

Trig Integrals

(1) Standard Integrals

$$\int \sin ax dx = -\frac{1}{a} \cos ax + c$$

$$\int \cos ax dx = \frac{1}{a} \sin ax + c$$

$$\int \sec^2 ax dx = \frac{1}{a} \tan ax + c$$

$$\int \tan ax dx = \int \frac{\sin ax}{\cos ax} dx$$

$$= -\frac{1}{a} \log \cos ax + c \quad \text{OR} \quad \frac{1}{a} \log \sec ax + c$$

(2) $\sin^n x$ or $\cos^n x$

$$\int \sin x dx = \underline{-\cos x + c}$$

$$\begin{aligned}\int \sin^2 x dx &= \frac{1}{2} \int (1 - \cos 2x) dx \\ &= \frac{1}{2} \left(x - \frac{1}{2} \sin 2x \right) + c\end{aligned}$$

$$\begin{aligned}\int \sin^3 x dx &= \int \sin x \sin^2 x dx \\ &= \int \sin x (1 - \cos^2 x) dx \\ &= -\int (1 - u^2) du \\ &= \frac{1}{3} u^3 - u + c \\ &= \frac{1}{3} \cos^3 x - \cos x + c\end{aligned}$$

$$u = \cos x$$

$$du = -\sin x dx$$

Odd Power

Factorise as $\sin x (\sin^2 x)^{\text{some power}}$

Substitute $\sin^2 x = 1 - \cos^2 x$

Use $u = \cos x$

$$\int \sin^4 x dx = \int (\sin^2 x)^2 dx$$

$$= \frac{1}{4} \int (1 - \cos 2x)^2 dx$$

$$= \frac{1}{4} \int (1 - 2 \cos 2x + \cos^2 2x) dx$$

$$= \frac{1}{4} \int \left[1 - 2 \cos 2x + \frac{1}{2} (1 + \cos 4x) \right] dx$$

$$= \frac{1}{4} \int \left(\frac{3}{2} - 2 \cos 2x + \frac{1}{2} \cos 4x \right) dx$$

$$= \frac{1}{4} \left(\frac{3}{2} x - \sin 2x + \frac{1}{8} \sin 4x \right) + c$$

Even Power

Factorise as $(\sin^2 x)^{\text{some power}}$

Substitute $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$$\int \sin^5 x dx = \int \sin x (\sin^2 x)^2 dx$$

$$= \int \sin x (1 - \cos^2 x)^2 dx$$

$$u = \cos x$$

$$du = -\sin x dx$$

$$= -\int (1 - u^2)^2 du$$

$$= -\int (1 - 2u^2 + u^4) du$$

$$= -\left(u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right) + c$$

$$= -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + c$$

(3) $\sin^n x$ and $\cos^n x$

Usually done by substitution $u = \sin x$ or $u = \cos x$

e.g. (i) $\int \cos^5 x \sin^3 x dx$

$$= \int \cos^5 x (1 - \cos^2 x) \sin x dx \quad u = \cos x$$

$$= -\int u^5 (1 - u^2) du \quad du = -\sin x dx$$

$$= \int (u^7 - u^5) du$$

$$= \frac{1}{8} u^8 - \frac{1}{6} u^6 + c$$

$$= \frac{1}{8} \cos^8 x - \frac{1}{6} \cos^6 x + c$$

Both powers odd

Choose either as u

Usually the higher power

$$(ii) \int \sin^6 x \cos^3 x dx$$

$$= \int \sin^6 x (1 - \sin^2 x) \cos x dx$$

$$u = \sin x$$

$$= \int u^6 (1 - u^2) du$$

$$du = \cos x dx$$

$$= \int (u^6 - u^8) du$$

$$= \frac{1}{7} u^7 - \frac{1}{9} u^9 + c$$

One power odd & one power even

Choose even as u

$$= \frac{1}{7} \sin^7 x - \frac{1}{9} \sin^9 x + c$$

$$(iii) \int \sin^2 x \cos^2 x dx = \int \sin^2 x (1 - \sin^2 x) dx$$

$$= \int (\sin^2 x - \sin^4 x) dx$$

Both powers even

Use $\sin^2 x = 1 - \cos^2 x$

or $\cos^2 x = 1 - \sin^2 x$

$$= \frac{1}{2} x - \frac{1}{4} \sin 2x - \frac{3}{8} x + \frac{1}{4} \sin 2x - \frac{1}{32} \sin 4x + c$$

$$= \frac{1}{8} x - \frac{1}{32} \sin 4x + c$$

(4) $\tan^n x$ or $\cot^n x$

$$\int \tan x dx = \underline{-\log \cos x + c}$$

$$\begin{aligned}\int \tan^2 x dx &= \int (\sec^2 x - 1) dx \\ &= \underline{\tan x - x + c}\end{aligned}$$

$$\begin{aligned}\int \tan^3 x dx &= \int \tan x (\sec^2 x - 1) dx \\ &= \int \tan x \sec^2 x dx - \int \tan x dx \\ &= \int u du - \int \tan x dx\end{aligned}$$

$$u = \tan x$$

$$du = \sec^2 x dx$$

$$= \frac{1}{2} u^2 + \log \cos x + c$$

$$= \underline{\frac{1}{2} \tan^2 x + \log \cos x + c}$$

$$\begin{aligned}\int \tan^4 x dx &= \int \tan^2 x (\sec^2 x - 1) dx \\ &= \int \tan^2 x \sec^2 x dx - \int \tan^2 x dx && u = \tan x \\ &= \int u^2 du - \int \tan^2 x dx && du = \sec^2 x dx \\ &= \frac{1}{3} u^3 - \tan x + x + c \\ &= \frac{1}{3} \tan^3 x - \tan x + x + c\end{aligned}$$

(5) $\sec^n x$ or $\operatorname{cosec}^n x$

$$\begin{aligned}\int \sec x dx &= \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} dx \\ &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx \\ &= \log(\sec x + \tan x) + c\end{aligned}$$

$$\int \sec^2 x dx = \tan x + c$$

$$\int \sec^3 x dx = \int \sec x \sec^2 x dx \quad u = \sec x \quad v = \tan x$$

$$= \sec x \tan x - \int \sec x \tan^2 x dx \quad du = \sec x \tan x dx \quad dv = \sec^2 x dx$$

$$= \sec x \tan x - \int \sec x (\sec^2 x - 1) dx$$

$$= \sec x \tan x - \int \sec^3 x dx + \int \sec x dx$$

$$= \sec x \tan x - \int \sec^3 x dx + \log(\sec x + \tan x)$$

$$\therefore 2 \int \sec^3 x dx = \sec x \tan x + \log(\sec x + \tan x) + c$$

$$\int \sec^3 x dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \log(\sec x + \tan x) + c$$

Odd powers

Done by parts

$$\int \sec^4 x dx = \int \sec^2 x (1 + \tan^2 x) dx$$

$$= \int (1 + u^2) du$$

$$= u + \frac{1}{3} u^3 + c$$

$$= \tan x + \frac{1}{3} \tan^3 x + c$$

$$u = \tan x$$

$$du = \sec^2 x dx$$

Even Power

Factorise as $\sec^2 x (\sec^2 x)^{\text{some power}}$

Substitute $\sec^2 x = 1 + \tan^2 x$

Use $u = \tan x$

Exercise 2C; 1, 2, 4, 5, 7, 8, 10, 12, 14, 16, 17, 18