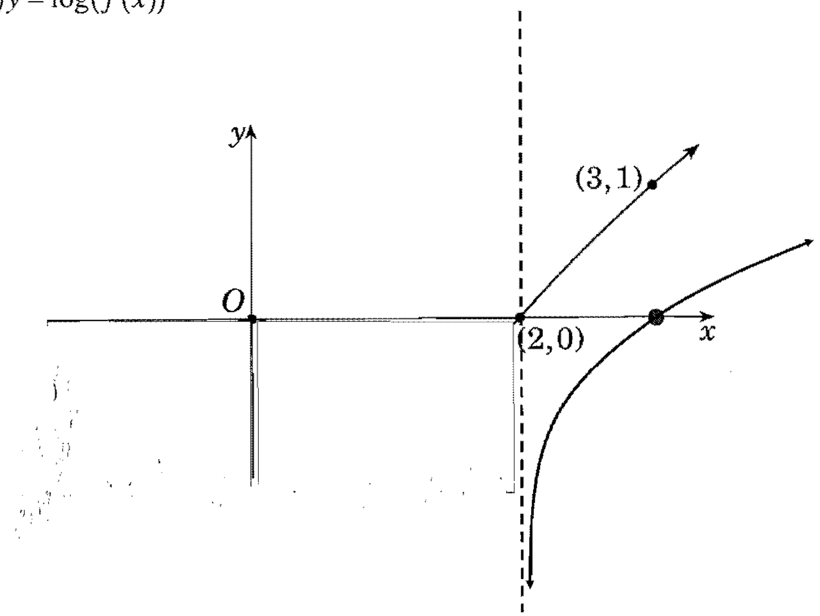
$(ii)\ y^2 = f(x)$



$(iii)\ y=|f(|x|)|$



$(iv)\ y=\log(f(x))$



Exercise 1A; 9, 10, 11a, 12

Exercise 1B; 2bd, 9egh, 11ace, 13af, 14abf, 30, 32e, 34

1.5 Review Exercise

1 E6 Draw careful sketches of the following curves, indicating clearly any asymptotes. Do not use Calculus.

a) $y=\dfrac{x}{x^2-1}$ b) $y=\dfrac{x-1}{x(x+1)}$ c) $y=\dfrac{x^2}{x^2-1}$ d) $y=\dfrac{x(x-1)}{x+1}$

e) $y=\dfrac{5-x}{(x-2)(x+1)}$ f) $y=\dfrac{x-2}{(x+1)(5-x)}$ g) $y=\dfrac{2x^2+4x+3}{x^2-1}$ h) $y=\dfrac{(2-x)(x+1)}{x^2+1}$

2 E6 Sketch the following curves, clearly showing any asymptotes, *x*-intercepts, *y*-intercept, the coordinates of any turning points and points of inflexion, where possible.

a) $y=2x^2-x^4$ b) $y=2x^5+5x^2-3$ c) $y=x^3+3x^2+3$ d) $y=x^5+x-1$

e) $y=x^5-30x^3+500x$ f) $y=x^4+4x-4$ g) $y=\dfrac{x^2-x+2}{x+1}$ h) $y=\dfrac{1}{x(x-1)^2}$

i) $y=\dfrac{2(x-1)^2}{x^2}$ j) $y=\dfrac{2(x-1)^2}{x^2+1}$ k) $y=\dfrac{x^2-1}{x^2+1}$ l) $y=\dfrac{x^3-1}{x}$

3 E6 a) Sketch the function $g(x)=xe^{-x}$ for $x\ge -1$.

b) Given *g*(*x*) as in (a) above, the function *f*(*x*) may be given by the rule:

$g(-x), x\le 1$

$g(x-2), x\ge 1$

Find the zeros of the function, and the maximum and minimum values.
Draw a sketch of the graph of *f*(*x*).

4 E6 The function *f*(*x*) has derivative $f'(x)$ whose graph appears below.
You are given that
$f'(-2)=f'(1)=0$ and $f'(x)\to\infty$ as $x\to-\infty$ and $f'(x)\to 0$ as $x\to\infty$.
Also, $f(0)=0$ and $f(3)>0$.
a) Describe the behaviour of *f*(*x*) as $x\to\infty$.
b) Sketch the graph of *f*(*x*) showing its behaviour at its stationary points.

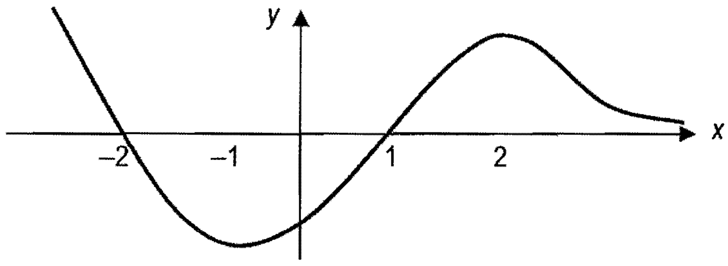


fig. 1.26

5 E6 a) Sketch the graph of $g(x)=x^4-4x^3+4x^2-\frac{1}{2}$, showing that it has four real zeros. You are not required to find the coordinates of any *x*-intercepts.
b) On different diagrams, sketch the curves

i) $y=|g(x)|$ ii) $y^2=g(x)$.

c) Calculate the slope of the curve $y^2=g(x)$ at any point *x* and describe the nature of the curve $y^2=g(x)$ at the zeros of $g(x)$.

6 E6 A function is defined by $f(x)=\dfrac{\ln x}{x}$, for $x>0$.

a) Prove that the graph of *f*(*x*) has a relative maximum point at $x=e$ and a point of inflexion at $x=e^{\frac{3}{2}}$.

b) Discuss the behaviour of $f(x)$ in the neighbourhood of $x=0$ and for large values of x.
c) Hence, draw a clear sketch of $f(x)$ indicating on it these features.
d) Without further calculation draw separate sketches of the graphs of

i) $y=\left|\frac{\ln x}{x}\right|$ ii) $y=\frac{x}{\ln x}$ iii) $y^2=\frac{\ln x}{x}$

7 E6 Let $f(x)=2-x-x^2$.
On separate diagrams, sketch the following graphs without using Calculus

a) $y=|f(x)|$ b) $y=f^2(x)$ c) $y=\sqrt{f(x)}$ d) $y=f(x-2)$
e) $y=f(-x)$ f) $y=f(|x|)$ g) $|y|=f(x)$ h) $y=\log_e f(x)$
i) $y=e^{f(x)}$ j) $y=\frac{1}{f(x)}$ k) $y=\frac{x}{f(x)}$ l) $y=f(\log_e x)$

8 E6 Sketch the following pairs of graphs without using Calculus

a) $y=\frac{x+1}{(x-2)^2}$ and $y=\frac{\sqrt{x+1}}{x-2}$ b) $y=\frac{(x-1)^2}{x+1}$ and $y=\frac{x-1}{\sqrt{x+1}}$

9 E6 Draw the following curves, showing any turning points, points of inflexion, and asymptotes, where possible.

a) $y=\frac{x+1}{e^x}$ b) $y=\frac{e^x}{x^2+1}$ c) $y=\frac{e^x}{(x+1)^2}$ d) $y=\frac{e^x}{x^2-3}$

10 E6 Given the curve $x^2-y^2+xy=5$.
a) Find $\frac{dy}{dx}$ hence, find the points on the curve whose tangents are vertical or horizontal.
b) Discuss the behaviour of the curve for large values of x.
c) Hence, sketch the curve.

11 E2 Given $y=2\sin x+\cos 2x, -2\pi\le x\le 2\pi$.
a) Sketch the curve, showing any turning points.
b) Show that there are three values of x that satisfy the equation $2\sin x+\cos 2x=\frac{1}{2}x$ and find approximations to these values.

12 E2 True or false? Explain.

a) $\int_0^{\pi/4}\tan x\,dx<\int_0^{\pi/4}\tan^2 x\,dx$ b) $\int_0^1\frac{1}{1+x}dx<\int_0^1\frac{1}{\sqrt{1+x}}dx$

13 E2 A is a point on the circumference of a circle of radius a. Using A as the centre, an arc of radius r is drawn, $r<2a$, to intercept the circle at two points B and C.
a) If $\angle BAC=2\theta$ and the arc length $BC=\ell$, show that $r=2a\cos\theta$, and $\ell=4a\theta\cos\theta$.
b) Hence, show that ℓ is maximum when $\theta=\cot\theta$.
c) By using a graphical means, find θ correct to the nearest degree.

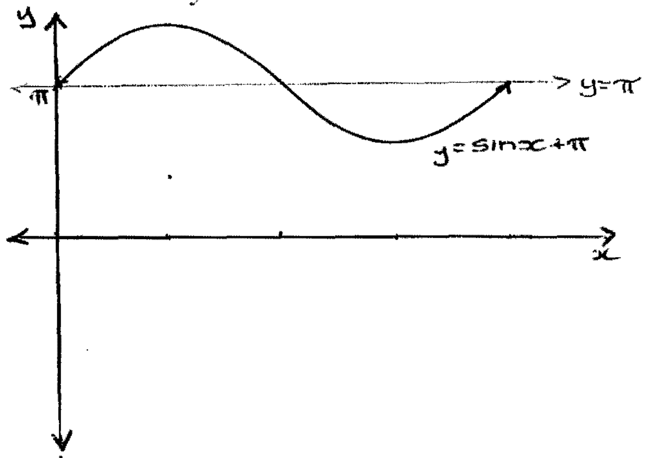
(C) TRANSFORMATIONS

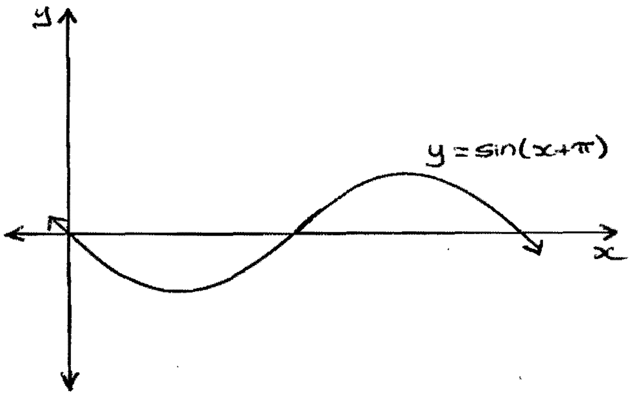
Given that the graph of $y = f(x)$ can be sketched, then it is possible to build other sketches through appropriate transformations :

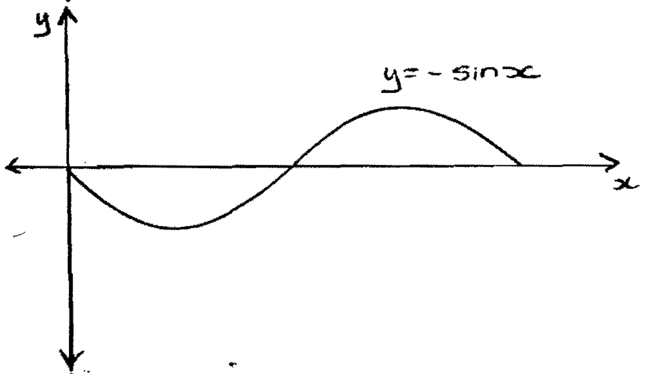
- $y = f(x) \pm a$ OR $(y \mp a) = f(x)$, a is grouped with y,(shift $f(x)$ up or down by a)
- $y = f(x \pm a)$ a is grouped with x,(shift $f(x)$ left or right by a)
- $y = -f(x)$ (reflect $f(x)$ in the x axis)
- $y = f(-x)$ (reflect $f(x)$ in the y axis)
- $y = |f(x)|$ (reflect the part of $f(x)$ where $f(x) < 0$ in the x axis)
- $y = f(|x|)$ (reflect the part of $f(x)$ where $x > 0$ in the y axis)
- $y = kf(x)$ (stretch $f(x)$ vertically, k<1 shallower, k>1 steeper)
- $y = f(kx)$ (stretch $f(x)$ horizontally, k<1 steeper, k>1 shallower)

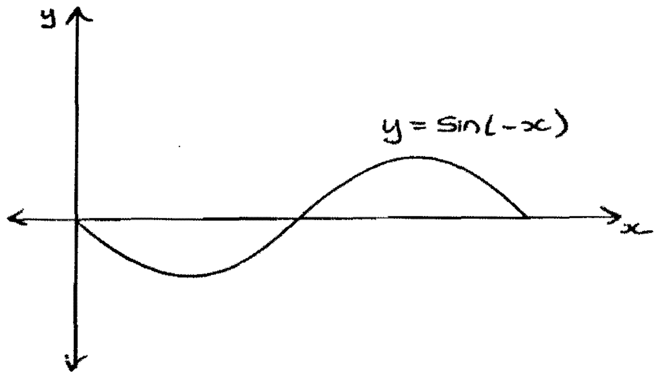
Exercises

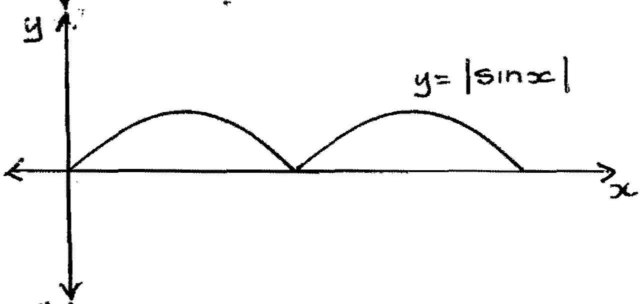
1. Sketch $y = \sin x$ and then sketch all of the above transformations where a = π and k = 2.

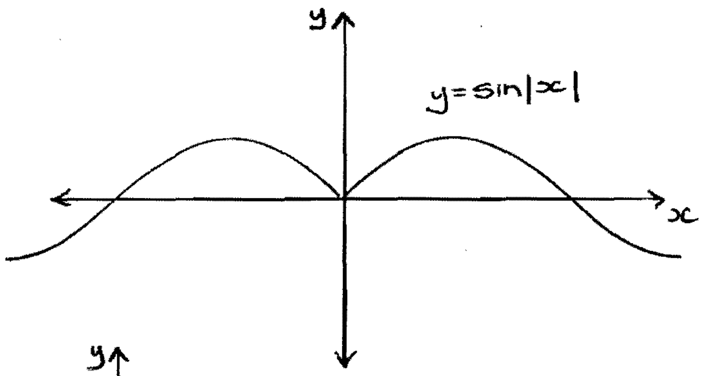


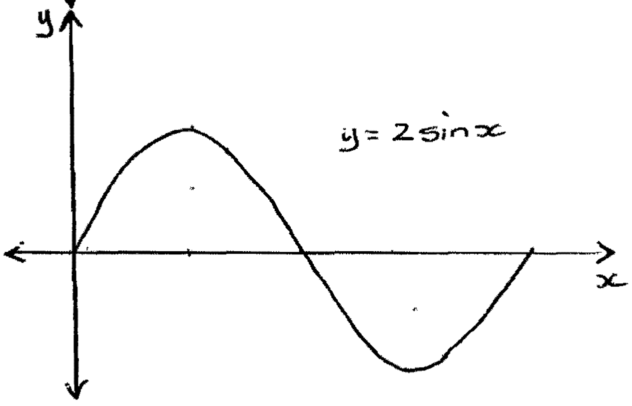


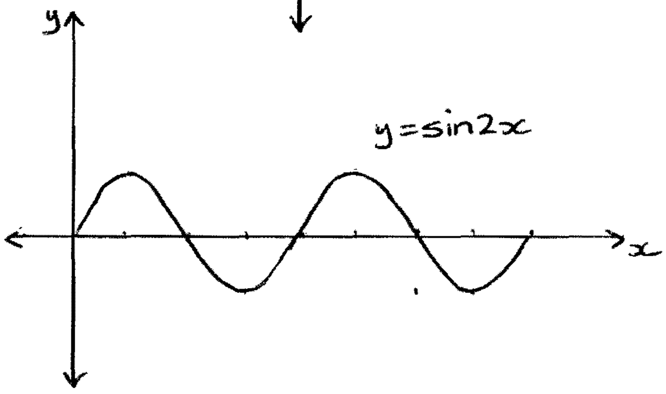












Sketch the following (start each sketch with the basic curve)

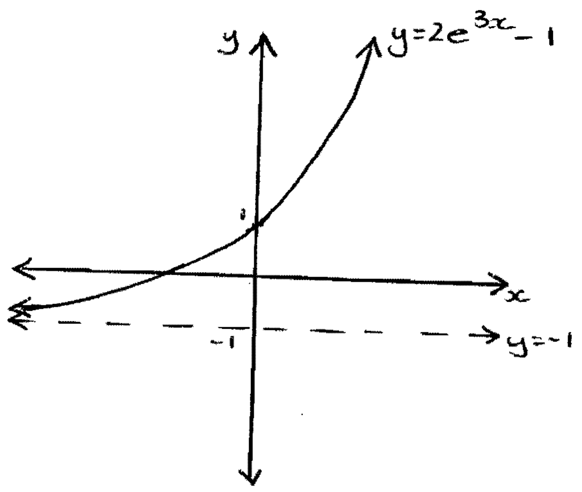
(a) $y = 2e^{3x} - 1$

basic curve $y = e^{x}$

$y = e^{3x}$ ← steeper

$y = 2e^{3x}$ ← steeper again

$y = 2e^{3x} - 1$ ← down 1



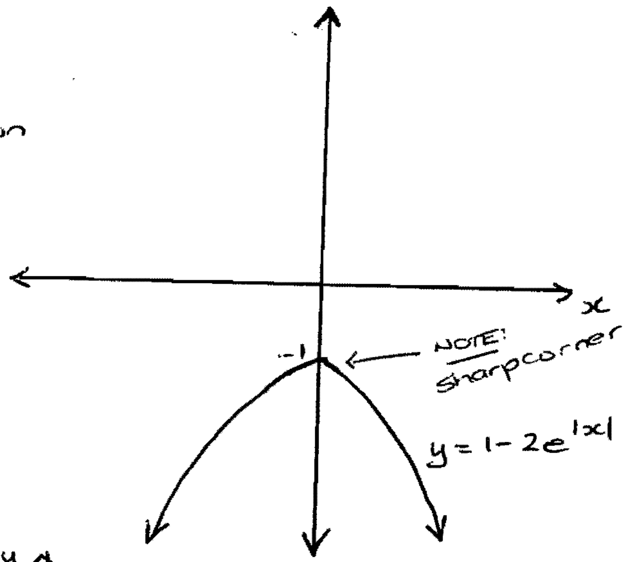
(b) $y = 1 - 2e^{|x|}$

basic curve $y = e^{x}$

$y = e^{|x|}$ ← relect in y axis

$y = -2e^{|x|}$ ← steeper, upsidedown

$y = 1 - 2e^{|x|}$ ← up 1



(c) $y = 3 + 2\sin(2x + \pi)$

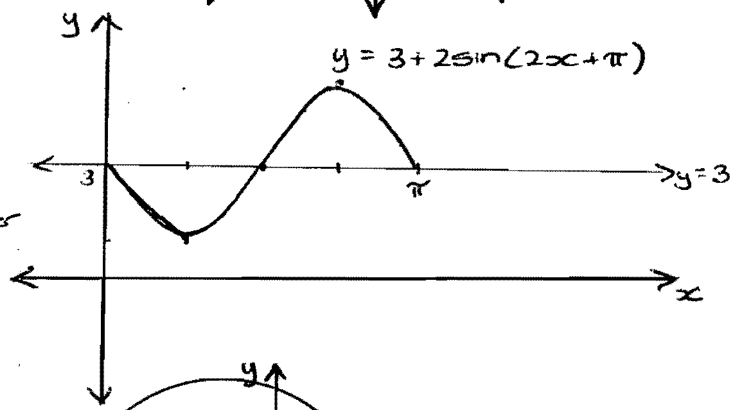
basic curve $y = \sin x$

$y = \sin 2x$ ← stretch ↔ steeper

$y = \sin(2x + \pi)$ ← across π

$y = 2\sin(2x+\pi)$ ← stretch ↑ steeper

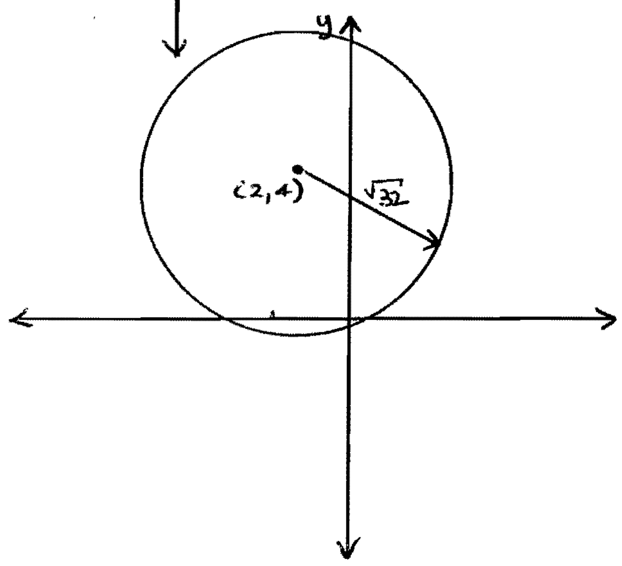
$y = 3 + 2\sin(2x+\pi)$ ← up 3



3. Sketch $x^2 + 4x + y^2 - 8y = 12$

$x^2 + 4x + 4 + y^2 - 8y + 16 = 32$

$(x+2)^2 + (y-4)^2 = 32$



(D) ADDITION AND SUBTRACTION OF ORDINATES

$y = f(x) + g(x)$ can be graphed by first graphing $y = f(x)$ and $y = g(x)$ separately and then adding their ordinates together.

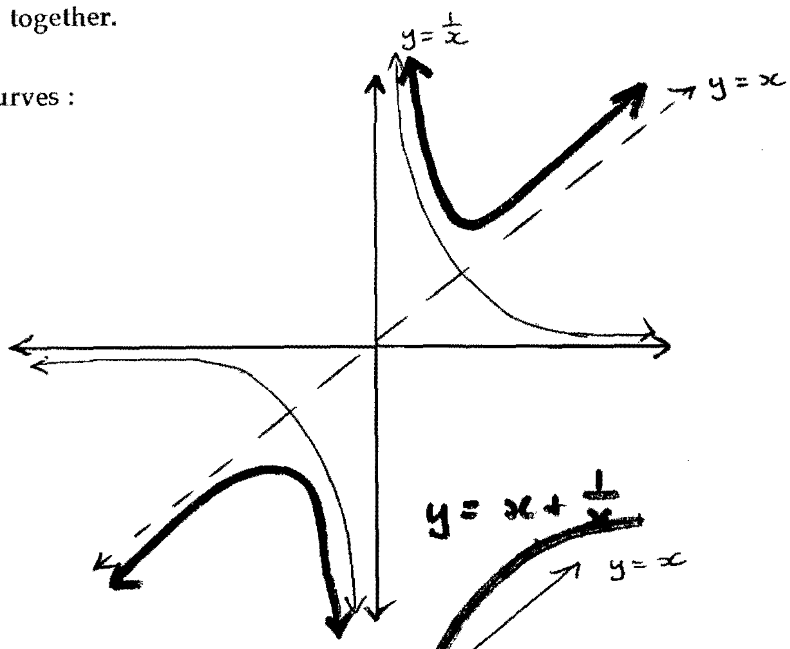
NOTE : First locate points on $y = f(x) + g(x)$ corresponding to $f(x) = 0$ and $g(x) = 0$, then plot further points by addition and subtraction of ordinates and finally locate the position of stationary points.

$y = f(x) - g(x)$ can be graphed by first graphing $y = f(x)$ and $y = -g(x)$ separately and then adding their ordinates together.

