Integration By Partial Fractions

To find; $\int \frac{A(x)}{P(x)} dx$

(1) If $\deg A(x) \geq \deg P(x)$, perform a division

(2) If $\deg A(x) < \deg P(x)$, factorise $P(x)$

a) for linear factor $(x-a)$, write $\frac{A}{x-a}$

b) for multiple linear factors $(x-a)^{\mathrm{n}}$, write

$$\frac{A}{(x-a)} + \frac{B}{(x-a)^2} + \ldots + \frac{C}{(x-a)^n}$$

c) for polynomial factors e.g. $ax^2 + bx + c$, write $\frac{Ax+B}{ax^2+bx+c}$

e.g. $(i)\int \frac{x^2}{x+1}dx$

$$= \int \left[x - 1 + \frac{1}{x+1} \right] dx$$

$$= \frac{1}{2}x^2 - x + \log(x+1) + c$$

$$\begin{array}{r} x \quad -1 \\ x+1 \overline{) x^2 + 0x + 0} \\ \underline{x^2 + x} \\ -x + 0 \\ \underline{-x - 1} \\ 1 \end{array}$$

$(ii)\int \frac{3dx}{x^2 - x}$

$$= \int \frac{3dx}{x(x-1)}$$

$$= \int \left[\frac{-3}{x} + \frac{3}{(x-1)} \right] dx$$

$$= -3\log x + 3\log(x-1) + c$$

$$= 3\log\left(\frac{x-1}{x} \right) + c$$

$$\frac{A}{x} + \frac{B}{x-1} = \frac{3}{x(x-1)}$$

$$A(x-1) + Bx = 3$$

$\underline{x = 0}$

$-A = 3$

$A = -3$

$\underline{x = 1}$

$B = 3$

$$(iii)\int \frac{x+5}{x^2-3x-10}\,dx$$

$$=\int \frac{x+5}{(x-5)(x+2)}\,dx$$

$$=\int \left[\frac{10}{7(x-5)}-\frac{3}{7(x+2)}\right]dx$$

$$=\frac{10}{7}\log(x-5)-\frac{3}{7}\log(x+2)+c$$

$$\frac{A}{(x-5)}+\frac{B}{(x+2)}=\frac{x+5}{(x-5)(x+2)}$$

$$A(x+2)+B(x-5)=x+5$$

$x=-2$ | $x=5$

$-7B=3$ | $7A=10$

$B=\frac{-3}{7}$ | $A=\frac{10}{7}$

$$(iv)\int \frac{dx}{x^3+x}$$

$$=\int \frac{dx}{x(x^2+1)}$$

$$=\int \left[\frac{1}{x}-\frac{x}{x^2+1}\right]dx$$

$$=\log x-\frac{1}{2}\log(x^2+1)+c$$

$$\frac{A}{x}+\frac{Bx+C}{x^2+1}=\frac{1}{x(x^2+1)}$$

$$A(x^2+1)+(Bx+C)x=1$$

$x=0$ | $x=i$

$A=1$ | $-B+Ci=1$

$B=-1$ $\quad C=0$

$$(v)\int \frac{x\,dx}{(x+1)^2(x^2+1)}$$

$$\frac{A}{(x+1)}+\frac{B}{(x+1)^2}+\frac{Cx+D}{(x^2+1)}=\frac{x}{(x+1)^2(x^2+1)}$$

$$A(x+1)(x^2+1)+B(x^2+1)+(Cx+D)(x+1)^2=x$$

$$x=-1$$

$$2B=-1$$

$$B=\frac{-1}{2}$$

$$x=i$$

$$-2C+2Di=i$$

$$C=0 \quad D=\frac{1}{2}$$

$$x=0$$

$$2A+B+D=0$$

$$2A-\frac{1}{2}+\frac{1}{2}=0$$

$$A=0$$

$$=\int\left[\frac{-1}{2(x+1)^2}+\frac{1}{2(x^2+1)}\right]dx$$

$$=\frac{1}{2}\int\left[-(x+1)^{-2}+\frac{1}{(x^2+1)}\right]dx$$

$$=\frac{1}{2}\left\{\frac{-(x+1)^{-1}}{-1}+\tan^{-1}x\right\}+c$$

$$=\frac{1}{2}\left(\frac{1}{x+1}+\tan^{-1}x\right)+c$$

Exercise 2G;
1, 3, 5, 7 to 21