# *Relationships Between Binomial Coefficients*

***Binomial Theorem***

$$(1+x)^n = \sum_{k=0}^{n} {}^nC_k x^k$$

$$= {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \ldots + {}^nC_k x^k + \ldots + {}^nC_n x^n$$

e.g. ($i$) Find the values of;

a) $\sum_{k=1}^{n} {}^nC_k$

$$(1+x)^n = \sum_{k=0}^{n} {}^nC_k x^k$$

let $x = 1$;

$$(1+1)^n = \sum_{k=0}^{n} {}^nC_k 1^k$$

$$2^n = \sum_{k=0}^{n} {}^nC_k$$

$$2^n = {}^nC_0 + \sum_{k=1}^{n} {}^nC_k$$

$$\sum_{k=1}^{n} {}^nC_k = 2^n - {}^nC_0$$

$$\sum_{k=1}^{n} {}^nC_k = 2^n - 1$$

b) ${}^nC_1+{}^nC_3+{}^nC_5+{}^nC_7+\ldots\ldots$

$$(1+x)^n=\sum_{k=0}^{n}{}^nC_k x^k$$

$$={}^nC_0+{}^nC_1x+{}^nC_2x^2+{}^nC_3x^3+{}^nC_4x^4+{}^nC_5x^5+\ldots\ldots$$

let $x$ = 1; $(1+1)^n={}^nC_0+{}^nC_1+{}^nC_2+{}^nC_3+{}^nC_4+{}^nC_5+\ldots\ldots$

$$2^n={}^nC_0+{}^nC_1+{}^nC_2+{}^nC_3+{}^nC_4+{}^nC_5+\ldots\ldots \qquad (1)$$

let $x$ = -1; $(1-1)^n={}^nC_0-{}^nC_1+{}^nC_2-{}^nC_3+{}^nC_4-{}^nC_5+\ldots\ldots$

$$0={}^nC_0-{}^nC_1+{}^nC_2-{}^nC_3+{}^nC_4-{}^nC_5+\ldots\ldots \qquad (2)$$

subtract (2) from (1)

$$2^n=2{}^nC_1+2{}^nC_3+2{}^nC_5+\ldots\ldots$$

$$2^{n-1}={}^nC_1+{}^nC_3+{}^nC_5+\ldots\ldots$$

c) $\sum_{k=1}^{n} k\,{}^{n}C_k$

$$(1+x)^n = \sum_{k=0}^{n} {}^{n}C_k x^k$$

Differentiate both sides

$$n(1+x)^{n-1} = \sum_{k=0}^{n} k\,{}^{n}C_k x^{k-1}$$

let $x = 1$;
$$n(1+1)^{n-1} = \sum_{k=0}^{n} k\,{}^{n}C_k$$

$$n(2)^{n-1} = (0)\,{}^{n}C_0 + \sum_{k=1}^{n} k\,{}^{n}C_k$$

$$\sum_{k=1}^{n} k\,{}^{n}C_k = n(2)^{n-1}$$

d) $\sum_{k=0}^{n} \frac{(-1)^k \, {}^nC_k}{k+1}$

$$(1+x)^n = \sum_{k=0}^{n} {}^nC_k x^k$$

Integrate both sides

$$\frac{(1+x)^{n+1}}{n+1} + K = \sum_{k=0}^{n} {}^nC_k \frac{x^{k+1}}{k+1}$$

let $x = 0$; $$\frac{(1+0)^{n+1}}{n+1} + K = \sum_{k=0}^{n} {}^nC_k \frac{0^{k+1}}{k+1}$$

$$K = \frac{-1}{n+1}$$

let $x = -1$; $$\frac{(1-1)^{n+1} - 1}{n+1} = \sum_{k=0}^{n} {}^nC_k \frac{(-1)^{k+1}}{k+1}$$

$$\sum_{k=0}^{n} {}^nC_k \frac{(-1)^{k+1}}{k+1} = \frac{-1}{n+1}$$

$$\sum_{k=0}^{n} {}^nC_k \frac{(-1)^k}{k+1} = \frac{1}{n+1}$$

$(ii)$ By equating the coefficients of $x^n$ on both sides of the identity;

$$(1+x)^n(1+x)^n \equiv (1+x)^{2n}$$

show that;

$$\sum_{k=0}^{n}\binom{n}{k}^2 = \frac{(2n)!}{(n!)^2}$$

$$(1+x)^n = \sum_{k=0}^{n} {}^nC_k x^k$$

$$= {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \ldots + {}^nC_{n-2}x^{n-2} + {}^nC_{n-1}x^{n-1} + {}^nC_n x^n$$

coefficient of $x^n$ in $(1+x)^n(1+x)^n$

$$\left({}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \ldots + {}^nC_{n-2}x^{n-2} + {}^nC_{n-1}x^{n-1} + {}^nC_n x^n\right)$$

$$\times\left({}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \ldots + {}^nC_{n-2}x^{n-2} + {}^nC_{n-1}x^{n-1} + {}^nC_n x^n\right)$$

$$= \binom{n}{0}\binom{n}{n}x^n + \binom{n}{1}x\binom{n}{n-1}x^{n-1} + \binom{n}{2}x^2\binom{n}{n-2}x^{n-2}$$

$$+ \ldots + \binom{n}{n-2}x^{n-2}\binom{n}{2}x^2 + \binom{n}{n-1}x^{n-1}\binom{n}{1}x + \binom{n}{n}x^n\binom{n}{0}$$

$$\text{coefficient of } x^n = \binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \binom{n}{2}\binom{n}{n-2} + \ldots + \binom{n}{n}\binom{n}{0}$$

$$\text{But} \binom{n}{k} = \binom{n}{n-k}$$

$$= \binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \ldots + \binom{n}{n}^2$$

$$= \sum_{k=0}^{n} \binom{n}{k}^2$$

$\text{coefficient of } x^n \text{ in } (1+x)^{2n}$

$$(1+x)^{2n} = \binom{2n}{0} + \binom{2n}{1}x + \binom{2n}{2}x^2 + \ldots + \binom{2n}{n}x^n + \ldots + \binom{2n}{2n}x^{2n}$$

$$\text{coefficient of } x^n = \binom{2n}{n}$$

Now

$$(1+x)^n(1+x)^n \equiv (1+x)^{2n}$$

$$\therefore \sum_{k=0}^{n}\binom{n}{k}^2 = \binom{2n}{n}$$

$$= \frac{(2n)!}{n!\,n!}$$

$$= \frac{(2n)!}{(n!)^2}$$

**Exercise 5F;**
**4, 5, 6, 8, 10,15**

**+ *worksheets***