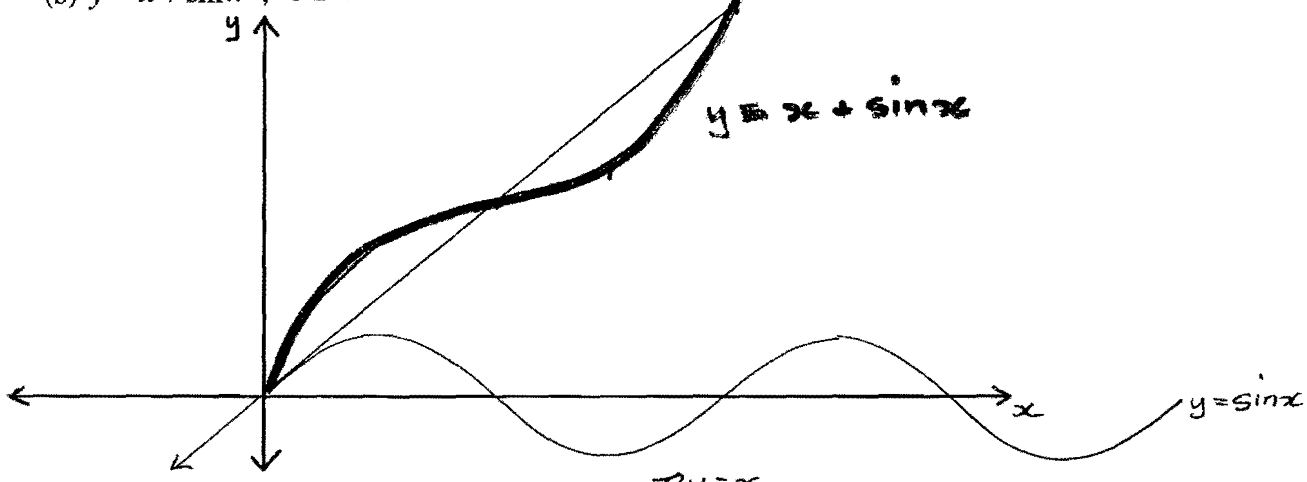
Exercises

Sketch the following curves :

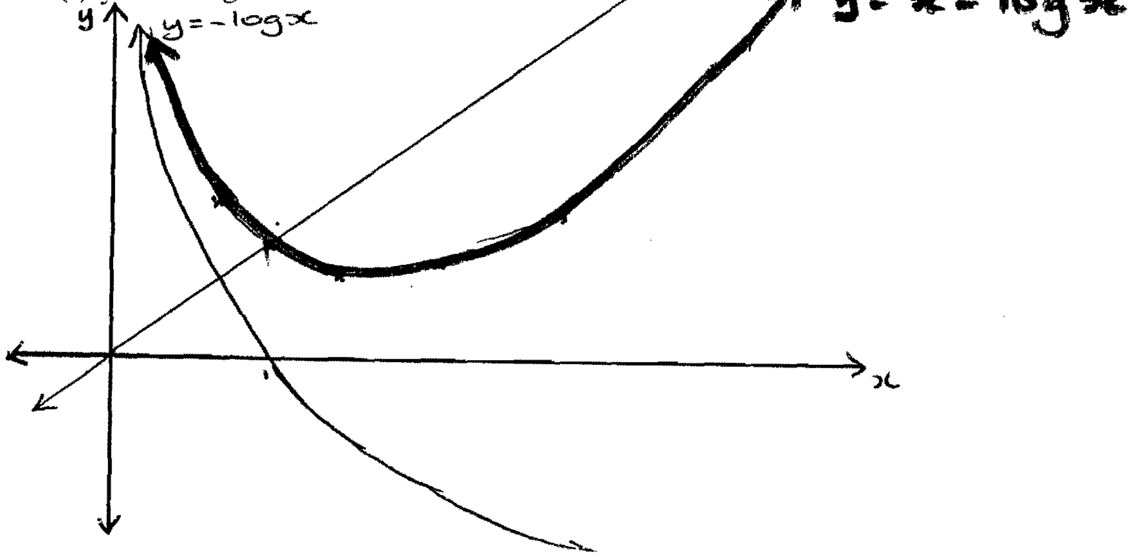
(a) $y = x + \dfrac{1}{x}$



(b) $y = x + \sin x$, $0 \le x \le 4\pi$



(c) $y = x - \log x$



(E) MULTIPLICATION OF FUNCTIONS

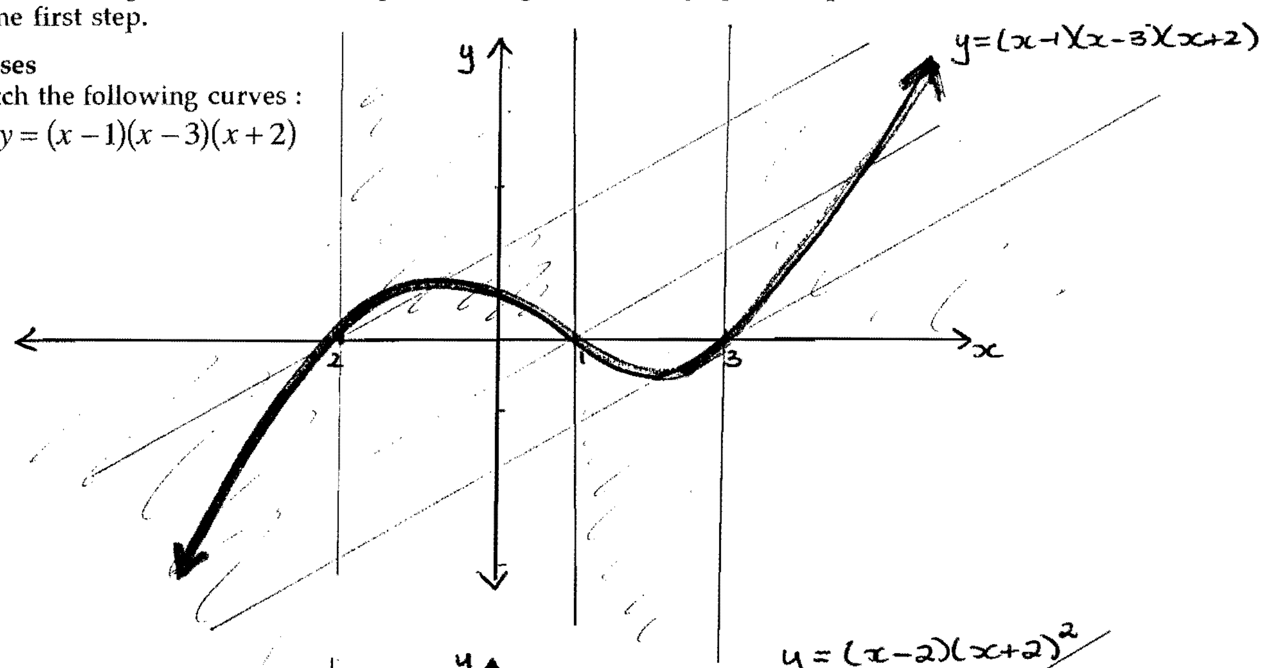
The graph of $y = f(x)\cdot g(x)$ can be graphed by first graphing $y = f(x)$ and $y = g(x)$ separately and then examining the sign of this product. Special note needs to be made of points where either $f(x) = 0$ or 1 or $g(x) = 0$ or 1.

NOTE : The regions on the number plane through which the graph must pass should be shaded in as the first step.

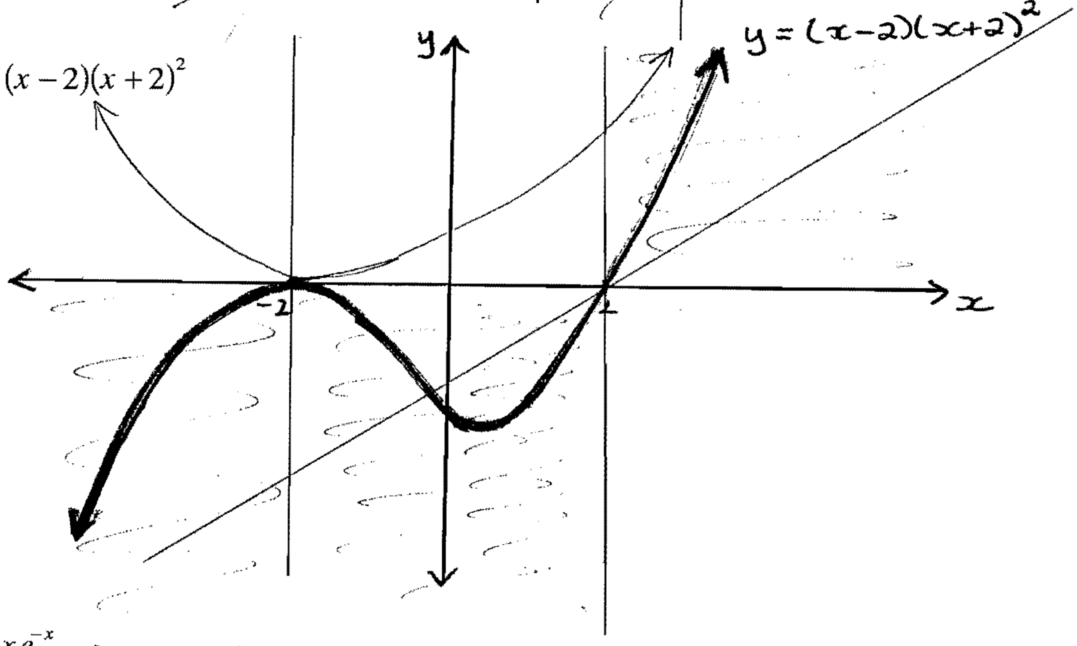
Exercises

Sketch the following curves :

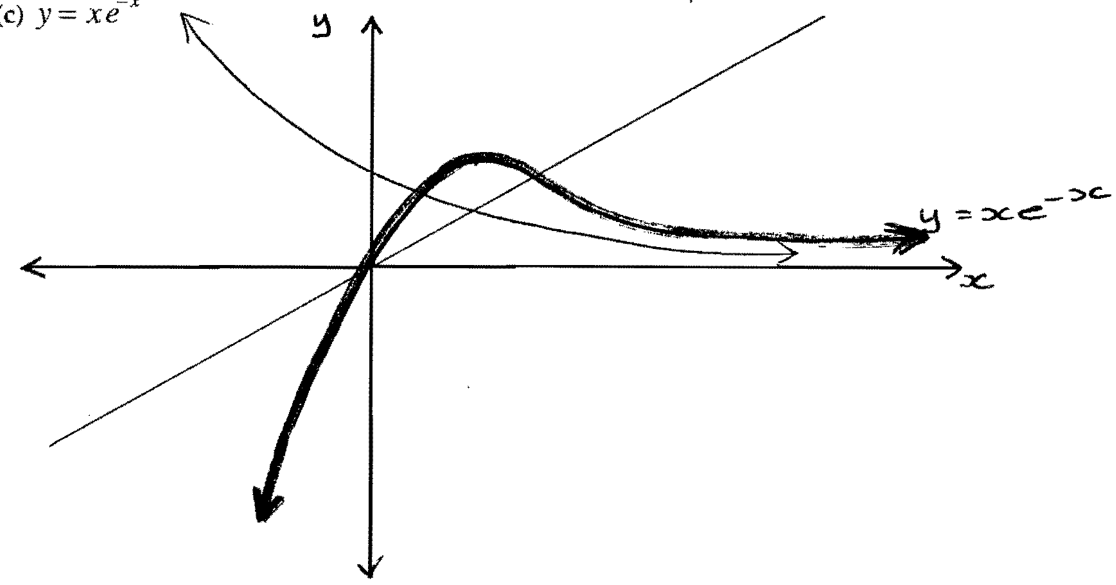
(a) $y = (x-1)(x-3)(x+2)$



(b) $y = (x-2)(x+2)^2$



(c) $y = xe^{-x}$



(F) DIVISION OF FUNCTIONS

The graph of $y=\dfrac{f(x)}{g(x)}$ can be graphed by:

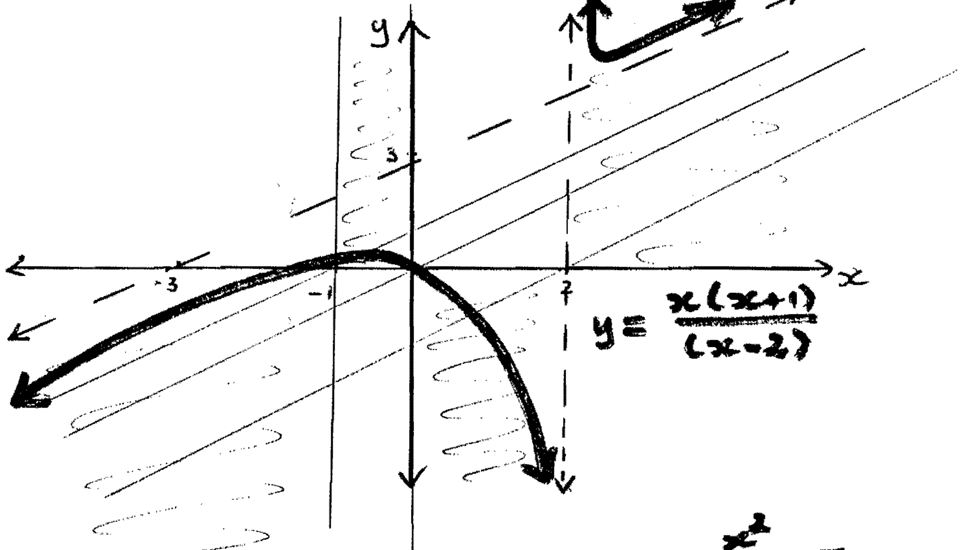
STEP 1 First graph $y=f(x)$ and $y=g(x)$ separately.
STEP 2 Mark in the vertical asymptotes.
STEP 3 Shade in the regions in which the curve must be (same as multiplication)
STEP 4 Investigate the behaviour of the function for large values of x (find horizontal/oblique asymptotes, look at dominance)

Exercises

Sketch the following curves :

(a) $y=\dfrac{x(x+1)}{(x-2)}$

$= (x+3) + \dfrac{6}{(x-2)}$



(b) $y=\dfrac{x^2}{(x+2)(x-1)}$

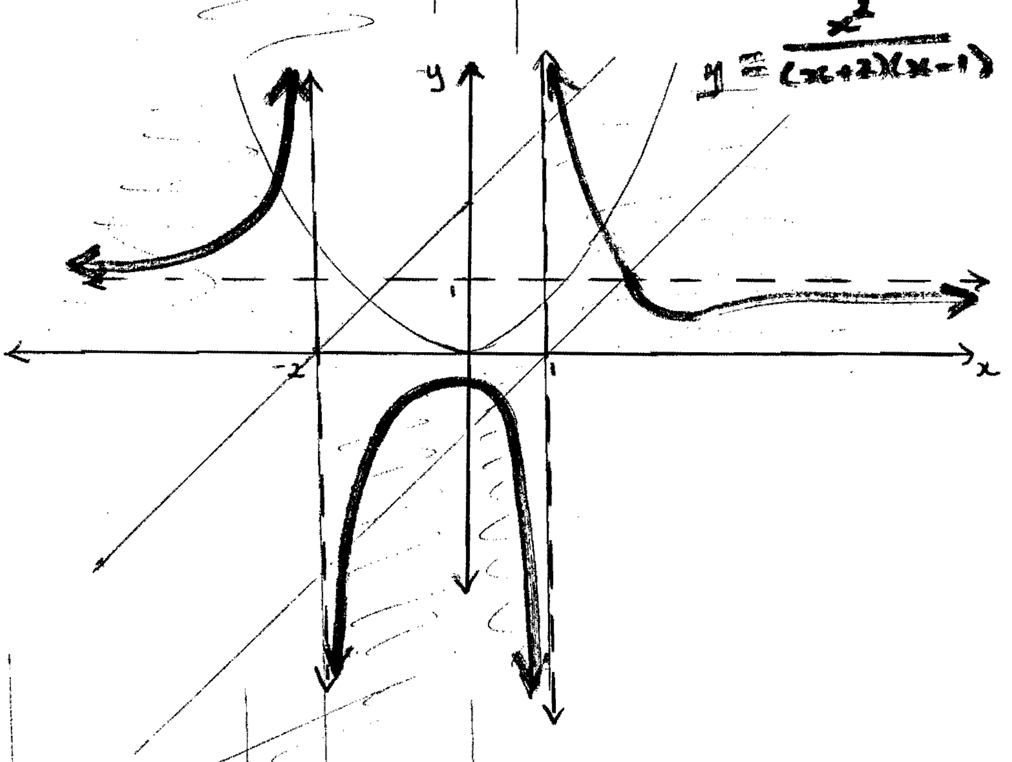
$\lim_{x\to\infty} y = 1$

$\dfrac{x^2}{(x+2)(x-1)} = 1$

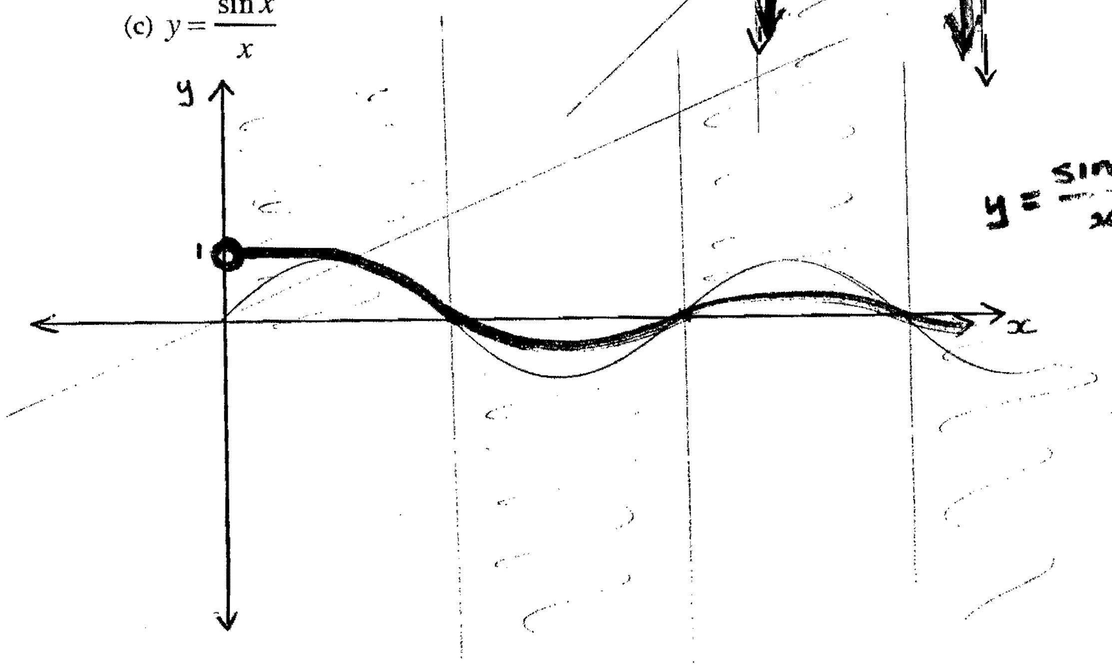
$x^2 = x^2 + x - 2$

$x - 2 = 0$

$x = 2$



(c) $y=\dfrac{\sin x}{x}$



(G) GRAPHS OF RECIPROCAL FUNCTIONS

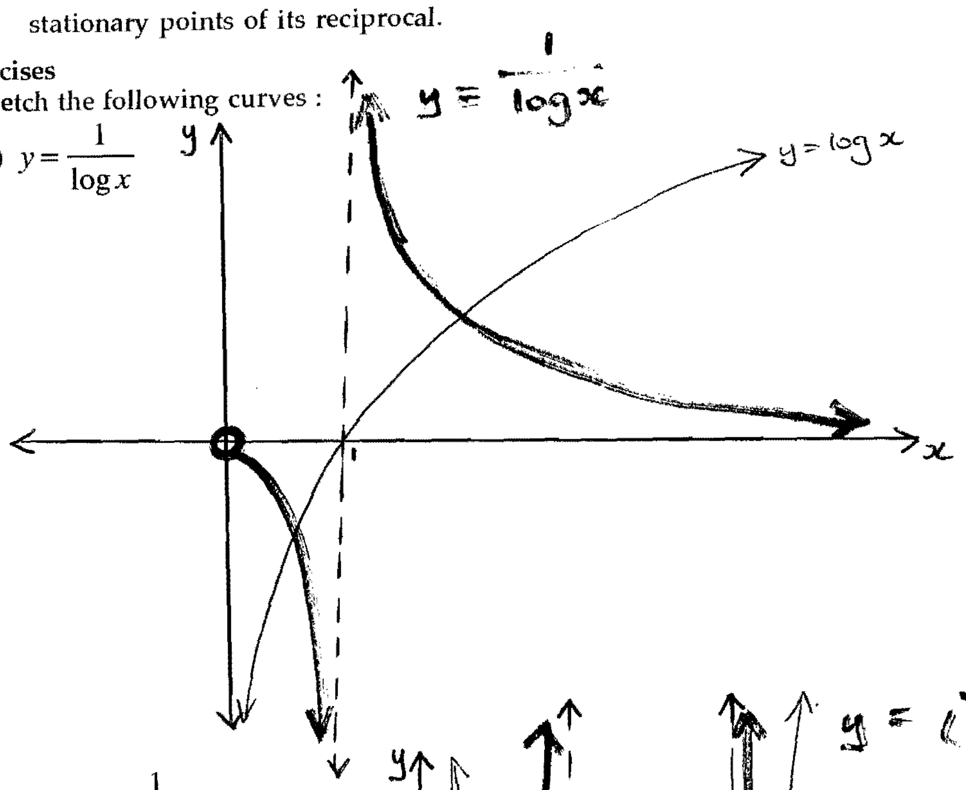
The graph of $y = \dfrac{1}{f(x)}$ can be sketched by first drawing $y = f(x)$ and noticing:

- when $f(x) = 0$, then $\dfrac{1}{f(x)}$ is undefined, (i.e. a vertical asymptote exists)
- when $f(x) \to \infty$, then $\dfrac{1}{f(x)} \to 0$, (i.e. asymptotes become x intercepts, unless undefined)
- when $f(x)$ is increasing, the reciprocal is decreasing, and visa-versa
- when $f(x)$ is positive, $\dfrac{1}{f(x)}$ is positive, etc.
- the derivative of $\dfrac{1}{f(x)}$ is $\dfrac{-f'(x)}{[f(x)]^2}$, hence stationary points of the original curve are stationary points of its reciprocal.

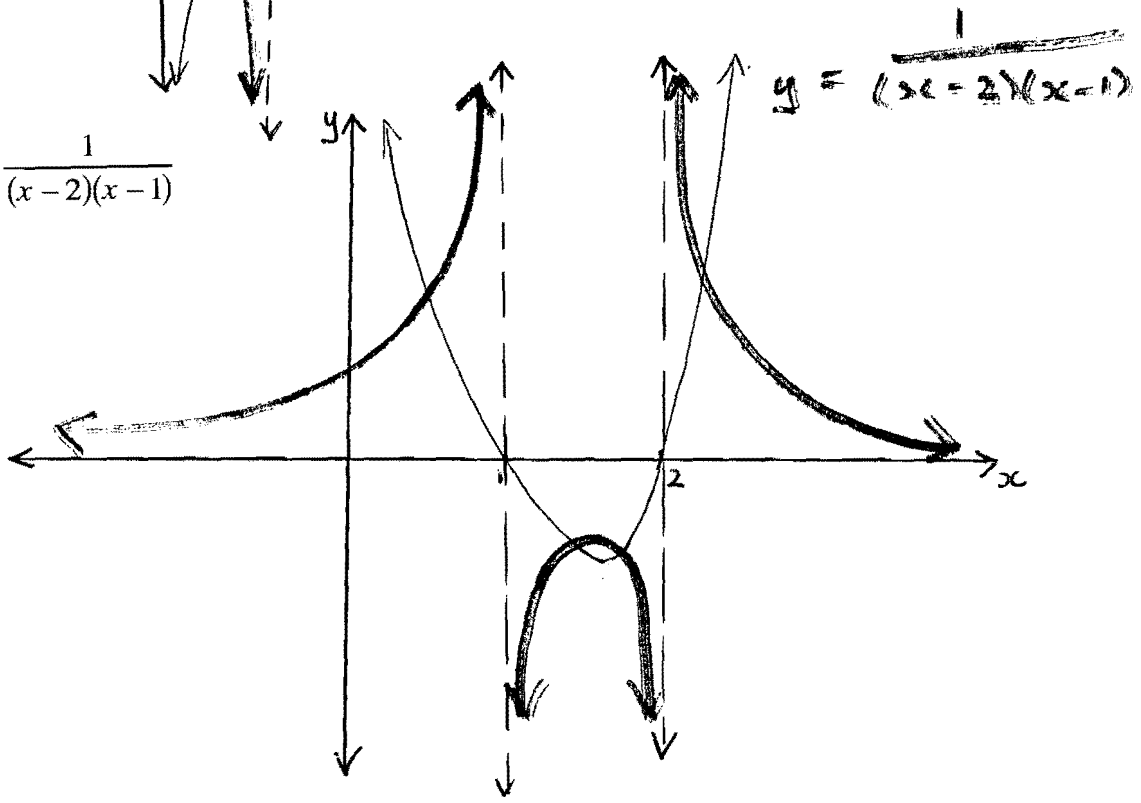
Exercises

Sketch the following curves :

(a) $y = \dfrac{1}{\log x}$



(b) $y = \dfrac{1}{(x-2)(x-1)}$

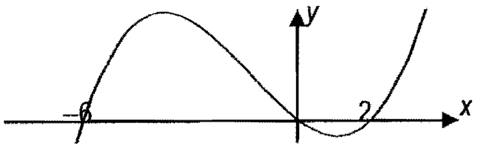


Worked Solutions

Chapter 1: Curve Sketching

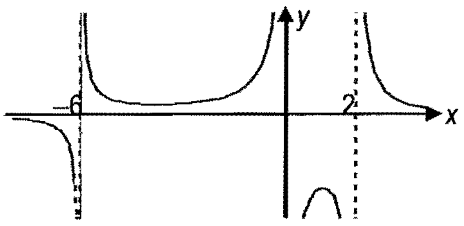
Exercise 1.1 (Rational functions)

1

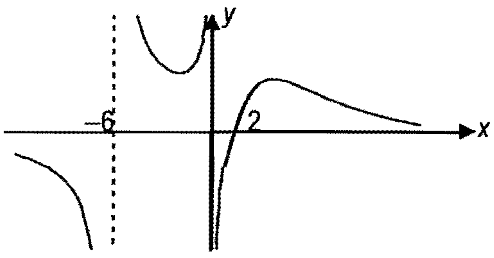


From this graph, $y < 0$ for $x < -6$ or $0 < x < 2$, and $y > 0$ for $-6 < x < 0$ or $x > 2$. This property holds true for the following curves.

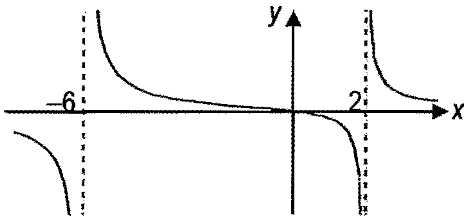
a) Asymptotes $x = -6, 0, 2$ and $y = 0$.



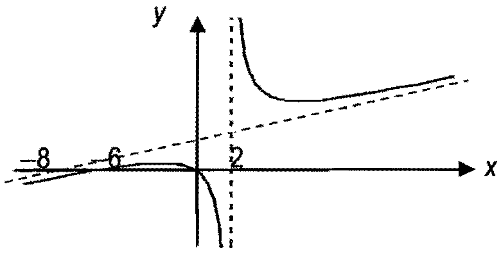
b) Asymptotes $x = -6$, 0 and $y = 0$.
The x-intercept (2,0).



c) Asymptotes $x = -6$, 2 and $y = 0$.
The x-intercept: (0,0).



d) By long division, $\dfrac{x^2+6x}{x-2} = x+8+\dfrac{16}{x-2}$, the curve has two asymptotes $y = x + 8$ and $x = 2$.
The x-intercepts are (0,0) and (–6,0).

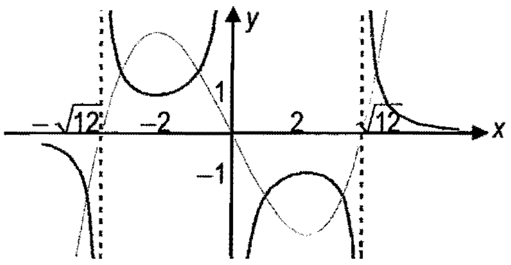


2 a) $y = \dfrac{16}{x^3-12x} = \dfrac{16}{x(x^2-12)}$.

Asymptotes $x = 0, \pm\sqrt{12}$ and $y = 0$.

$y' = \dfrac{-16(3x^2-12)}{(x^3-12x)^2}$, $y' = 0$ when $x = \pm 2$.

$\therefore$ Turning points: (–2, 1) and (2, –1).
(The guide graph $x^3 - 12x$ is added in purple.)



b) $y = \dfrac{4x+5}{x^2-1}$.

Asymptotes $x = -1$, 1 and $y = 0$.

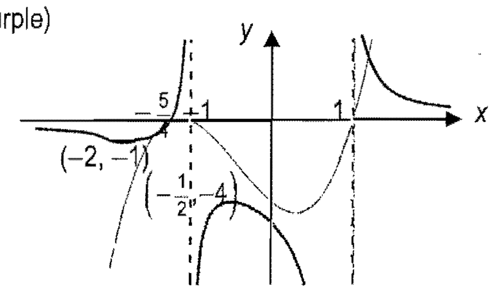
The x-intercept is $(-\frac{5}{4},0)$, the y-intercept is (0,–5).

$y' = \dfrac{4(x^2-1)-2x(4x+5)}{(x^2-1)^2} = \dfrac{-4x^2-10x-4}{(x^2-1)^2}$

$= \dfrac{-2(2x+1)(x+2)}{(x^2-1)^2}$. $y'=0$ when $x = -\frac{1}{2}$ or -2.

$\therefore$ Turning points $\left(-\frac{1}{2},-4\right)$ and $(-2,-1)$.

(The guide graph $(4x+5)(x-1)(x+1)$ is added in purple)



c) By long division, $\dfrac{(x+1)^2}{(x-1)(x-3)} = 1 + \dfrac{6x-2}{(x-1)(x-3)}$.

Asymptotes $x = 1, 3$ and $y = 1$.

The x-intercept is $(-1,0)$, the y-intercept is $(0,\frac{1}{3})$.

$$y' = \frac{6(x-1)(x-3)-(2x-4)(6x-2)}{(x-1)^2(x-3)^2}$$

$$= \frac{6x^2-24x+18-12x^2+28x-8}{(x-1)^2(x-3)^2}$$

$$= \frac{-6x^2+4x+10}{(x-1)^2(x-3)^2}$$

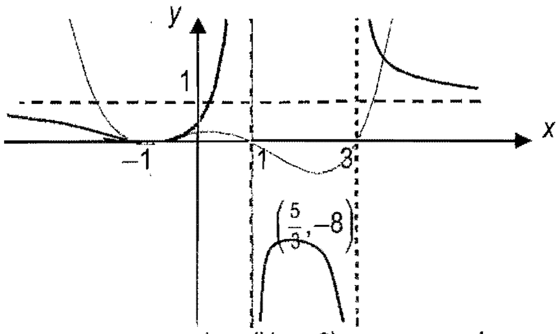
$$= \frac{-2(3x-5)(x+1)}{(x-1)^2(x-3)^2}.$$

$y'=0$ when $x=-1$, or $\frac{5}{3}$.

$\therefore$ Turning points $(-1, 0)$ and $\left(\frac{5}{3},-8\right)$.

As $x \to +\infty, y \simeq 1+\dfrac{6}{x} \to 1^+$; As $x \to -\infty, y \to 1^-$.

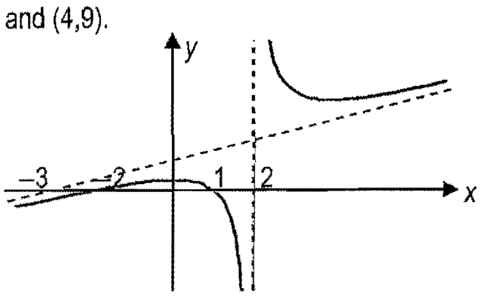
The guide graph $(x+1)^2(x-1)(x-3)$ is added in purple.



d) By long division, $\dfrac{(x-1)(x+2)}{x-2} = x+3+\dfrac{4}{x-2}$, the curve has two asymptotes $y = x + 3$ and $x = 2$.

The x-intercepts are $(-2,0)$ and $(1,0)$.

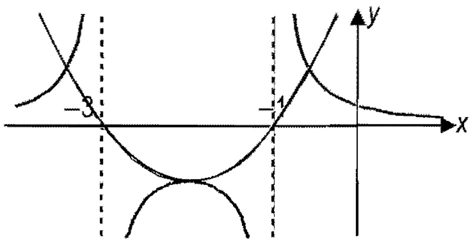
$y' = \dfrac{x(x-4)}{(x-2)^2}$, $y'=0$ when $x = 0, 4$. $\therefore$ Turning points $(0,1)$ and $(4,9)$.



3 a) Asymptotes $x = -3, -1$ and $y = 0$.

$y' = \dfrac{-2(x+2)}{[(x+1)(x+3)]^2}$, $y'=0$ when $x=-2$. $\therefore$ Turning point $(-2,-1)$.

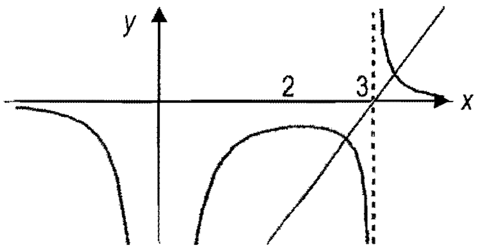
(The guide graph $(x + 1)(x + 3)$ is added in pink)



b) Asymptotes $x = 0, 3$ and $y = 0$.

$y' = \dfrac{-3(x-2)}{x^3(x-3)^2}$, $y'=0$ when $x = 2$. $\therefore$ TP $(2,-\frac{1}{4})$.

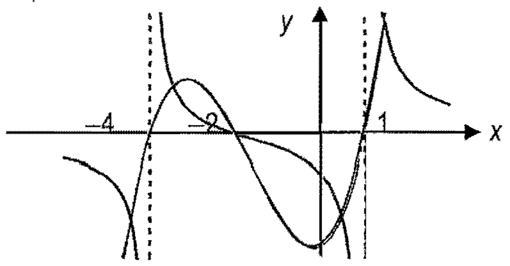
(The guide graph $y = x-3$, noting that the positive factor x^2 can be ignored, is added in pink.)



c) Asymptotes $x = -4$, 1 and $y = 0$, y-intercept $(0,-\frac{1}{2})$.

$y' = \dfrac{-(x^2+4x+10)}{[(x-1)(x+4)]^2}$, $y' < 0$ always.

(The guide graph $(x+2)(x-1)(x+4)$ is added in pink).



d) Asymptotes $x = 0, 4$ and $y = 0$.

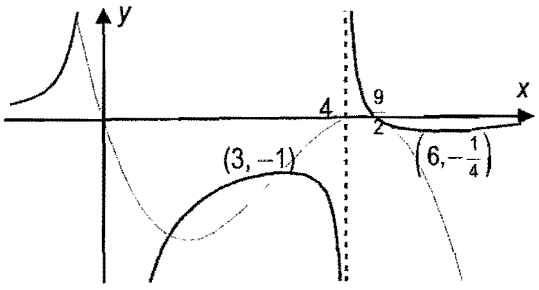
$$y' = \frac{-2(x^2-4x)-(2x-4)(9-2x)}{x^2(x-4)^2}$$

$$= \frac{2(x^2-9x+18)}{x^2(x-4)^2}$$

$$= \frac{2(x-3)(x-6)}{x^2(x-4)^2}.$$

$y' = 0$ when $x = 3, 6$.∴ Turning points $(3, -1)$, $\left(6, -\frac{1}{4}\right)$.

(The guide graph $(9-2x)x(x-4)$ is added in purple)

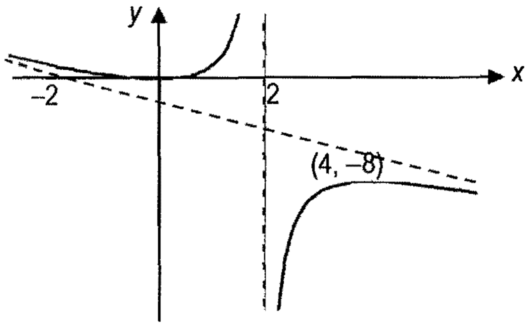


e) By long division, $\frac{x^2}{2-x} = -x-2+\frac{4}{2-x}$

Asymptotes $x = 2$ and $y = -x - 2$.

$y' = -1 + \frac{4}{(2-x)^2}$, $y' = 0$ when $x = 0$ or 4.

∴ Turning points $(0, 0)$ and $(4, -8)$.

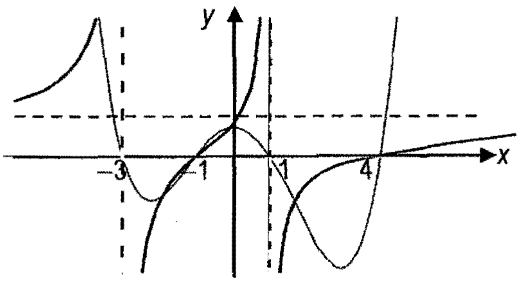


f) Asymptotes $x = -3$, 1 and $y = 1$.

The x-intercepts $(-1,0)$ and $(4,0)$, the y-intercept $\left(0, \frac{4}{3}\right)$

The graph shows no turning points (here, $y' > 0$)

The guide graph $(x + 1)(x - 4)(x - 1)(x + 3)$ is added in pink.



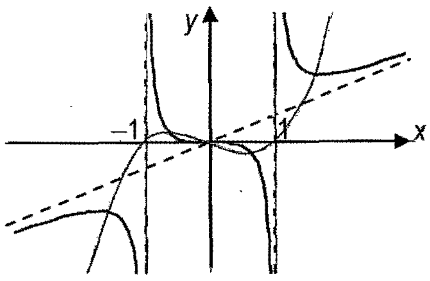
g) Asymptotes $x = \pm 1$ and $y = x$. The x-intercept is $(0,0)$

$y' = \frac{x^2(x^2-3)}{(x^2-1)^2}$, $y' = 0$ when $x = 0, \pm\sqrt{3}$

∴ Turning points $\pm\left(\sqrt{3}, \frac{3\sqrt{3}}{2}\right)$ and $(0,0)$.

The point $(0,0)$ is a horizontal point of inflexion.

The guide graph $x(x^2-1)$ is added in pink.

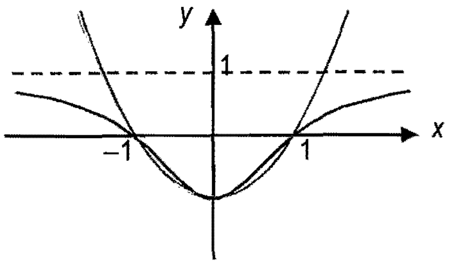


h) Asymptote $y = 1$. The x-intercepts $(\pm 1, 0)$

(The function is even, so it's symmetrical about the y-axis)

$y' = \frac{4x}{(x^2+1)^2}$, $y' = 0$ when $x = 0$, ∴ TP $(0, -1)$

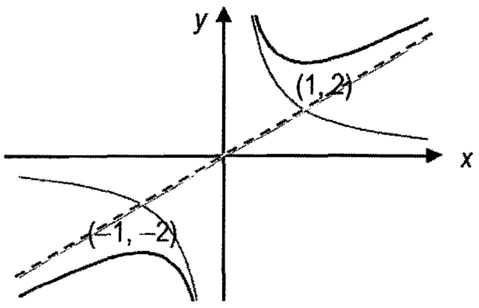
The guide graph $x^2 - 1$ is added in pink.



4 a) When $x \to \infty, y \to x : y = x$ is the asymptote.

When $x \to 0, y \to \frac{1}{x} : y = \frac{1}{x}$ (or $x = 0$) is the asymptote

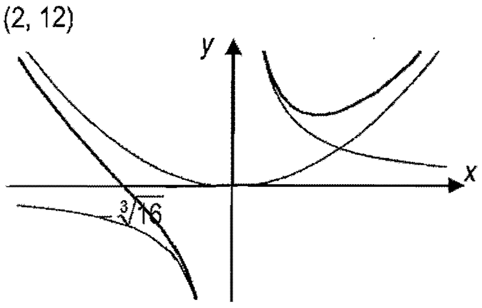
$y' = 1 - \frac{1}{x^2}$, $y' = 0$ when $x^2 = 1$,∵ $x = \pm 1$, ∴ Turning points $(1, 2)$ and $(-1, -2)$.



b) When $x \to \infty, y \to x^2 : y = x^2$ is the asymptote.

When $x \to 0, y \to \frac{16}{x} : y = \frac{16}{x}$ (or $x = 0$) is the asymptote. The x-intercept is $(-\sqrt[3]{16}, 0)$.

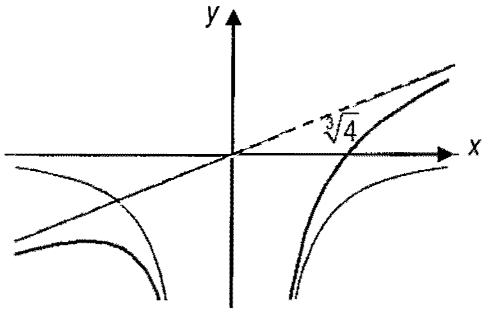
$y' = 2x - \frac{16}{x^2}, y' = 0$ when $x^3 = 8, \therefore x = 2$, $\therefore$ Turning point (2, 12)



c) When $x \to \infty, y \to x : y = x$ is the asymptote.

When $x \to 0, y \to \frac{-4}{x^2} : y = \frac{-4}{x^2}$ (or $x = 0$) is the asymptote. The x-intercept is $(\sqrt[3]{4}, 0)$.

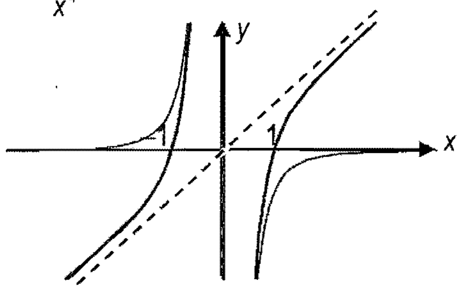
$y' = 1 + \frac{8}{x^3}, y' = 0$ when $x^3 = -8, \therefore x = -2$, $\therefore$ Turning point (–2, –3)



d) When $x \to \infty, y \to x : y = x$ is the asymptote.

When $x \to 0, y \to \frac{-1}{x^3} : y = \frac{-1}{x^3}$ (or $x = 0$) is the asymptote. The x-intercepts are (1,0) and (–1, 0).

$y' = 1 + \frac{3}{x^4} > 0$, $\therefore$ No turning points.

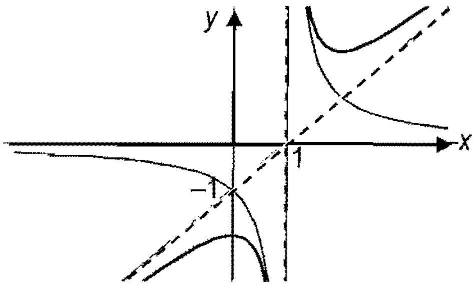


5 a) When $x \to \infty, y \to x - 1 : y = x - 1$ is the asymptote.

When $x \to 1, y \to \frac{1}{x-1} \therefore x = 1$ is the asymptote.

$y' = 1 - \frac{1}{(x-1)^2}, y' = 0$ when $(x-1)^2 = 1, \therefore x - 1 = \pm 1$

$\therefore x = 0$ or 2, $\therefore$ Turning points (0, –2) and (2, 2)

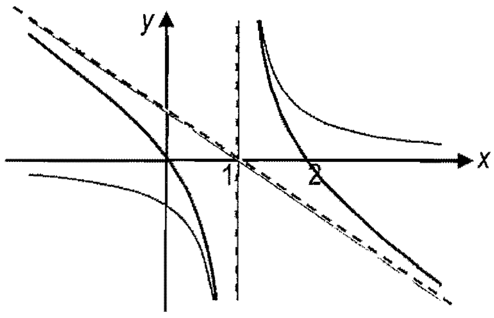


b) When $x \to \infty, y \to 1 - x : y = 1 - x$ is the asymptote.

When $x \to 1, y \to \frac{1}{x-1} \therefore x = 1$ is the asymptote.

The x-intercepts are (0,0) and (2,0)

$y' = -1 - \frac{1}{(x-1)^2} < 0$, $\therefore$ No turning points.

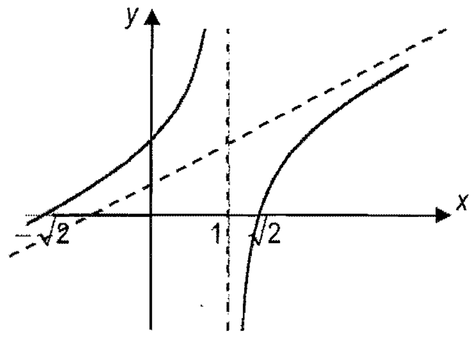


c) When $x \to \infty, y \to x + 1 : y = x + 1$ is the asymptote.

When $x \to 1, y \to \frac{1}{1-x} \therefore x = 1$ is the asymptote.

For the x-intercepts: Let $y = 0$, $x + 1 + \frac{1}{1-x} = 0$,

$\therefore 1 - x^2 + 1 = 0, \therefore x = \pm\sqrt{2}$



d) When $x \to \infty, y \to x - 1 : y = x - 1$ is the asymptote.

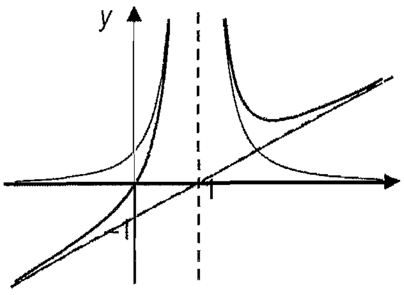
When $x \to 1, y \to \frac{1}{(x-1)^2}$ $\therefore x = 1$ is the asymptote.

$y' = 1 - \frac{2}{(x-1)^3}, y' = 0$ when $(x-1)^3 = 2, \therefore x = \sqrt[3]{2} + 1$

$\therefore$ Turning point $\left(\sqrt[3]{2}+1, \sqrt[3]{2}+\frac{1}{\sqrt[3]{4}}\right)$.

For the x-intercepts: Let $y = 0$, $x - 1 + \frac{1}{(x-1)^2} = 0$,

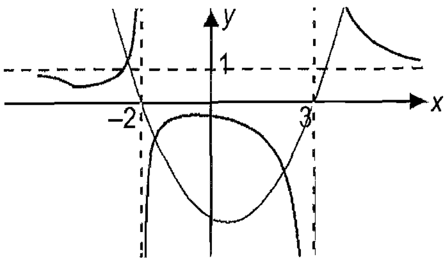
$\therefore (x-1)^3 + 1 = 0, \therefore (x-1)^3 = -1, \therefore x = 0$.



6 $y = \frac{x^2+2x+4}{x^2-x-6} = 1 + \frac{3x+10}{x^2-x-6}$.

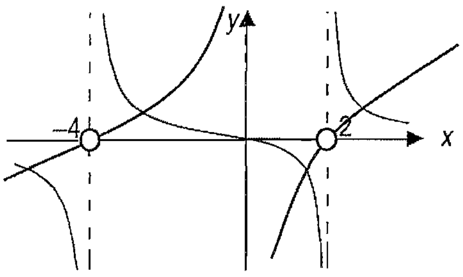
As $x \to +\infty, y \approx 1 + \frac{3}{x} \to 1^+$, as $x \to -\infty, y \to 1^-$.

The asymptotes are $x = -2$, 3 and $y = 1$.

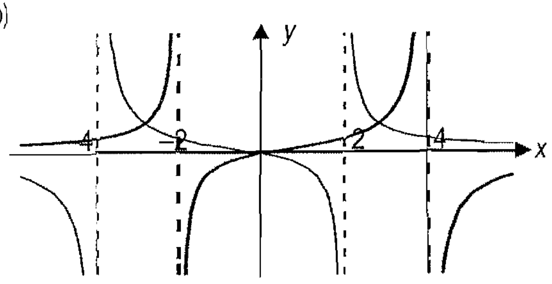


7

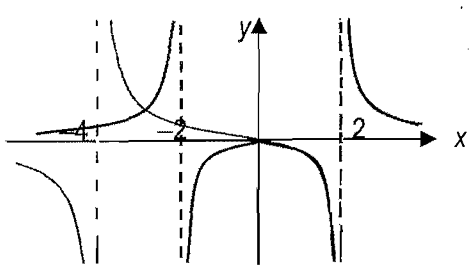
a)



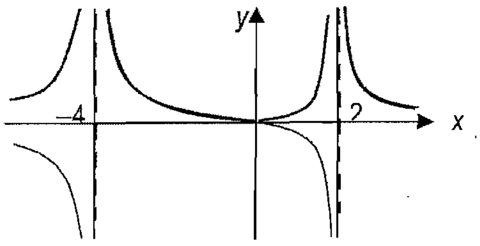
b)



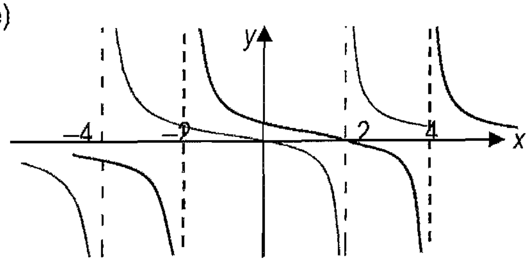
c)



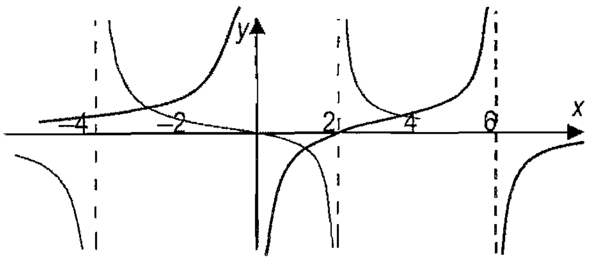
d)



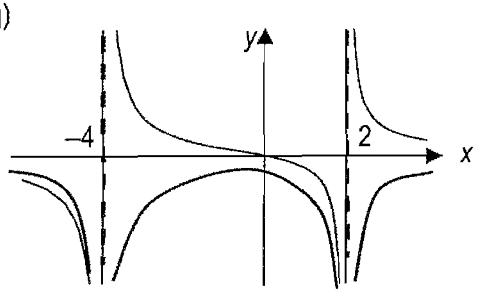
e)



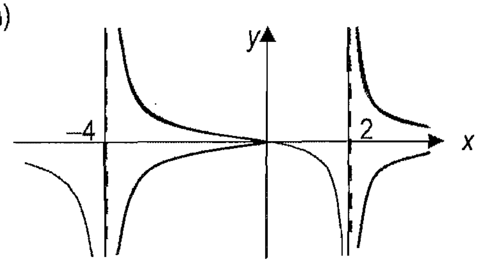
f)



g)



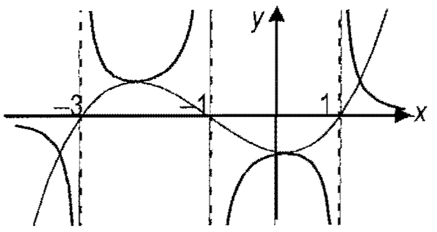
h)



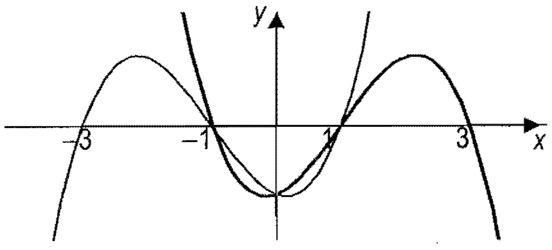
Note: In (b) the curve is reflected about the y-axis, in (c) if $x < 0, f(|x|) = f(-x)$, in (e) the curve is shifted to the right 2 units, in (f) it's the curve in (b) that is shifted to the right 2 units, in (h) where $f(x) > 0, f(x) = \pm y$.

8

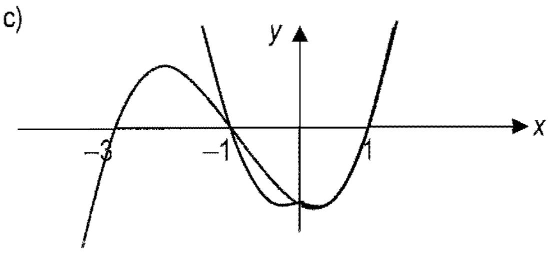
a)



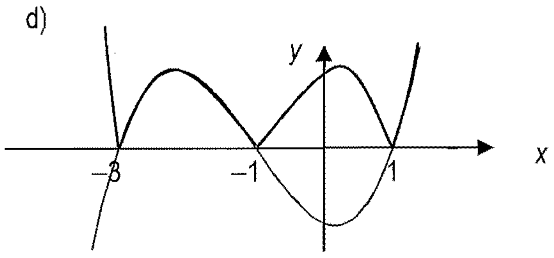
b)



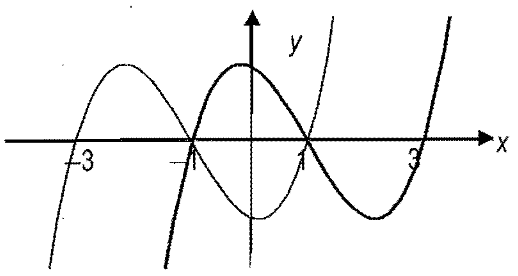
c)



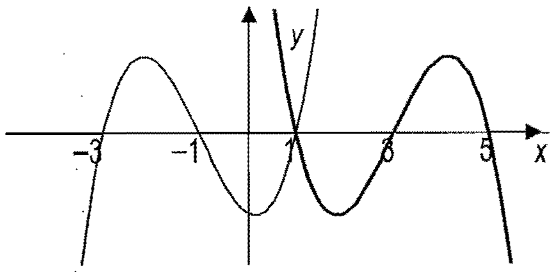
d)



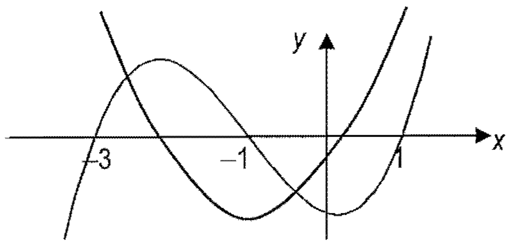
e)



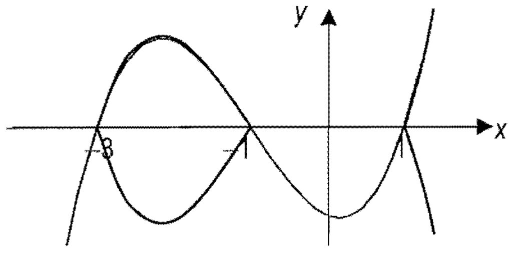
f)



g)

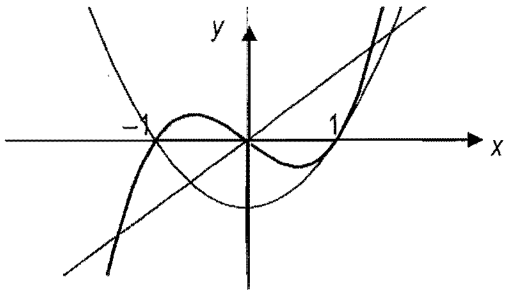


h)

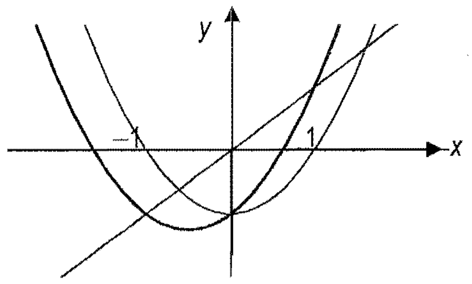


9

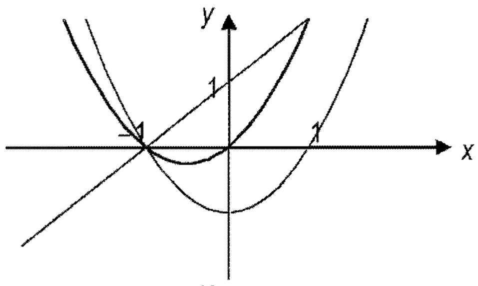
a)



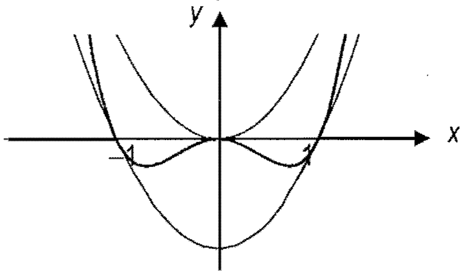
b)



c)

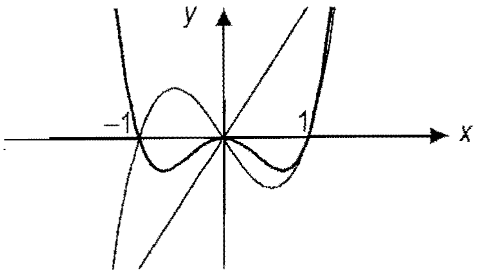


d)

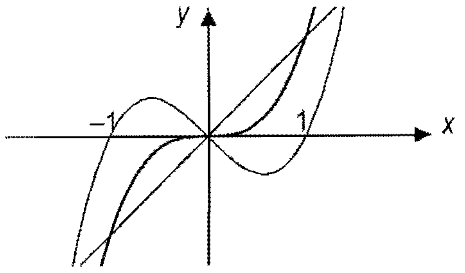


10

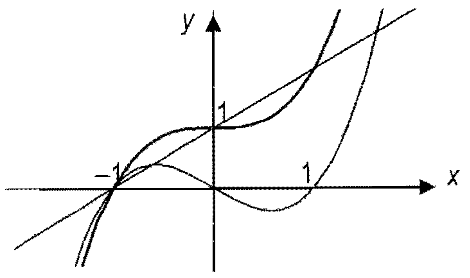
a)



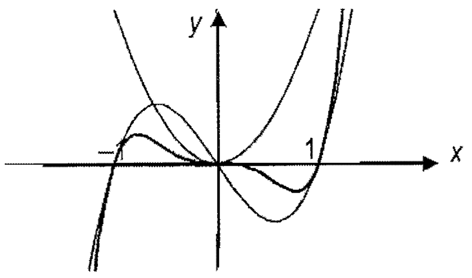
b)



c)



d)



11

a) $(1-x)^2 \geq 0$

$1+x^2-2x \geq 0$

$\therefore 1+x^2 \geq 2x$

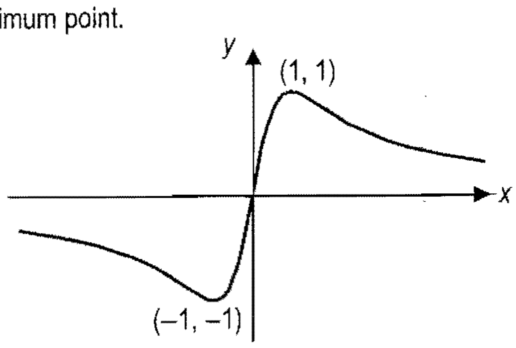
$\therefore 1 \geq \dfrac{2x}{1+x^2}$

b) Let $f(x)=\dfrac{2x}{1+x^2}$, $f(-x)=\dfrac{2(-x)}{1+x^2}=\dfrac{-2x}{1+x^2}=-f(x)$

$\therefore f(x)$ is odd, $\therefore$ Its curve is symmetrical about the origin

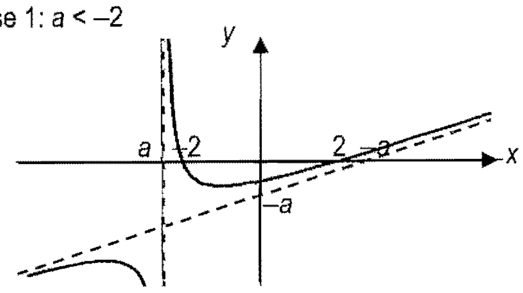
c) As $x \to \infty, \dfrac{2x}{1+x^2} \to 0 : y=0$ is the asymptote.

Further, $\dfrac{2x}{1+x^2}=1$ when $x=1$, thus, (1, 1) is the maximum point.

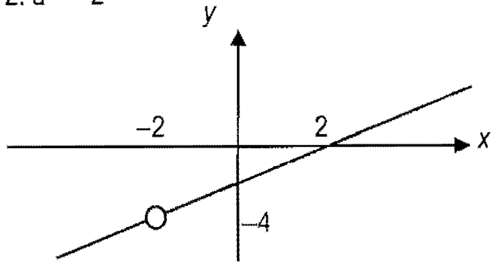


12

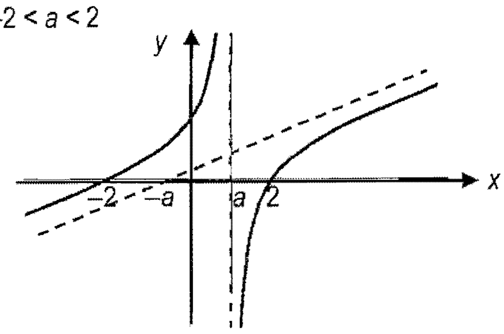
Case 1: $a<-2$



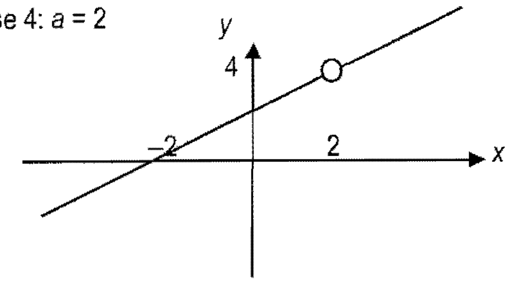
Case 2: $a=-2$



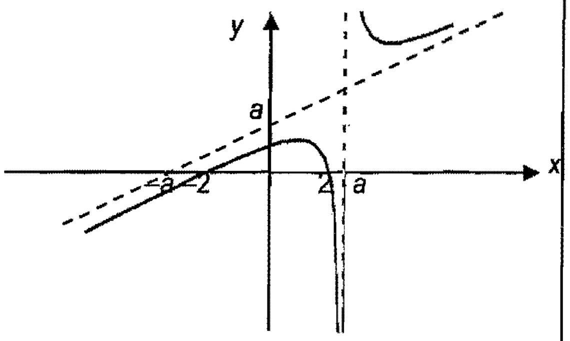
Case 3: $-2<a<2$



Case 4: $a=2$



Case 5: $a > 2$



Note: When $a \neq 2, y = \frac{x^2-4}{x-a} = x + a + \frac{a^2-4}{x-a}$: the curve has a vertical asymptote at $x = a$ and an oblique asymptote of $y = x + a$. Its x-intercepts are $(\pm 2, 0)$.

When $a = -2, y = \frac{x^2-4}{x+2} = x - 2$ if $x \neq -2$: The curve is the straight line $y = x - 2$ with a hole at $(-2, -4)$.

When $a = 2, y = \frac{x^2-4}{x-2} = x + 2$ if $x \neq 2$: The curve is the straight line $y = x + 2$ with a hole at $(2, 4)$.

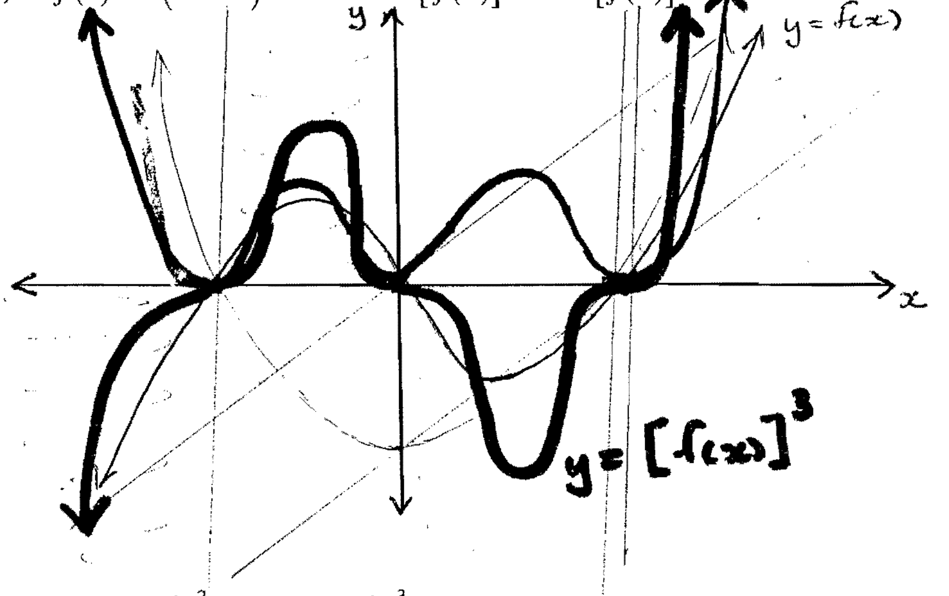
(H) GRAPHS OF THE FORM $y=[f(x)]^n$, where $n>1$ and an integer

The graph of $y=[f(x)]^n$ can be sketched by first drawing $y=f(x)$ and noticing:

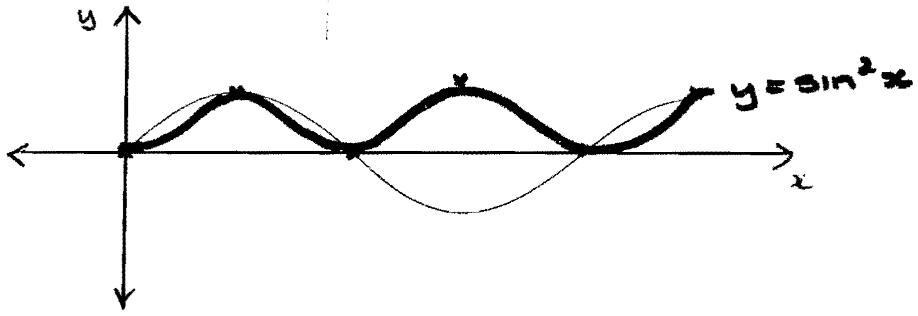
- All stationary points must still be stationary points
- All points where the curve cuts the x-axis are also stationary points on the x-axis
- If $|f(x)|>1$ then $|[f(x)]^n|>f(x)$
- If $|f(x)|<1$ then $|[f(x)]^n|<f(x)$
- If n is even then $[f(x)]^n \geq 0$
- If n is odd then the sign of $[f(x)]^n$ is the same as the sign of $f(x)$ for any given value of x

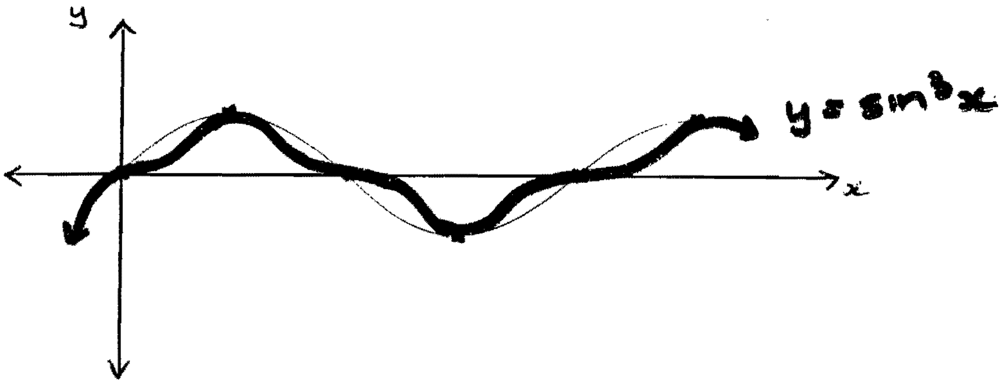
Exercises

(a) If $f(x)=x(x^2-3)$ then sketch $[f(x)]^2$ and $[f(x)]^3$



(b) Sketch $y=\sin^2 x$ and $y=\sin^3 x$

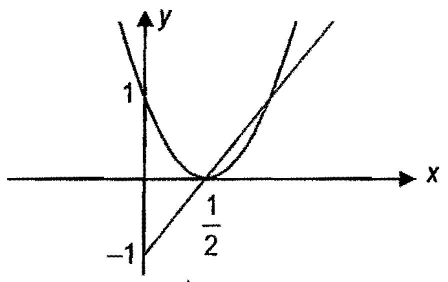




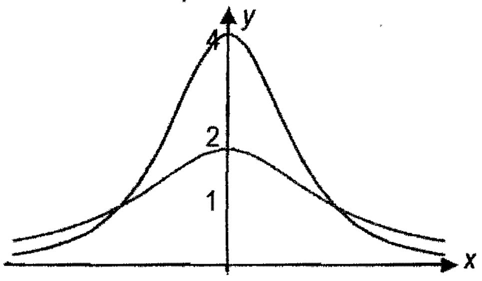
Exercise 1.2 (The graph of $y = f^n(x)$)

1

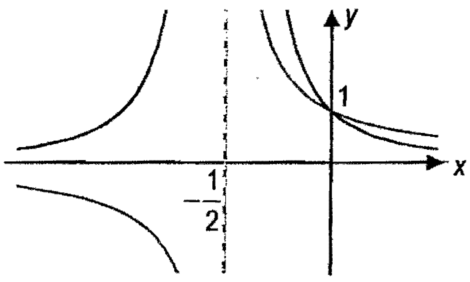
a)



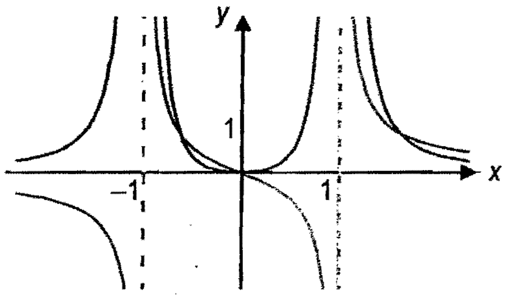
b)



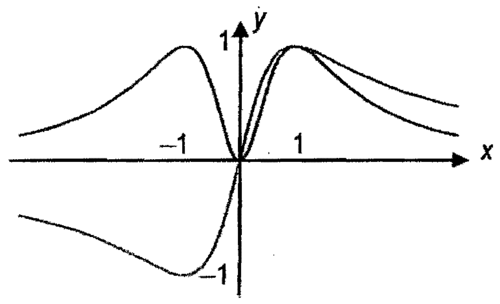
c)



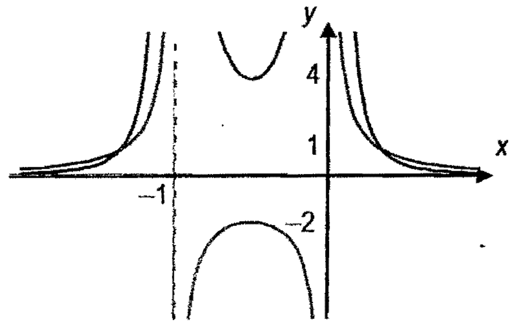
d)



e)



f)



Note: The purple curve is $f(x)$ and the blue curve is $f^2(x)$.

2 a) Let $f(x) = x^2 - 2, g(x) = f^3(x) = (x^2-2)^3$.

$g'(x) = 6x(x^2-2)^2$.

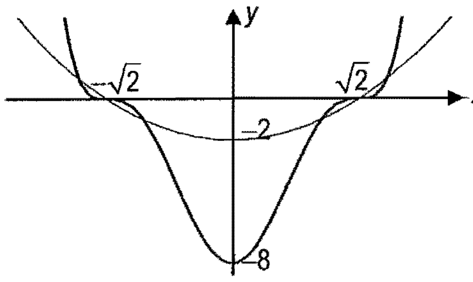
$\therefore g'(x) = 0$ when $x = \pm\sqrt{2}, 0$.

Turning points $(-\sqrt{2},0), (0,-8), (\sqrt{2},0)$.

Determining the nature of the turning points:

x	-3	$-\sqrt{2}$	-1	0	1	$\sqrt{2}$	3
g'	–	0	–	0	+	0	+
	↘	→	↘	→	↗	→	↗

$\therefore (-\sqrt{2},0)$ and $(\sqrt{2},0)$ are horizontal points of inflexion.



b) Let $f(x)=\dfrac{2x}{x^2+1}, g(x)=f^3(x)=\dfrac{8x^3}{(x^2+1)^3}$.

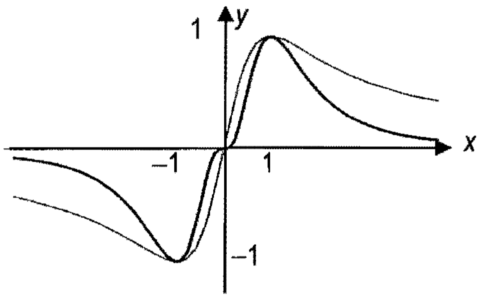
$g'(x)=\dfrac{8(3x^2(x^2+1)^3-6x^4(x^2+1)^2)}{(x^2+1)^6}=\dfrac{24x^2(1-x^2)}{(x^2+1)^4}$.

$\therefore g'(x)=0$ when $x=\pm1,0$.

Turning points $(-1,-1),(0,0),(1,1)$.

x	−2	−1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2
g'	−	0	+	0	+	0	−
	↘	→	↗	→	↗	→	↘

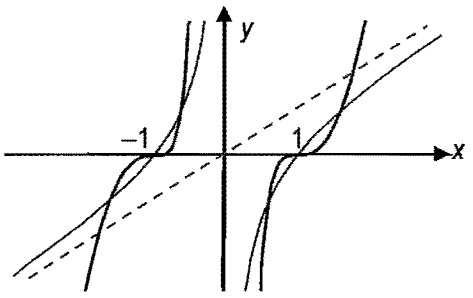
$\therefore(-1,-1)$ is minimum, $(1,1)$ maximum and $(0,0)$ horizontal point of inflexion.



c) Let $f(x)=\dfrac{x^2-1}{x}, g(x)=\dfrac{(x^2-1)^3}{x^3}$

$g'(x)=\dfrac{6x^4(x^2-1)^2-3x^2(x^2-1)^3}{x^6}=\dfrac{3(x^2-1)^2(x^2+1)}{x^4}$

$\therefore g'(x)=0$ when $x=\pm1$.



Turning point $(\pm1,0)$. But since $g'(x)>0$ for all $x\neq0$ or ±1, these are the horizontal points of inflexion.

Notice that both curves have a vertical asymptote at $x=0$, but due to $\dfrac{x^2-1}{x}=x-\dfrac{1}{x}$, $f(x)$ has an oblique asymptote of equation $y=x$, which becomes the asymptote $y=x^3$ in $g(x)$.

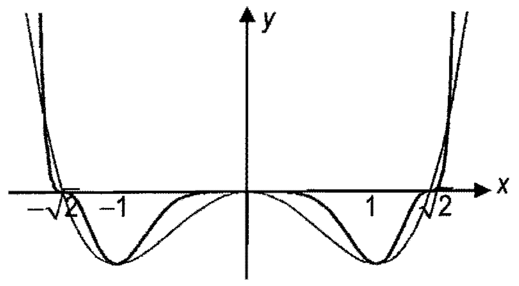
d) Let $f(x)=x^2(x^2-2), g(x)=x^6(x^2-2)^3$.

$g'(x)=6x^5(x^2-2)^3+6x^7(x^2-2)^2$

$\quad=12x^5(x^2-2)^2(x^2-1)$.

$\therefore$ Turning points are $(0,0),(\pm\sqrt{2},0),(\pm1,-1)$.

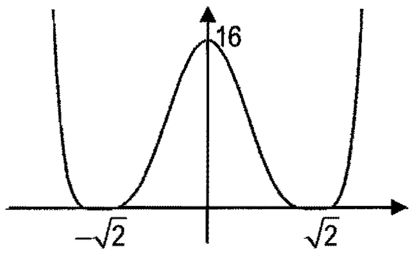
Notice that the x-intercepts $(\pm\sqrt{2},0)$ of $f(x)$ become horizontal points of inflexion, but since this is an even function $(0,0)$ becomes the maximum point. This confirms that not all x-intercepts of $f(x)$ become points of inflexion in the graph of $g(x)=f^3(x)$.



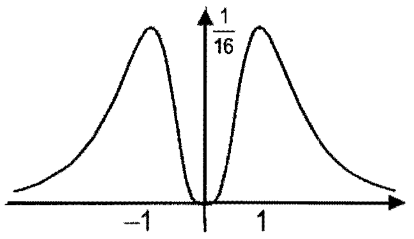
Note: The purple curve is $f(x)$ and the blue curve is $g(x)=f^3(x)$.

3

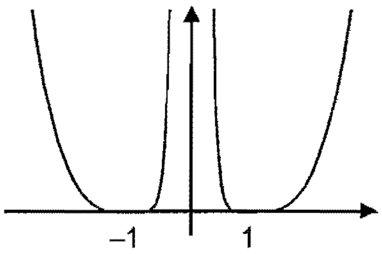
a)



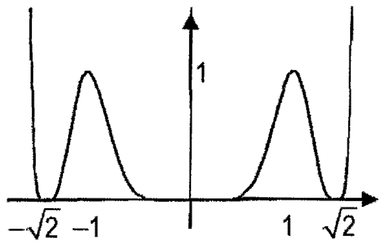
b)



c)

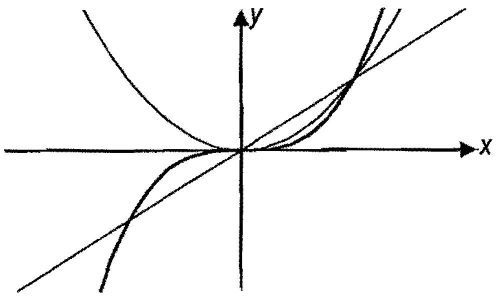


d)

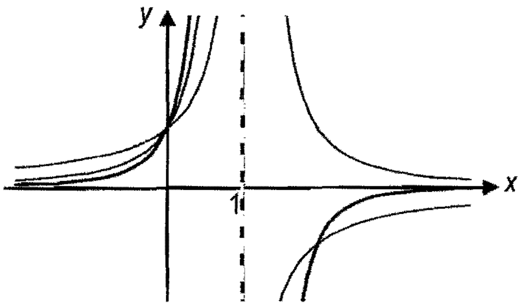


4

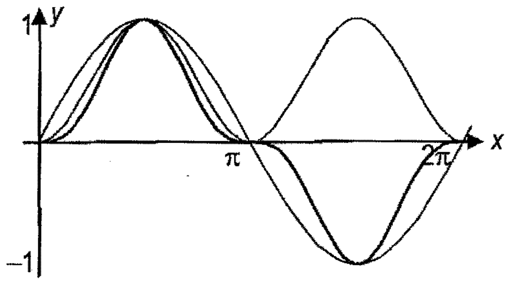
a)



b)



c) Since $|\sin x| \le 1, |\sin^3 x| \le \sin^2 x \le |\sin x|$.

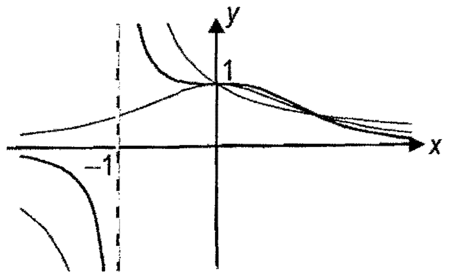


d) $\frac{1}{x+1}$ is a hyperbola with asymptotes $x = -1, y = 0$.

$\frac{1}{x^2+1}$ is the reciprocal of the parabola x^2+1. Since $\infty > x^2+1 \ge 1, 0 < \frac{1}{x^2+1} \le 1$.

Let $g(x) = \frac{1}{x^3+1}, g'(x) = \frac{-3x^2}{(x^3+1)^2}$,

$g'(x) = 0$ when $x = 0$, $\therefore$ Turning point (0, 1). However, as $g'(x) < 0$ for all $x \ne 0$, this is a horizontal point of inflexion.



Note: The first curve in each question is in green, the second curve is in purple and the third curve is in blue.

α) For $0 < x < \frac{\pi}{2}, \sin x < 1, \therefore \sin^2 x > \sin^3 x$,

$\therefore \int_0^{\pi/2} \sin^2 x dx > \int_0^{\pi/2} \sin^3 x dx$. $\therefore$ False.

β) For $2 < x < 3, |1-x| > 1, \frac{1}{|1-x|} < 1, \frac{1}{|1-x|} > \frac{1}{(1-x)^4}$

$\therefore \int_2^3 \frac{dx}{1-x} > \int_2^3 \frac{dx}{(1-x)^4}$. $\therefore$ False.

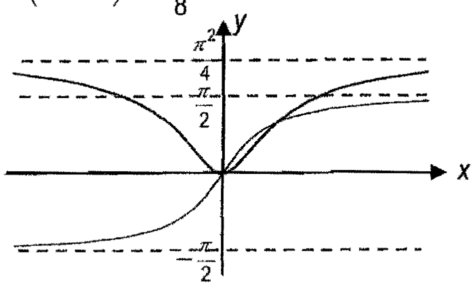
γ) For $0 < x < 1, x > x^2, \therefore x^{2000} > x^{2001}$

$\therefore \int_0^1 x^{2000} dx > \int_0^1 x^{2001} dx$. $\therefore$ False.

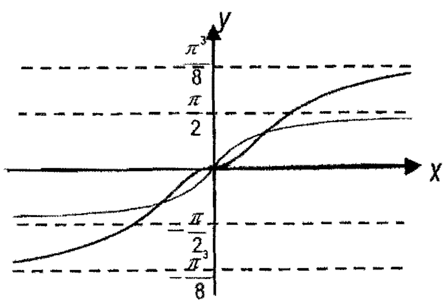
δ) For $0 < x < 1, x^{2000} > x^{2001}, \frac{1}{x^{2000}+1} < \frac{1}{x^{2001}+1}$

$\therefore \int_0^1 \frac{dx}{x^{2000}+1} < \int_0^1 \frac{dx}{x^{2001}+1}$. $\therefore$ True.

5 a) $-\frac{\pi}{2} < \tan^{-1} x < \frac{\pi}{2}, \therefore 0 \le \left(\tan^{-1} x\right)^2 < \frac{\pi^2}{4}$ and $-\frac{\pi^3}{8} < \left(\tan^{-1} x\right)^3 < \frac{\pi^3}{8}$.

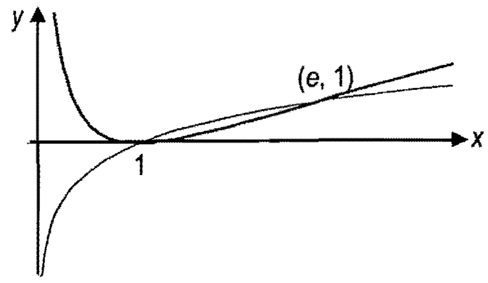


b)

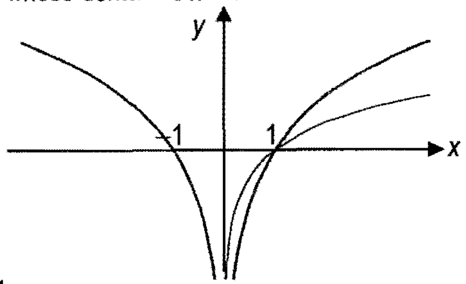


6

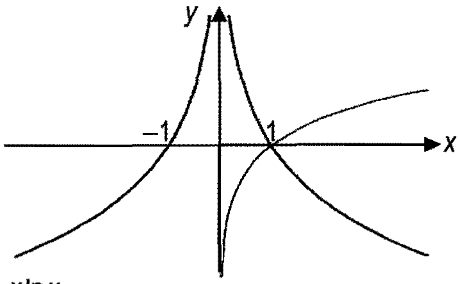
a)



b) $\ln(x^2)$, whose domain is $x \neq 0$, is not the same as $2\ln x$, whose domain is $x > 0$.



c) $\ln\frac{1}{x^2} = \ln 1 - \ln(x^2) = -\ln x^2$. Its graph is (b) reflected about the x-axis.



d) $y = x\ln x$

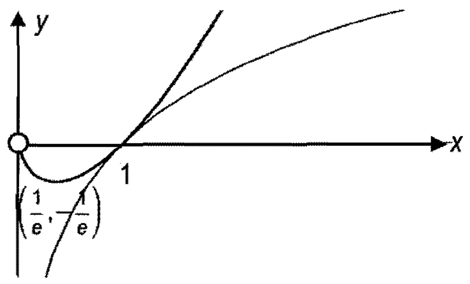
$y' = \ln x + 1$

$y' = 0$ when $\ln x = -1, \therefore x = \frac{1}{e}, \therefore \text{TP}\left(\frac{1}{e}, -\frac{1}{e}\right)$.

y' is a monotonously increasing curve, so this is a minimum point (its gradient changes from negative to positive).

Let $y = 0, x = 0^+$ or $\ln x = 0, \therefore x = 0^+, 1$.

As $x \to 0^+, \ln x \to -\infty, y = x.\ln x \to 0^-$ (because $\ln x$ is a weaker function, compared with x^n).



e) $y = x(\ln x)^2$

$y' = (\ln x)^2 + 2\ln x = \ln x(\ln x + 2)$

$y' = 0$ when $\ln x = 0, -2, \therefore x = \frac{1}{e^2}, 1$.

$\therefore$ Turning points: $\left(\frac{1}{e^2}, \frac{4}{e^2}\right), (1,0)$.

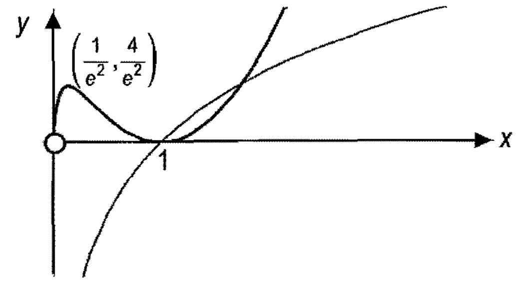
$y'' = 2(\ln x)\frac{1}{x} + \frac{2}{x} = \frac{2(\ln x + 1)}{x}$.

When $x = \frac{1}{e^2}, y'' = -2e^2 < 0, \therefore \left(\frac{1}{e^2}, \frac{4}{e^2}\right)$ is maximum.

When $x = 1, y'' = 2 > 0, \therefore (1,0)$ is a minimum point.

Let $y = 0, x = 0^+$ or $\ln x = 0, \therefore x = 0^+, 1$.

As $x \to 0^+, (\ln x)^2 \to \infty, y = x.\ln x \to 0^+$ (because $\ln x$ is a weaker function, compared with x^n). Also, notice that $y' = (\ln x)^2 \to +\infty$ as $x \to 0^+$.



f) $y = x^2\ln x$

$y' = 2x\ln x + x = x(2\ln x + 1)$

$y' = 0$ when $\ln x = -\frac{1}{2}, \therefore x = \frac{1}{\sqrt{e}}$.

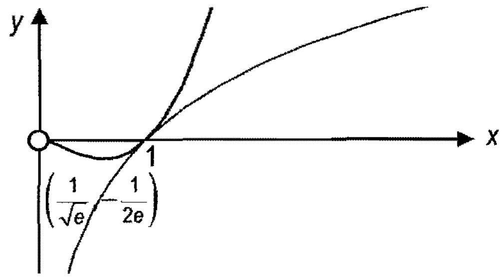
$\therefore$ Turning point: $\left(\frac{1}{\sqrt{e}}, \frac{-1}{2e}\right)$.

$y'' = 2\ln x + 3$.

When $x = \frac{1}{\sqrt{e}}, y'' = 2 > 0, \therefore \left(\frac{1}{\sqrt{e}}, \frac{-1}{2e}\right)$ is minimum.

Let $y = 0, x = 0^+$ or $\ln x = 0, \therefore x = 0^+, 1$.

As $x \to 0^+, \ln x \to -\infty, y = x^2.\ln x \to 0^-$ (because x^2 dominates the function). Also, $y' \to 0$ as $x \to 0^+$.



g) $y = x^2(\ln x)^2$

$y' = 2x(\ln x)^2 + 2x\ln x = 2x\ln x(\ln x + 1)$

$y' = 0$ when $x = 0^+$ or $\ln x = 0, -1, \therefore x = 0^+, 1$ or $\frac{1}{e}$.

$\therefore$ Turning points: $\left(\frac{1}{e}, \frac{1}{e^2}\right), (1,0)$.

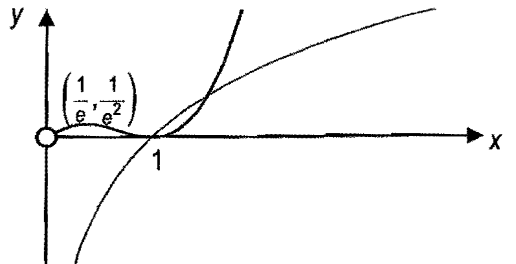
$y'' = 2(\ln x)^2 + 4\ln x + 2\ln x + 2 = 2(\ln x)^2 + 6\ln x + 2.$

When $x = \frac{1}{e}, y'' = -2 < 0, \therefore \left(\frac{1}{e}, \frac{1}{e^2}\right)$ is maximum.

When $x = 1, y'' = 2 > 0, \therefore (1,0)$ is minimum.

Let $y = 0, x = 0^+$ or $\ln x = 0, \therefore x = 0^+, 1$.

As $x \to 0^+, (\ln x)^2 \to +\infty, y = x^2.\ln x \to 0^+$ (because x^2 dominates the function). Also, $y' \to 0$ as $x \to 0^+$.



h) $y = x^2(\ln x^2)$

$y' = 2x(\ln x^2) + x^2\frac{2}{x} = 2x(\ln x^2) + 2x = 2x(\ln x^2 + 1)$

$y' = 0$ when $x = 0$ or $\ln x^2 = -1, \therefore x = 0$ or $\pm\frac{1}{\sqrt{e}}$.

$\therefore$ Turning points: $\left(\pm\frac{1}{\sqrt{e}}, -\frac{1}{e}\right)$ and $(0,0)$ although the curve is undefined when $x = 0$.

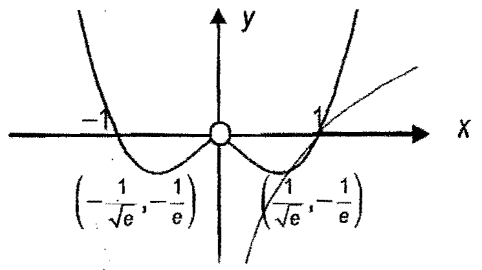
$y'' = 2(\ln x^2) + 4 + 2 = 2\ln x^2 + 6.$

When $x = \pm\frac{1}{\sqrt{e}}, y'' = 4 > 0, \therefore \left(\pm\frac{1}{\sqrt{e}}, -\frac{1}{e}\right)$ is min.

When $x \to 0, y'' \to -\infty < 0, \therefore (0,0)$ is maximum.

Let $y = 0, x = 0^+$ or $\ln x = 0, \therefore x = 0^+, 1$.

As $x \to 0^+$ or $0^-, (\ln x^2) \to -\infty, y = x^2.\ln x^2 \to 0^-$ (because x^2 dominates the function).



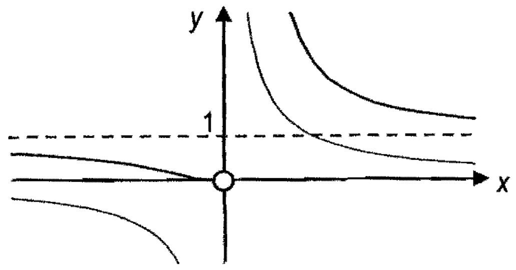
i) $y = e^{1/x}$

$y' = e^{1/x}\left(-\frac{1}{x^2}\right) < 0, \therefore$ No turning points. However, as $x \to 0^-, e^{1/x} \to 0, \therefore y' \to 0.$

When $x \to 0^-, y \to 0^+$; when $x \to 0^+, y \to +\infty$.

When $x \to +\infty, y \to 1^+$; when $x \to -\infty, y \to 1^-$.

Note: This question can be done by drawing $\frac{1}{x}$ then the composite function $e^{1/x}$ (Refer to section 1.4.2)



j) $y = \frac{e^x}{x}$

$y' = \frac{e^x x - e^x}{x^2} = \frac{e^x(x-1)}{x^2}$

$y' = 0$ when $x = 1, \therefore$ Turning point: $(1,e)$.

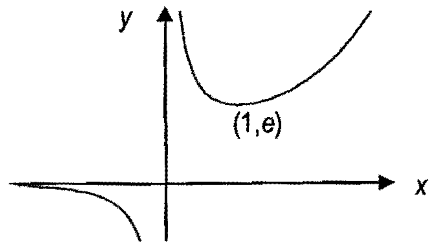
Consider the graph of y', as e^x and $x^2 > 0$ always, the sign of y' is determined by $x - 1$.

For $x < 1, y' < 0$, for $x > 1, y' > 0, \therefore (1,e)$ is minimum.

When $x \to 0, y \to \infty$.

When $x \to +\infty, y \to +\infty$ (e^x dominates the function)

When $x \to -\infty, y \to 0^-$.



k) $y = \frac{e^x}{x^2}$

$y' = \frac{e^x x^2 - 2xe^x}{x^4} = \frac{e^x(x-2)}{x^3}$

$y' = 0$ when $x = 2, \therefore$ Turning point: $\left(2, \frac{e^2}{4}\right)$.

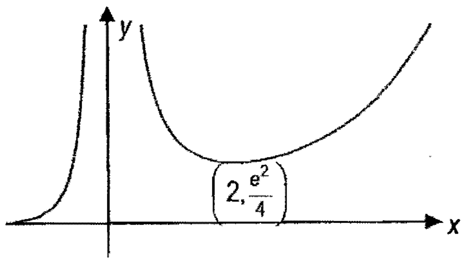
x	1	2	3
y'	–	0	+
	↘	→	↗

$\therefore \left(2, \frac{e^2}{4}\right)$ is a minimum point.

When $x \to 0, y \to \infty$.

When $x \to +\infty, y \to +\infty$.

When $x \to -\infty, y \to 0^+$ (e^x dominates the function)

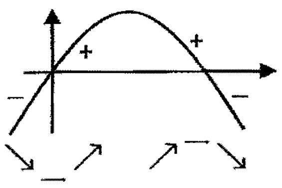


I) $y = \frac{x^2}{e^x}$

$y' = \frac{2xe^x - x^2e^x}{e^{2x}} = \frac{x(2-x)}{e^x}$

$y' = 0$ when $x = 0, 2, \therefore$ Turning point: $(0,0), \left(2, \frac{4}{e^2}\right)$.

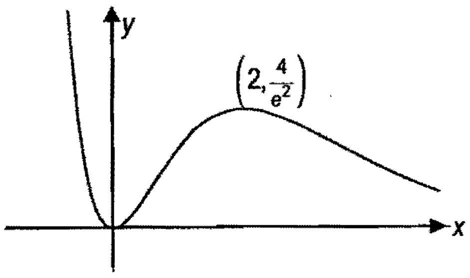
Consider the graph of y', as $e^x > 0$ always, the sign of y' is determined by the graph of $x(2-x)$.



$\therefore x = 0$ is the minimum point, and $x = 2$ is the maximum point.

When $x \to +\infty, y \to 0$ (e^x dominates the function)

When $x \to -\infty, y \to +\infty$.



Notice that the purple curve in each of parts (a) to (h) is $\ln x$, while in part (i) is $\frac{1}{x}$.

(I) GRAPHS OF THE FORM $y = \sqrt{f(x)}$

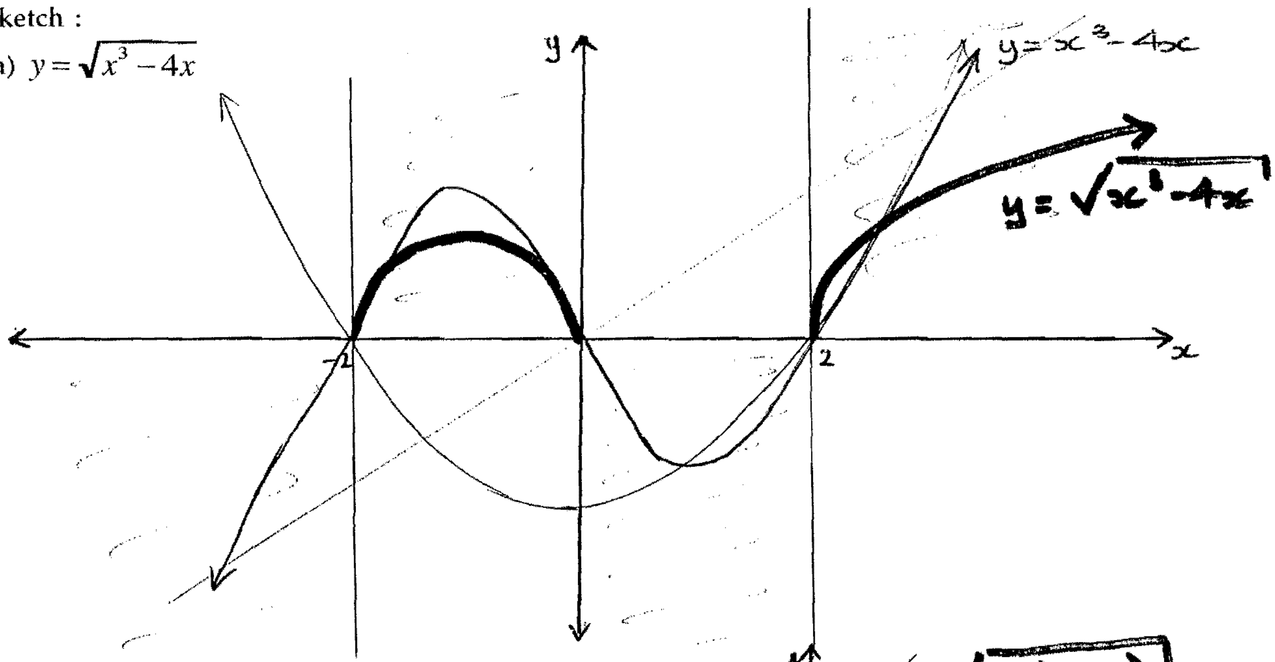
The graph of $y = \sqrt{f(x)}$ can be sketched by first drawing $y = f(x)$ and noticing :

- $\sqrt{f(x)}$ is only defined if $f(x) \geq 0$
- $\sqrt{f(x)} \geq 0$ for all x in the domain
- $\sqrt{f(x)} < f(x)$ if $f(x) > 1$, and $\sqrt{f(x)} > f(x)$ if $0 < f(x) < 1$
- $\dfrac{dy}{dx} = \dfrac{f'(x)}{2\sqrt{f(x)}}$ implies there are critical points where $f(x) = 0$

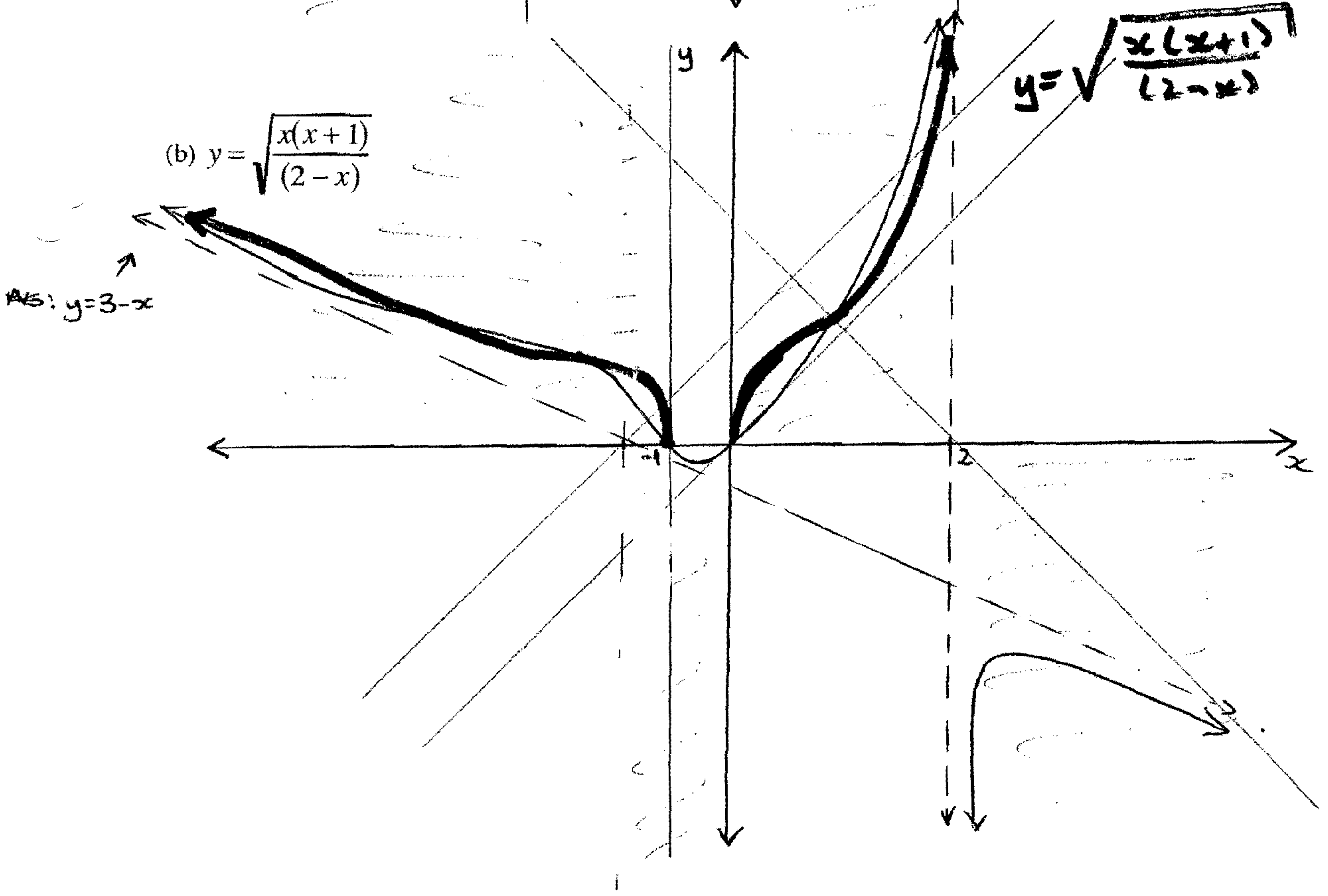
Exercises

Sketch :

(a) $y = \sqrt{x^3 - 4x}$



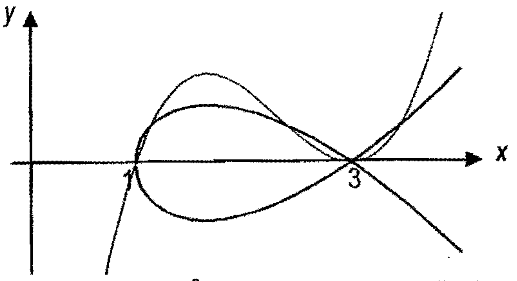
(b) $y = \sqrt{\dfrac{x(x+1)}{(2-x)}}$



Exercise 1.3 (The graph of $y^2 = f(x)$)

1

a)



The curve $y=(x-3)^2(x-1)$ is in purple, while the curve $y^2=(x-3)^2(x-1)$ is in blue.

b) $y^2=(x-3)^2(x-1)$.

$2yy'=2(x-3)(x-1)+(x-3)^2=(x-3)(3x-5)$.

$\therefore y'=\dfrac{(x-3)(3x-5)}{\pm 2(x-3)\sqrt{x-1}}=\dfrac{3x-5}{\pm 2\sqrt{x-1}}$ if $x\neq 3$.

$y'=0$ when $x=\dfrac{5}{3}$, Turning points $\left(\dfrac{5}{3},\pm\dfrac{4\sqrt{6}}{9}\right)$.

$y'\to\infty$ when $x\to 1$.

$y'\to\dfrac{4}{\pm 2\sqrt{2}}=\pm\sqrt{2}$ when $x\to 3$.

2 a) i) Solving $x(x-1)\geq 0$ gives $x\leq 0$ or $x\geq 1$

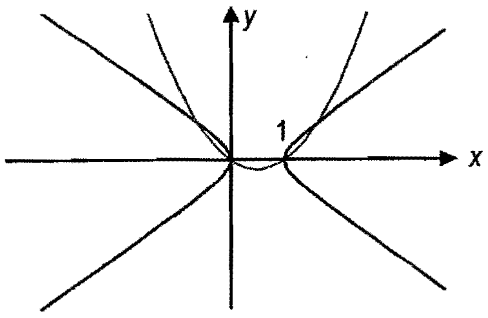
ii) $y^2=x(x-1)=x^2-x$.

$2yy'=2x-1$.

$\therefore y'=\dfrac{2x-1}{\pm 2\sqrt{x(x-1)}}$.

$y'=0$ when $x=\frac{1}{2}$, but this point does not belong to the domain. When $x\to 0$ or 1, $y'\to\infty$.

iii)



b) i) Solving $4-x^4\geq 0$ gives $-\sqrt{2}\leq x\leq\sqrt{2}$

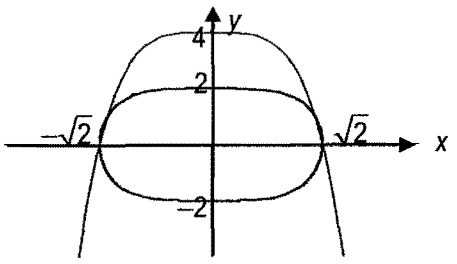
ii) $2yy'=-4x^3$.

$\therefore y' = \dfrac{-4x^3}{\pm 2\sqrt{4-x^4}} = \dfrac{-2x^3}{\pm\sqrt{4-x^4}}$.

$y' = 0$ when $x = 0$, Turning points: $(0, \pm 2)$.

When $x \to \pm\sqrt{2}$, $y' \to \infty$.

iii)



c) i) Solving $(x+5)(x-1)(x-4) \geq 0$ gives $-5 \leq x \leq 1$ or $x \geq 4$.

ii) $y^2 = (x+5)(x-1)(x-4) = x^3 - 21x + 20$.

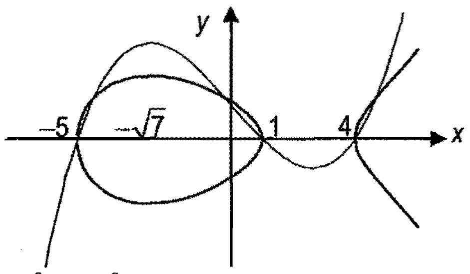
$2yy' = 3x^2 - 21 = 3(x^2 - 7)$.

$\therefore y' = \dfrac{3(x^2-7)}{\pm 2\sqrt{(x+5)(x-1)(x-4)}}$.

$y' = 0$ when $x = \pm\sqrt{7}$, but $x = \sqrt{7}$ does not belong to the domain. Turning points are $\left(-\sqrt{7}, \pm\sqrt{20+14\sqrt{7}}\right)$

When $x \to -5, 1$ or 4, $y' \to \infty$.

iii)



d) i) $(x^2-1)(x^2-4) \geq 0$ gives $x \leq -2, -1 \leq x \leq 1$ or $x \geq 2$.

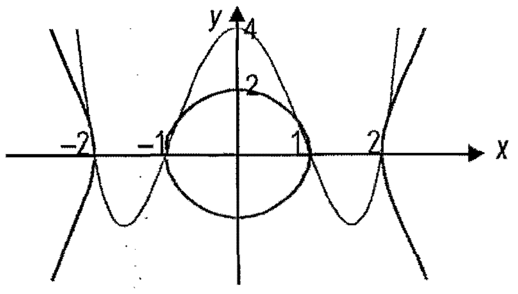
ii) $y^2 = (x^2-1)(x^2-4)$.

$2yy' = 2x(x^2-4) + 2x(x^2-1) = 2x(2x^2-5)$.

$y' = \dfrac{2x(2x^2-5)}{\pm 2(x^2-1)(x^2-4)} = \dfrac{x(2x^2-5)}{\pm(x^2-1)(x^2-4)}$.

When $x \to \pm 1$ or ± 2, $y' \to \infty$. When $x = 0$, $y' = 0$.

iii)



e) i) $x(x-1)^2 \geq 0$ gives $x \geq 0$.

ii) $y^2 = x(x-1)^2$.

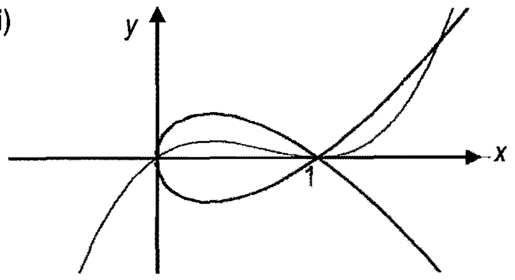
$2yy' = (x-1)^2 + 2x(x-1) = (x-1)(3x-1)$.

$\therefore y' = \dfrac{(x-1)(3x-1)}{\pm 2(x-1)\sqrt{x}} = \dfrac{3x-1}{\pm 2\sqrt{x}}$ if $x \neq 1$.

$y' = 0$ when $x = \frac{1}{3}$, Turning points $\left(\frac{1}{3}, \pm\frac{2}{3\sqrt{3}}\right)$.

When $x \to 0, y' \to \infty$. When $x \to 1, y' \to \pm 1$.

iii)



f) i) $x^2(1-x^2) \geq 0 \Leftrightarrow 1-x^2 \geq 0 \Leftrightarrow -1 \leq x \leq 1$.

ii) $y^2 = x^2(1-x^2) = x^2 - x^4$

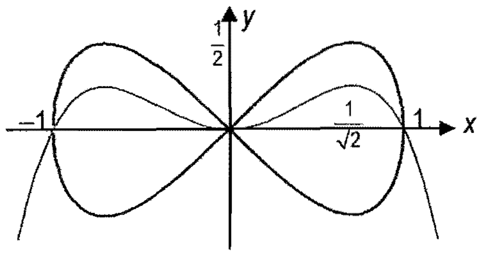
$2yy' = 2x - 4x^3 = 2x(1-2x^2)$.

$\therefore y' = \dfrac{2x(1-2x^2)}{\pm 2x\sqrt{1-x^2}} = \pm\dfrac{1-2x^2}{\sqrt{1-x^2}}$ if $x \neq 0$.

$y' = 0$ when $x = \pm\frac{1}{\sqrt{2}}$, $\therefore$ Turning points $\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{2}\right)$.

When $x \to \pm 1, y' \to \infty$. When $x \to 0, y' \to \pm 1$.

iii)



g) i) $(x-1)^2(x-3) \geq 0$ gives $x \geq 3$ or $x = 1$.

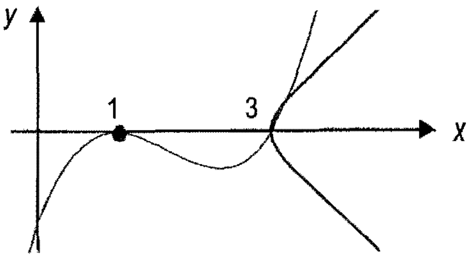
ii) $y^2 = (x-1)^2(x-3)$.

$2yy' = 2(x-1)(x-3) + (x-1)^2 = (x-1)(3x-7)$.

$y' = \dfrac{(x-1)(3x-7)}{\pm 2(x-1)\sqrt{x-3}} = \dfrac{3x-7}{\pm 2\sqrt{x-3}}$, if $x \neq 1$.

When $x \to 3, y' \to \infty$.

iii)



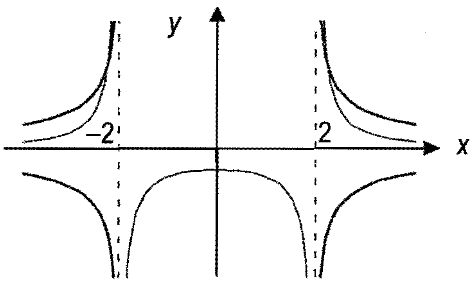
h) i) $\frac{1}{x^2-4} \geq 0 \Leftrightarrow x^2-4>0 \Leftrightarrow x<-2$ or $x>2$.

ii) $y^2=\frac{1}{x^2-4}$ then $2yy'=\frac{-2x}{(x^2-4)^2}$.

$y'=\frac{-x\sqrt{x^2-4}}{(x^2-4)^2}=\frac{-x}{\sqrt{(x^2-4)^3}}$ if $x \neq \pm 2$.

$y'=0$ when $x=0$, but $x=0$ does not belong to the domain. When $x \to \pm 2$, $y' \to \infty$.

iii)



Note: In each question, the purple curve is $y=f(x)$ while the blue curve is $y^2=f(x)$.

3

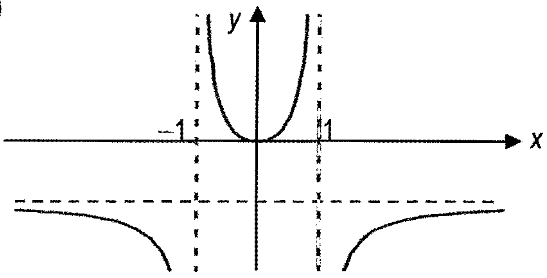
a) Let $f(x)=\frac{x^2}{1-x^2}$ then $f(-x)=\frac{(-x)^2}{1-(-x)^2}=\frac{x^2}{1-x^2}=f(x)$, $\therefore$ This is an even function. Even functions are symmetrical about the y-axis.

b) As $x \to \pm\infty, \frac{x^2}{1-x^2}=-1+\frac{1}{1-x^2} \to -1^-$.

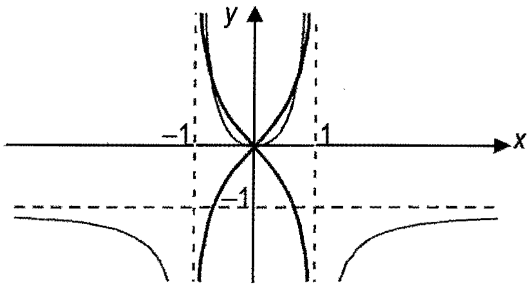
c) $y'=\frac{2x(1-x^2)+2x^3}{(1-x^2)^2}=\frac{2x}{(1-x^2)^2}$.

$y'=0$ when $x=0, \therefore$ Turning point is (0,0).

d)



e)



4 i)

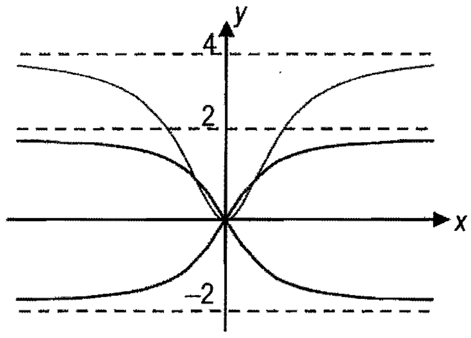
a) Let $f(x)=\frac{4x^2}{1+x^2}$ then $f(-x)=\frac{4(-x)^2}{1+(-x)^2}=\frac{4x^2}{1+x^2}=f(x)$, $\therefore$ Even.

b) As $x \to \pm\infty, \frac{4x^2}{1+x^2}=4-\frac{4}{1+x^2} \to 4^-$.

c) $y'=\frac{8x(1+x^2)-8x^3}{(1+x^2)^2}=\frac{8x}{(1+x^2)^2}$.

$y'=0$ when $x=0, \therefore$ Turning point is (0,0).

d) It's the purple curve, and e) it's the blue curve.



ii)

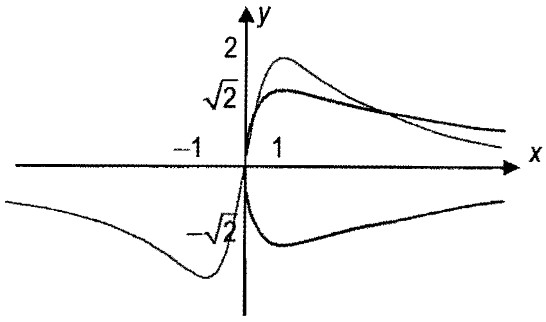
a) Let $f(x)=\frac{4x}{1+x^2}$ then $f(-x)=\frac{4(-x)}{1+(-x)^2}=\frac{-4x}{1+x^2}=-f(x)$, $\therefore$ Odd. Odd functions are symmetrical about the origin (point of symmetry).

b) As $x \to +\infty, \frac{4x}{1+x^2} \to 0^+$. As $x \to -\infty, \frac{4x}{1+x^2} \to 0^-$.

c) $y'=\frac{4(1+x^2)-8x^2}{(1+x^2)^2}=\frac{4(1-x^2)}{(1+x^2)^2}$.

$y'=0$ when $x=\pm 1, \therefore$ Turning points are $\pm(1,2)$.

d) It's the purple curve, and e) it's the blue curve.



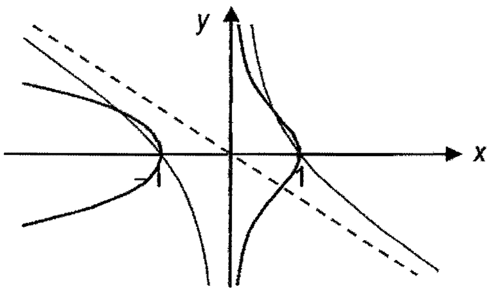
iii)

a) Let $f(x)=\frac{1-x^2}{x}$ then $f(-x)=\frac{1-(-x)^2}{(-x)}=\frac{1-x^2}{-x}=-f(x)$, $\therefore$ Odd.

b) As $x \to +\infty, \frac{1-x^2}{x}=\frac{1}{x}-x \to -x$.

c) $y=\frac{1}{x}-x, \therefore y'=-\frac{1}{x^2}-1<0. \therefore$ No turning points.

d) It's the purple curve, and e) it's the blue curve.



iv)

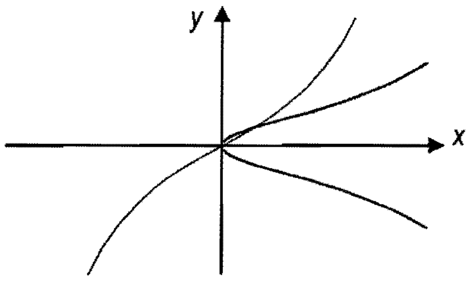
a) Let $f(x)=e^x-e^{-x}$ then $f(-x)=e^{-x}-e^x=-f(x)$

$\therefore$ Odd.

b) As $x \to +\infty, e^{-x} \to 0, \therefore e^x-e^{-x} \to \infty$.

c) $y'=e^x+e^{-x}>0. \therefore$ No turning points.

d) It's the purple curve, and e) it's the blue curve.



v)

a) Let $f(x)=\frac{1}{e^x+e^{-x}}$ then $f(-x)=\frac{1}{e^{-x}+e^x}=f(x)$

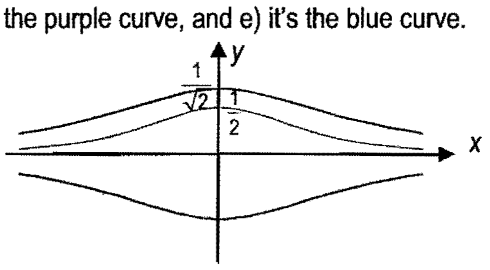
$\therefore$ Even, $\therefore$ The curve is symmetrical about the y-axis.

b) As $x \to +\infty, e^x+e^{-x} \to \infty, \therefore \frac{1}{e^x+e^{-x}} \to 0$.

c) $y'=\frac{-(e^x-e^{-x})}{(e^x+e^{-x})^2}$.

$y'=0$ when $e^x=e^{-x}, \therefore e^{2x}=1, \therefore x=0, \therefore$ TP: $(0,\frac{1}{2})$.

d) It's the purple curve, and e) it's the blue curve.



vi)

a) Let $f(x)=\frac{e^x-e^{-x}}{e^x+e^{-x}}$ then $f(-x)=\frac{e^{-x}-e^x}{e^{-x}+e^x}=-f(x)$

$\therefore$ Odd. $\therefore$ The curve is symmetrical about the origin.

b) As $x \to +\infty$, dividing both top and bottom by e^x,

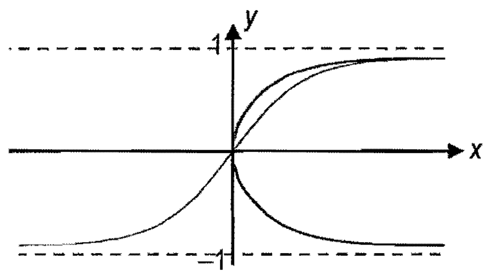
$\frac{e^x-e^{-x}}{e^x+e^{-x}}=\frac{1-e^{-2x}}{1+e^{-2x}} \to \frac{1-0}{1+0}=1$.

c) $y'=\frac{(e^x+e^{-x})^2-(e^x-e^{-x})^2}{(e^x+e^{-x})^2}$

$=\frac{(e^{2x}+e^{-2x}+2)-(e^{2x}+e^{-2x}-2)}{(e^x+e^{-x})^2}=\frac{4}{(e^x+e^{-x})^2}$.

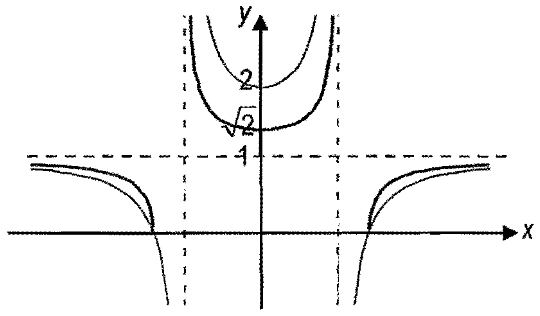
$y''>0 \therefore$ No turning points.

d) It's the purple curve, and e) it's the blue curve.

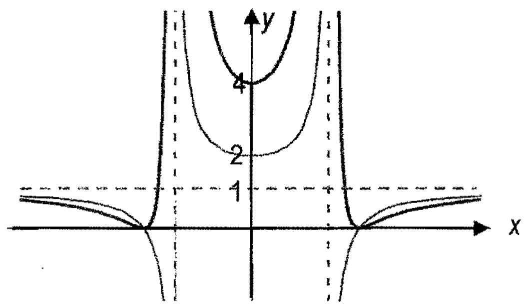


5

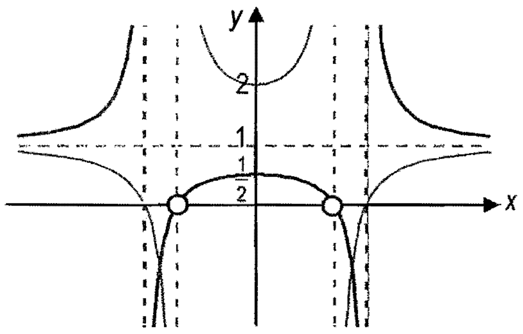
a)



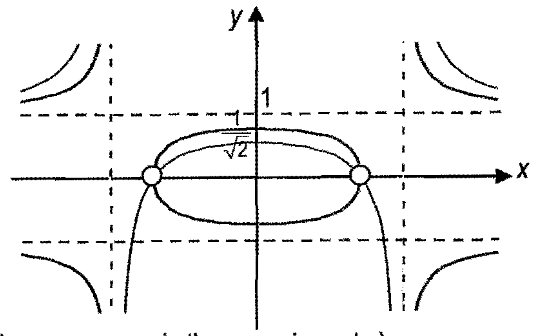
b)



c)



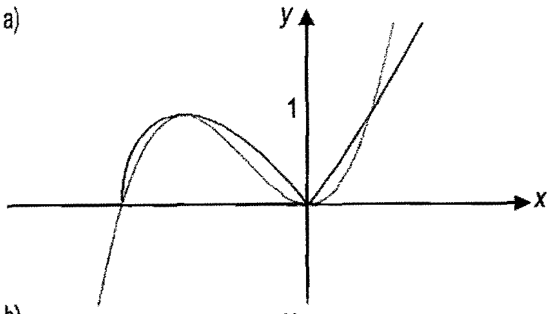
d)



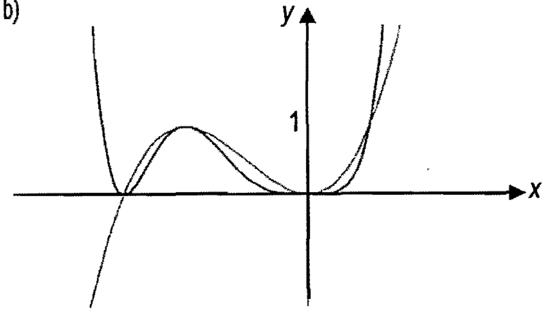
(The green curve is the curve in part c)

6

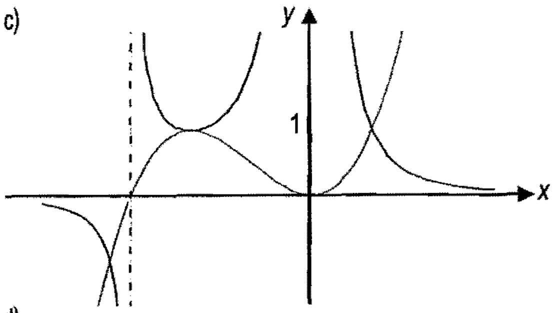
a)



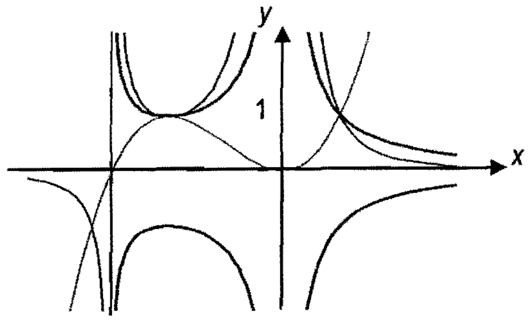
b)



c)



d)



(The green curve is the curve in part c)

7 a) $x^2y^2 = x^2 + y^2 \Leftrightarrow y^2(x^2-1) = x^2$

$\therefore y^2 = \dfrac{x^2}{x^2-1}, \therefore y = \pm\dfrac{x}{\sqrt{x^2-1}}$

Domain: $x < -1, x > 1$ or $x = 0$

Let's draw the curve $y = \dfrac{x^2}{x^2-1} = 1 + \dfrac{1}{x^2-1}$ first.

Asymptotes: $x = \pm 1, y = 1$.

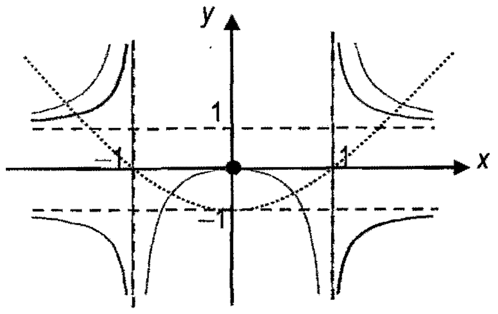
$y' = \dfrac{2x(x^2-1)-2x^3}{(x^2-1)^2} = \dfrac{-2x}{(x^2-1)^2}, y' = 0$ when $x = 0$

$\therefore$ Turning point (0,0). This curve is drawn in red.
(The guide graph $x^2 - 1$ is added in green).

The curve $y^2 = \dfrac{x^2}{x^2-1}$ is drawn in blue, noting that for

$-1 < x < 1, \dfrac{x^2}{x^2-1} < 0$ except when $x = 0$, so its square roots cannot be found. For $x < -1$ or $x > 1$, since

$\dfrac{x^2}{x^2-1} > 1, \sqrt{\dfrac{x^2}{x^2-1}} < \dfrac{x^2}{x^2-1}$.



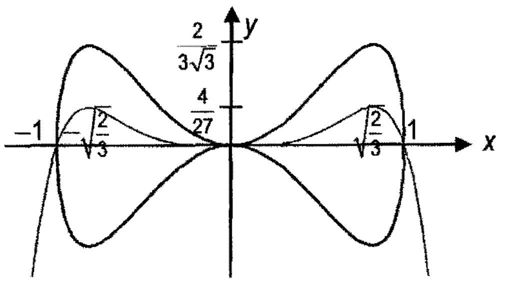
b) Domain: $x^4 - x^6 \geq 0 \Leftrightarrow 1 - x^2 \geq 0 \Leftrightarrow -1 \leq x \leq 1$.

$2yy' = 4x^3 - 6x^5$.

$y' = \dfrac{4x^3 - 6x^5}{\pm 2\sqrt{x^4 - x^6}} = \dfrac{x^3(4-6x^2)}{\pm 2x^2\sqrt{1-x^2}} = \dfrac{x(4-6x^2)}{\pm 2\sqrt{1-x^2}}$ if $x \neq 0$

When $x \to \pm 1, y' \to \infty$. When $x \to 0, y' \to 0$.

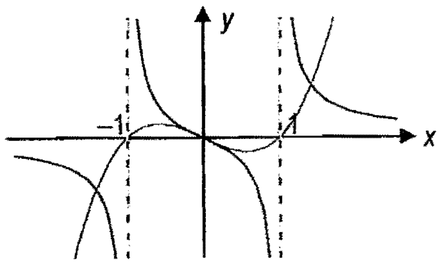
Turning points (0,0), $\left(\pm\sqrt{\dfrac{2}{3}}, \pm\dfrac{2}{3\sqrt{3}}\right)$.



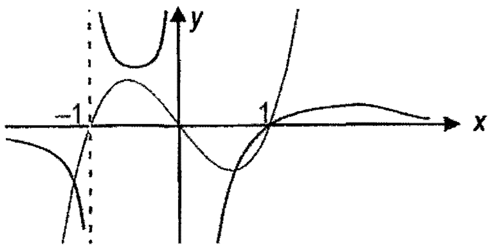
The red curve is $y = x^4 - x^6$ and the blue curve is $y^2 = x^4 - x^6$.

Review Exercise 1.5

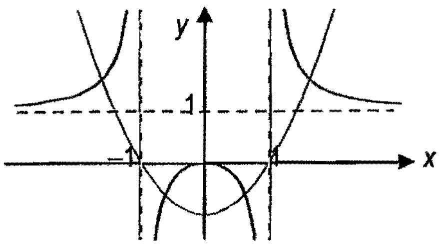
1 a) Asymptotes $x = \pm 1$ and $y = 0$.
The x-intercept (0,0).
(The guide graph $x(x^2 - 1)$ is added in purple)



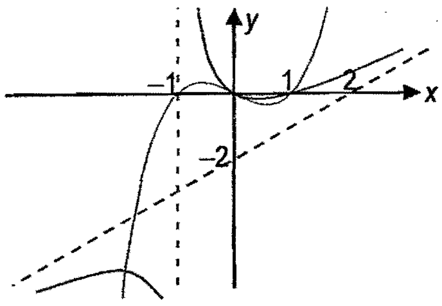
b) Asymptotes $x = -1$, 0 and $y = 0$.
The x-intercept (1,0).
(The guide graph $(x-1)x(x+1)$ is added in purple)



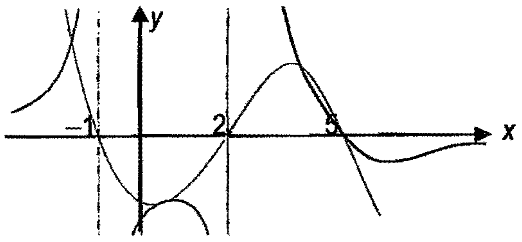
c) Asymptotes $x = \pm 1$ and $y = 1$.
(The guide graph $x^2 - 1$ is added in purple)



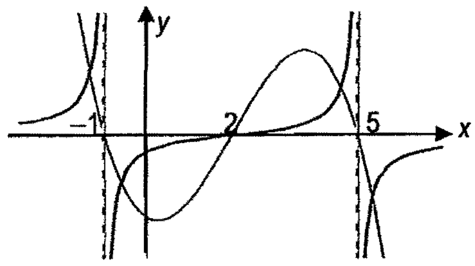
d) Asymptotes $x = -1$ and $y = x - 2$.
(The guide graph $(x-1)x(x+1)$ is added in purple)



e) Asymptotes $x = 2, -1$ and $y = 0$.
Note: The guide graph $(5-x)(x-2)(x+1)$ is added in purple in parts (e) and (f).



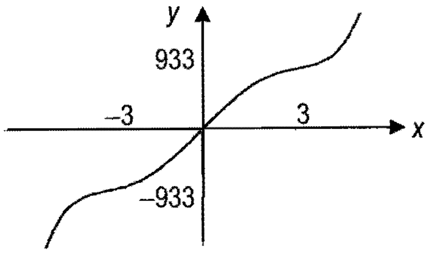
f) Asymptotes $x = -1, 5$ and $y = 0$.



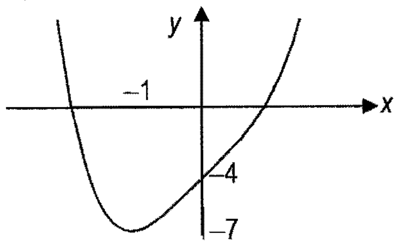
g) $\dfrac{2x^2+4x+3}{x^2-1} = 2 + \dfrac{4x+5}{x^2-1}$.

As $2x^2+4x+3>0$ (its discriminant is –8), the guide graph is x^2-1, i.e. $y>0$ for $x<-1$ or $x>1$; and $y<0$ for $-1<x<1$.

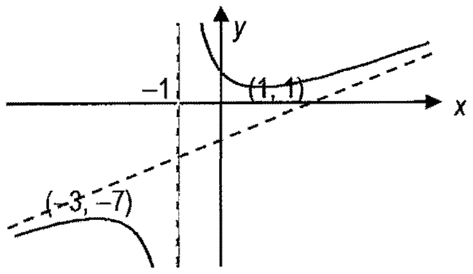
For $x < -3$ or $0 < x < 3, y'' < 0$: concave down.
For $x > 3$ or $-3 < x < 0, y'' > 0$: concave up.



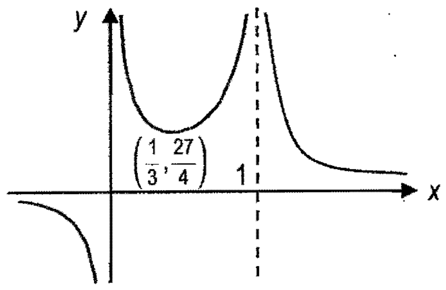
f) $y' = 4x^3 + 4$.
$y' = 0$ when $x = -1$, $\therefore$ Turning point: (–1, –7).
$y'' = 12x^2$.
When $x = -1, y'' = 12 > 0, \therefore$ minimum point.
$y'' = 0$ when $x = 0$.
However, since $y'' = 12x^2 \geq 0$, the curve does not change its concavity at the neighbourhood of x = 0, $\therefore$(0,–4) is not a point of inflexion.



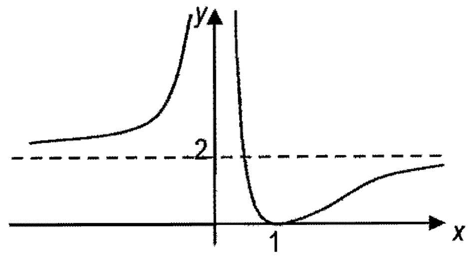
g) $\frac{x^2 - x + 2}{x+1} = x - 2 + \frac{4}{x+1}$.
Asymptotes: $x = -1$, and $y = x - 2$.
$y' = 1 - \frac{4}{(x+1)^2}$.
$y' = 0$ when $(x+1)^2 = 4, \therefore x = -3, 1$
$\therefore$Turning points: (–3, –7) and (1, 1).



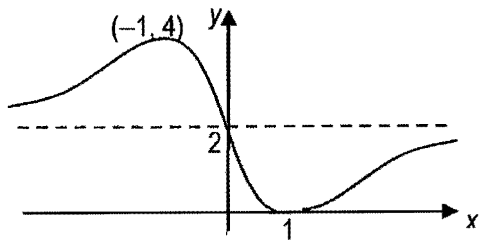
h) Asymptotes: x = 0, 1 and y = 0.
$y' = \frac{-(3x^2 - 4x + 1)}{x^2(x-1)^4} = \frac{-(3x-1)(x-1)}{x^2(x-1)^4} = \frac{1-3x}{x^2(x-1)^3}$ if $x \neq 1$.
$y' = 0$ when $x = \frac{1}{3}$. $\therefore$ Turning point: $\left(\frac{1}{3}, \frac{27}{4}\right)$.



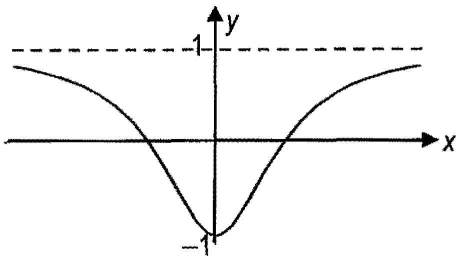
i) $\frac{2(x-1)^2}{x^2} = \frac{2x^2 - 4x + 2}{x^2} = 2 - \frac{4}{x} + \frac{2}{x^2}$.
Asymptotes: x = 0 and y = 2.
As $x \to +\infty, y \approx 2 - \frac{4}{x} \to 2^-$; as $x \to -\infty, y \to 2^+$.
$y' = \frac{4}{x^2} - \frac{4}{x^3} = \frac{4(x-1)}{x^3}, y' = 0$ when $x = 1$.
$\therefore$Turning point: (1, 0).



j) $\frac{2(x-1)^2}{x^2+1} = \frac{2x^2 - 4x + 2}{x^2+1} = 2 - \frac{4x}{x^2+1}$.
As $x \to +\infty, y \approx 2 - \frac{4}{x} \to 2^-$; as $x \to -\infty, y \to 2^+$.
$y' = -\frac{4(x^2+1) - 8x^2}{(x^2+1)^2} = \frac{4(x^2-1)}{(x^2+1)^2}, y' = 0$ when $x = \pm 1$.
$\therefore$ Turning points: (1, 0) and (–1, 4).



k) $\frac{x^2 - 1}{x^2 + 1} = 1 - \frac{2}{x^2+1}$.
As $x \to \pm\infty, y \to 1^-$.
$y' = \frac{4x}{(x^2+1)^2}, y' = 0$ when $x = 0$.
$\therefore$ Turning point: (0, –1).
This is an even function, so, the curve is symmetrical about the y-axis.



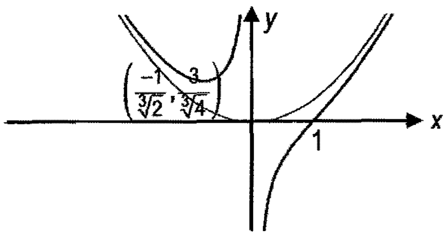
l) $\frac{x^3-1}{x}=x^2-\frac{1}{x}$

As $x\to 0, y\to\infty$; as $x\to\pm\infty, y\to x^2$.

$y'=2x+\frac{1}{x^2}, y'=0$ when $x^3=-\frac{1}{2}, \therefore x=\frac{-1}{\sqrt[3]{2}}$.

$\therefore$ Turning point: $\left(\frac{-1}{\sqrt[3]{2}},\frac{3}{\sqrt[3]{4}}\right)$.

(Note: The purple curve is x^2.)



3 a) $g(x)=xe^{-x}=0$ when $x=0$, positive when $x>0$ and negative when $x<0$.

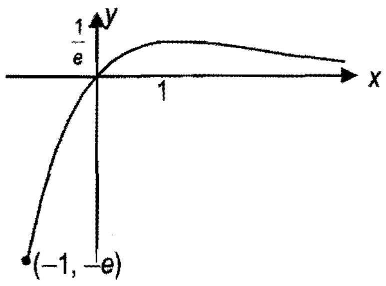
$g'(x)=-xe^{-x}+e^{-x}=e^{-x}(1-x)$.

$g'(x)=0$ when $x=1, \therefore$ Turning point $\left(1,\frac{1}{e}\right)$.

$g''(x)=-e^{-x}-e^{-x}(1-x)=e^{-x}(x-2)$.

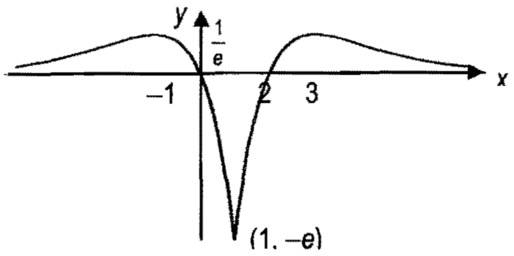
$g''(x)=0$ when $x=2, \therefore$ Point of inflexion $\left(2,\frac{2}{e^2}\right)$.

When $x\to+\infty, g(x)\to 0$. When $x=-1, g(-1)=-e$.



b) $g(x-2)$ is the graph of $g(x)$ having translated to the right 2 units, $\therefore g(x-2)$ starts from $(1,-e)$ goes up to a maximum at $\left(3,\frac{1}{e}\right)$, then decreases to zero as $x\to+\infty$.

For $x\le 1, g(-x)$ is the reflection of $g(x)$ about the y-axis.



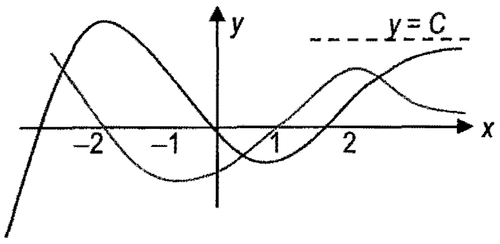
4 a) As $x\to\infty, f'(x)\to 0, \therefore f(x)\to C$. Since $f(3)>0$ (given), C must be positive.

b) Stationary points at $x=-2$ and 1.

At $x=-2$, as its gradient changes from positive to negative, it's a maximum point.

At $x=1$, as its gradient changes from negative to positive, it's a minimum point.

The gradient is most negative at $x=-1$ and most positive at $x=2$, hence, these points are points of inflexion.



5 a) Consider $h(x)=x^4-4x^3+4x^2$

$=x^2(x^2-4x+4)=x^2(x-2)^2$.

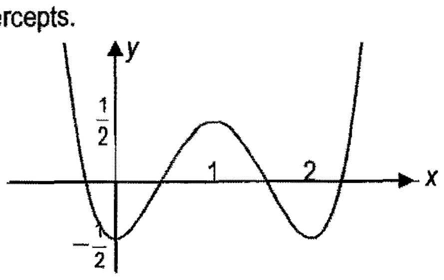
Since $h(x)$ is the square of the parabola $x(x-2)$, $h(x)$ has two minimum points at $x=0$ and 2, and a maximum point at $x=1$.

$\therefore$ Turning points of $h(x)$: (0, 0), (2, 0) and (1, 1).

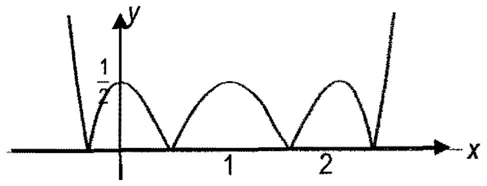
$\therefore$ The graph of $x^4-4x^3+4x^2-\frac{1}{2}$ (it's the graph of $h(x)$ shifted down $\frac{1}{2}$ units) cuts the x-axis at 4 points.

$\therefore$ The coordinates of the turning points are $\left(0,-\frac{1}{2}\right)$, $\left(1,\frac{1}{2}\right)$ and $\left(2,-\frac{1}{2}\right)$.

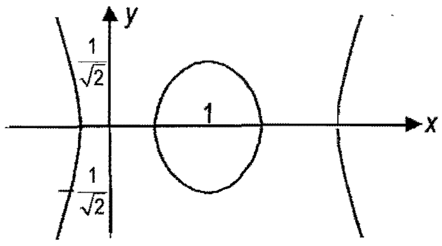
Note: We cannot tell the exact values of the x-intercepts.



b)
i)



ii) The curve $y^2 = g(x)$ is equivalent to $y = \pm\sqrt{g(x)}$. It is defined on the domain where $g(x) \geq 0$ only.



c) $2yy' = g'(x)$.

$\therefore y' = \frac{g'(x)}{2y} = \frac{g'(x)}{\pm 2\sqrt{g(x)}}$.

y' is undefined when $g(x) = 0$, $\therefore$ At the zeros of $g(x)$, the curve of $y^2 = g(x)$ has vertical tangents.

6 a) $f'(x) = \frac{1-\ln x}{x^2}$.

$f'(x) = 0$ when $\ln x = 1, \therefore x = e, \therefore$ TP: $\left(e, \frac{1}{e}\right)$.

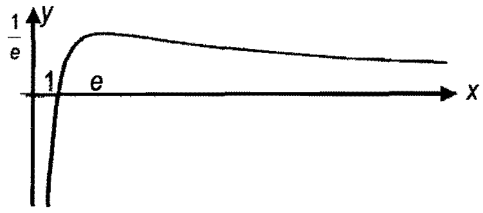
$f''(x) = \frac{-x - 2x(1-\ln x)}{x^4} = \frac{-3+2\ln x}{x^3}$.

$f''(x) = 0$ when $\ln x = \frac{3}{2}, \therefore x = e^{\frac{3}{2}}, \therefore$ POI: $\left(e^{\frac{3}{2}}, \frac{3}{2e^{3/2}}\right)$

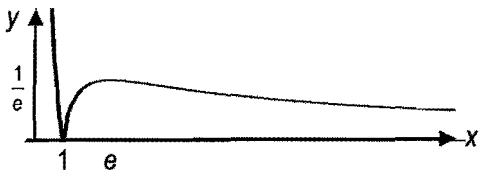
When $x = e, f''(e) = -\frac{1}{e^3} < 0, \therefore \left(e, \frac{1}{e}\right)$ is max.

b) As $x \to +\infty, f(x) \to 0^+$ (since the function is dominated by x). As $x \to 0, f(x) \to -\infty$.

c)

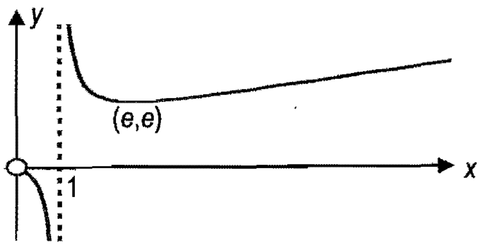


d)
i)



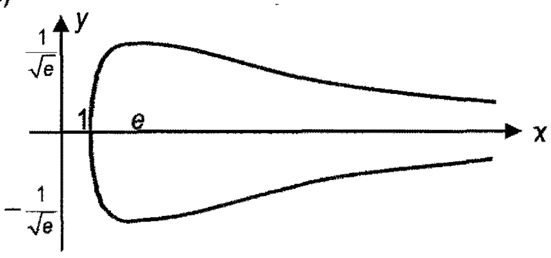
Note: The graph of $\left|\frac{\ln x}{x}\right|$ has a sharp point at $x = 1$.

ii)



Note: As the question does not ask for further calculation, there is no need to calculate the point of inflexion, which occurs at $x = e^2$.

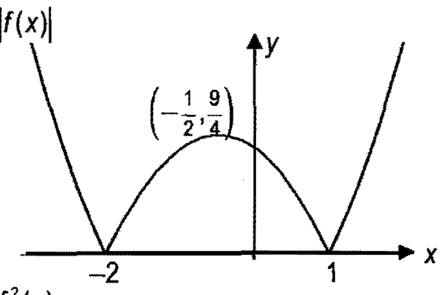
iii)



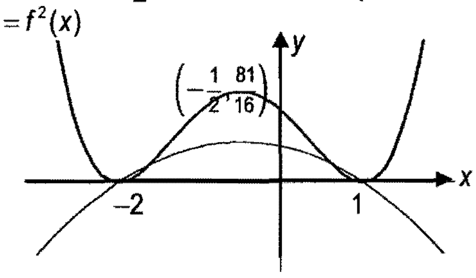
7 For $f(x) = 2 - x - x^2 = (2+x)(1-x)$, the x-intercepts are $(-2,0)$ and $(1,0)$.

Vertex: $x = \frac{-2+1}{2} = -\frac{1}{2}, y = 2 - \left(-\frac{1}{2}\right) - \left(-\frac{1}{2}\right)^2 = \frac{9}{4}$.

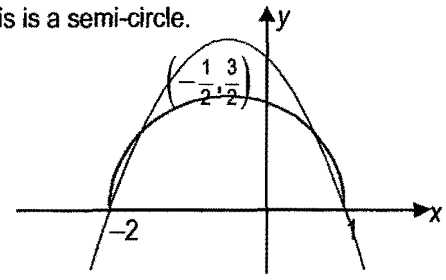
a) $y = |f(x)|$



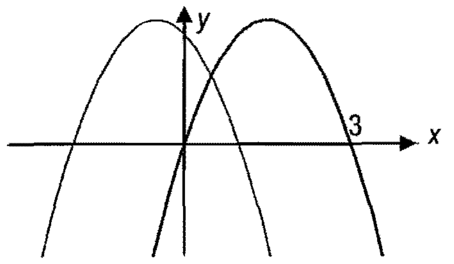
b) $y = f^2(x)$



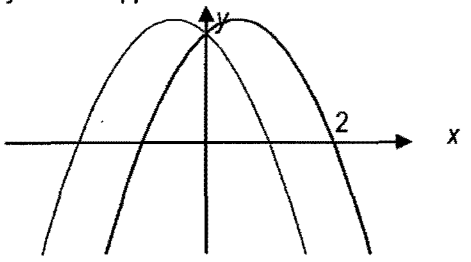
c) This is a semi-circle.



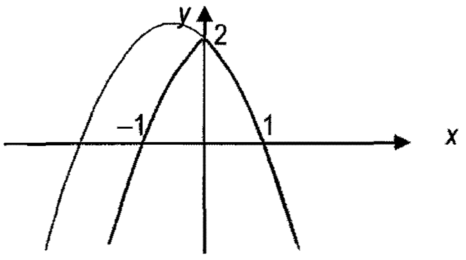
d) It's the graph of $f(x)$ shifted to the right 2 units.



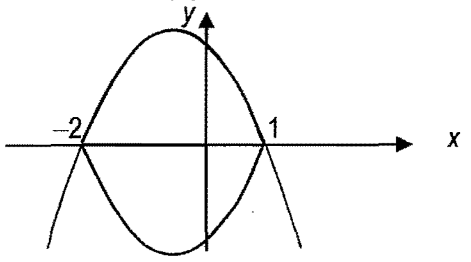
e) It's the graph of $f(x)$ but its left and right sides of the y-axis swapped.



f) If $x > 0$, it's $f(x)$; If $x < 0$, it's the same as part (e).



g) Where $f(x) > 0, |y| = f(x) \Leftrightarrow \pm y = f(x)$.

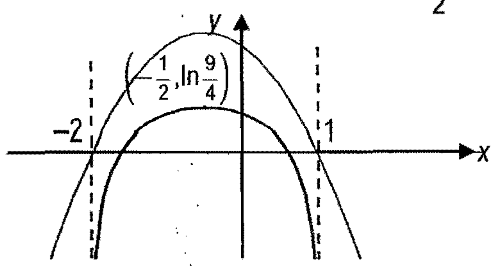


h) As $x \to -2$ or $1, f(x) \to 0, \log(f(x)) \to -\infty$.

The maximum value is $\left(-\frac{1}{2}, \ln\frac{9}{4}\right)$.

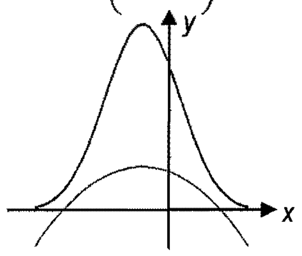
The x-intercepts are found by solving $f(x) = 1$,

$\therefore 2 - x - x^2 = 1 \Rightarrow x^2 + x - 1 = 0, \therefore x = \frac{-1 \pm \sqrt{5}}{2}$.



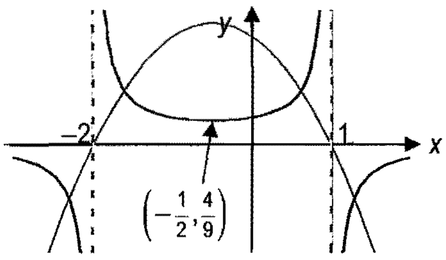
i) As $x \to \pm\infty, e^{f(x)} \to 0^+$.

The maximum value is $\left(-\frac{1}{2}, e^{\frac{9}{4}}\right)$.



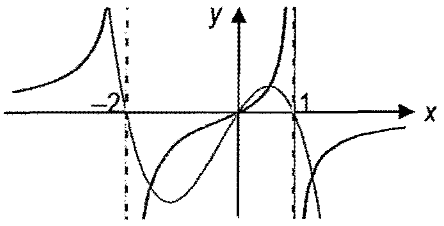
j) As $x \to -2$ or $1, f(x) \to 0, \frac{1}{f(x)} \to \infty$.

The minimum value is $\left(-\frac{1}{2}, \frac{4}{9}\right)$.



k) This curve has equation $y = \frac{x}{(1-x)(2+x)}$.

The guide graph is $x(1-x)(2+x)$ (shown in purple)



l) Beginning with ln x, critical y-values are $-\infty$, 0 and $+\infty$, which occur when $x \to 0, x = 1$ and $x \to +\infty$ respectively.

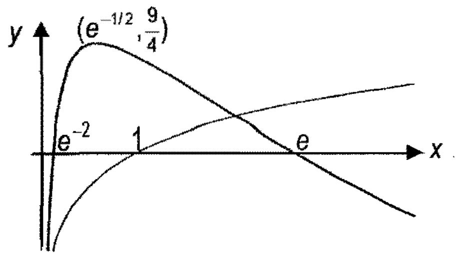
When $x \to 0, \ln x \to -\infty, f(\ln x) \to f(-\infty) = -\infty$.

When $x = 1, \ln x = 0, f(\ln x) = f(0) = 2$.

When $x \to +\infty, \ln x \to +\infty, f(\ln x) \to f(+\infty) = -\infty$.

$f(x)$ is max. when $x = -\frac{1}{2}, \therefore f(\ln x)$ is max. when

$\ln x = -\frac{1}{2}. \therefore x = e^{-1/2}$.

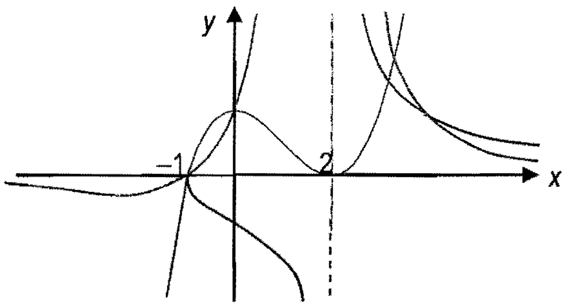


Finding the x-intercepts:
$f(x)=0$ when $x=-2$ or $1, \therefore f(\ln x)=0$ when
$\ln x=-2$ or $1, \therefore x=e^{-2}$ or e.

8 a) For $y=\dfrac{x+1}{(x-2)^2}$, asymptotes $x = 2$ and $y = 0$.

Also, the curve is positive for $x \geq -1$. The curve is shown in purple colour (the guide graph is in green).

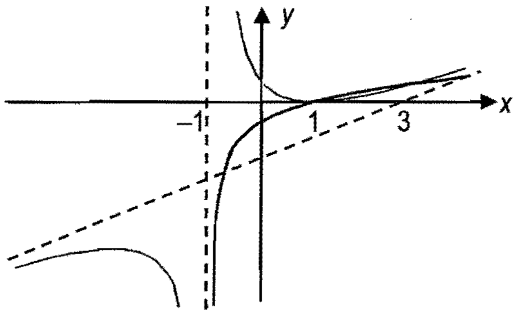
The curve $y=\dfrac{\sqrt{x+1}}{x-2}$ is not simply the positive square root of the other curve. The reason: for $-1 < x < 2$, $x-2$ is negative, therefore, this branch is below the x-axis. For $x > 2$: it's the positive square root of the other curve.



b) $\dfrac{(x-1)^2}{x+1}=x-3+\dfrac{4}{x+1}$.

Asymptotes $x=-1$ and $y=x-3$.
One of the x-intercept, which is also a turning point is $(1,0)$.

The curve $y=\dfrac{x-1}{\sqrt{x+1}}$ is not simply the positive square root of the other curve. The reason: for $-1 < x < 1$, $x-1$ is negative, therefore, this branch is below the x-axis. Also, note that the vertical asymptote $x = -1$ holds true for both curves, but the oblique asymptote disappears in the second curve.



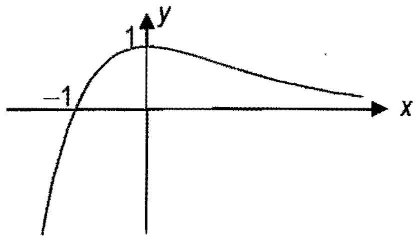
9 a) $y=\dfrac{x+1}{e^x}$.

$y'=\dfrac{e^x-e^x(x+1)}{e^{2x}}=\dfrac{-x}{e^x}$.

$y'=0$ when $x=0, \therefore$ Turning point: $(0,1)$.

As $x \to \infty, \dfrac{x+1}{e^x} \to 0$ (e^x dominates the function)

As $x \to -\infty, \dfrac{x+1}{e^x} \to -\infty$.



b) $y=\dfrac{e^x}{x^2+1}$.

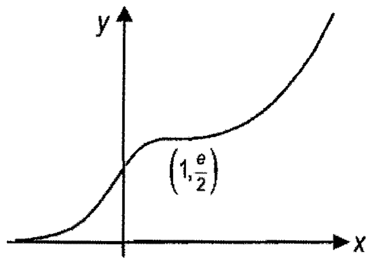
$y'=\dfrac{e^x(x^2+1)-2xe^x}{(x^2+1)^2}=\dfrac{e^x(x-1)^2}{(x^2+1)^2}$.

$y'=0$ when $x=1, \therefore$ Turning point: $(1,\frac{e}{2})$.

Consider the sign of y', which is always positive except when $x = 1$. $\therefore \left(1,\frac{e}{2}\right)$ is a horizontal point of inflexion.

As $x \to \infty, \dfrac{e^x}{x^2+1} \to \infty$ (as e^x dominates the function)

As $x \to -\infty, \dfrac{e^x}{x^2+1} \to 0$.



c) $y=\dfrac{e^x}{(x+1)^2}$.

$y'=\dfrac{e^x(x+1)^2-2(x+1)e^x}{(x+1)^4}=\dfrac{e^x(x-1)}{(x+1)^3}$.

$y'=0$ when $x=1, \therefore$ Turning point: $(1,\frac{e}{4})$.

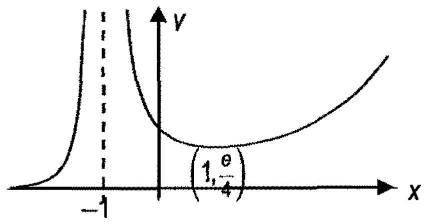
x	0	1	2
y'	$-e$	0	$\frac{e^2}{8}$
	↘	→	↗

$\therefore$ This is a minimum point.

As $x \to \infty, \dfrac{e^x}{(x+1)^2} \to \infty$ (e^x dominates the function)

As $x \to -\infty, \dfrac{e^x}{(x+1)^2} \to 0.$

As $x \to -1, \dfrac{e^x}{(x+1)^2} \to +\infty.$



d) $y = \dfrac{e^x}{x^2-3}$.

$y' = \dfrac{e^x(x^2-3)-2xe^x}{(x^2-3)^2} = \dfrac{e^x(x+1)(x-3)}{(x^2-3)^2}.$

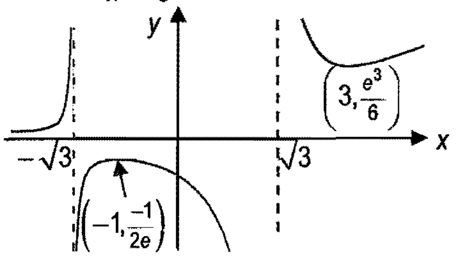
$y' = 0$ when $x = -1, 3$.

$\therefore$ Turning points: $(-1, \frac{-1}{2e})$ and $(3, \frac{e^3}{6})$.

As $x \to \infty, \dfrac{e^x}{x^2-3} \to \infty$ (e^x dominates the function)

As $x \to -\infty, \dfrac{e^x}{x^2-3} \to 0.$

As $x \to \pm\sqrt{3}, \dfrac{e^x}{x^2-3} \to +\infty.$



10 a) $x^2 - y^2 + xy = 5$.

$2x - 2y\dfrac{dy}{dx} + y + x\dfrac{dy}{dx} = 0$

$(x-2y)\dfrac{dy}{dx} = -2x - y.$

$\therefore \dfrac{dy}{dx} = \dfrac{-2x-y}{x-2y} = \dfrac{2x+y}{2y-x}.$

When $\dfrac{dy}{dx} = 0, y = -2x$. Put $y = -2x$ into given eqn,

$x^2 - 4x^2 - 2x^2 = 5$

$-x^2 = 1$. Impossible!

$\therefore$ The curve has no turning points.

When $\dfrac{dy}{dx} \to \infty, y = \dfrac{x}{2}$. Put $y = \dfrac{x}{2}$ into given eqn,

$x^2 - \dfrac{x^2}{4} + \dfrac{x^2}{2} = 5$

$\therefore x^2 = 4$

$\therefore x = \pm 2$

$\therefore$ The curve has vertical tangents at $\pm(2, 1)$.

b) Dividing both sides of the equation by x^2,

$1 - \dfrac{y^2}{x^2} + \dfrac{y}{x} = \dfrac{5}{x^2}$

$\dfrac{5}{4} - \left(\dfrac{y}{x} - \dfrac{1}{2}\right)^2 = \dfrac{5}{x^2}$

When $x \to \infty, \dfrac{5}{x^2} \to 0, \therefore \left(\dfrac{y}{x} - \dfrac{1}{2}\right)^2 \to \dfrac{5}{4}.$

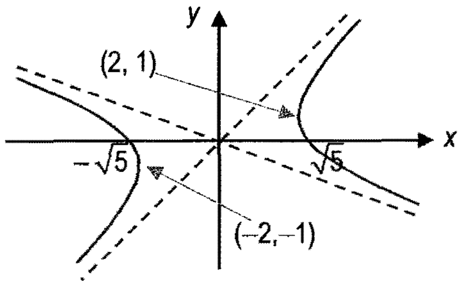
$\dfrac{y}{x} - \dfrac{1}{2} \to \pm\dfrac{\sqrt{5}}{2}, \therefore \dfrac{y}{x} \to \dfrac{1 \pm \sqrt{5}}{2}.$

$\therefore$ Asymptotes: $y = \dfrac{1 \pm \sqrt{5}}{2}x$.

Let $y = 0$, $x^2 = 5, \therefore x = \pm\sqrt{5}$.

Let $x = 0$, $-y^2 = 5$, $\therefore$ No solution.

c)



11 a) $y = 2\sin x + \cos 2x, -2\pi \le x \le 2\pi$.

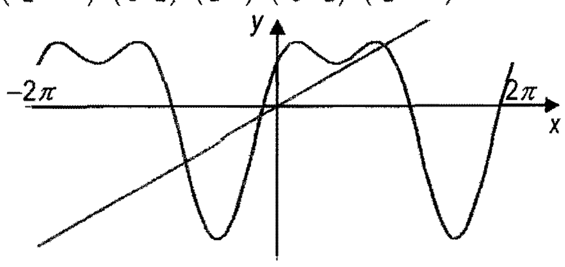
$\dfrac{dy}{dx} = 2\cos x - 2\sin 2x = 2\cos x - 4\sin x\cos x$

$= 2\cos x(1 - 2\sin x).$

$\dfrac{dy}{dx} = 0$ gives $\cos x = 0$ or $\sin x = \frac{1}{2}$.

$\therefore x = \dfrac{-3\pi}{2}, \dfrac{-\pi}{2}, \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{-11\pi}{6}, \dfrac{-7\pi}{6}, \dfrac{\pi}{6}, \dfrac{5\pi}{6}.$

$\therefore$ Turning points: $\left(\frac{-11\pi}{6}, \frac{3}{2}\right), \left(\frac{-3\pi}{2}, 1\right), \left(\frac{-7\pi}{6}, \frac{3}{2}\right),$
$\left(\frac{-\pi}{2}, -3\right), \left(\frac{\pi}{6}, \frac{3}{2}\right), \left(\frac{\pi}{2}, 1\right), \left(\frac{5\pi}{6}, \frac{3}{2}\right), \left(\frac{3\pi}{2}, -3\right).$



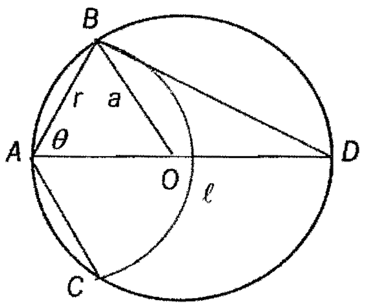
b) This equation cannot be solved algebraically. By adding the line $y=\frac{x}{2}$ to the above graph, there are three points of intersection, whose x-coordinates approximately are -2.4, -0.5 and 2.8.

12 a) For $0<x<\frac{\pi}{4}$, $\tan x<1$, $\therefore \tan x>\tan^2 x$ $\therefore$ false.

b) For $0<x<1$, $1+x>1$, $\therefore 1+x>\sqrt{1+x}$,

$\therefore \frac{1}{1+x}<\frac{1}{\sqrt{1+x}}$, $\therefore$ true.

13 a)



Refer to the diagram, let AD be the diameter, $\angle ABD=90°$ (semi-circle angle)

$\therefore \cos\theta=\frac{AB}{AD}=\frac{r}{2a}$, $\therefore r=2a\cos\theta$.

Arc length $BC=\ell=r.2\theta=2a\cos\theta.2\theta=4a\theta\cos\theta$.

b) $\frac{d\ell}{d\theta}=4a(\cos\theta-\theta\sin\theta)$.

$\frac{d\ell}{d\theta}=0$ when $\cos\theta=\theta\sin\theta$, $\therefore \frac{\cos\theta}{\sin\theta}=\theta$.

$\therefore \cot\theta=\theta$.

$\frac{d^2\ell}{d\theta^2}=4a(-\sin\theta-(\sin\theta+\theta\cos\theta))$

$=-4a(2\sin\theta+\theta\cos\theta)$

$=-4a\sin\theta(2+\theta\cot\theta)$.

When $\cot\theta=\theta$, $\frac{d^2\ell}{d\theta^2}=-4a\sin\theta(2+\theta^2)<0$, $\therefore$ max.

c) From the graphs of θ and $\cot\theta$, $0\le\theta<\frac{\pi}{2}$, the point of intersection occurs when $\theta\approx 0.86$ ($\approx 49°$).