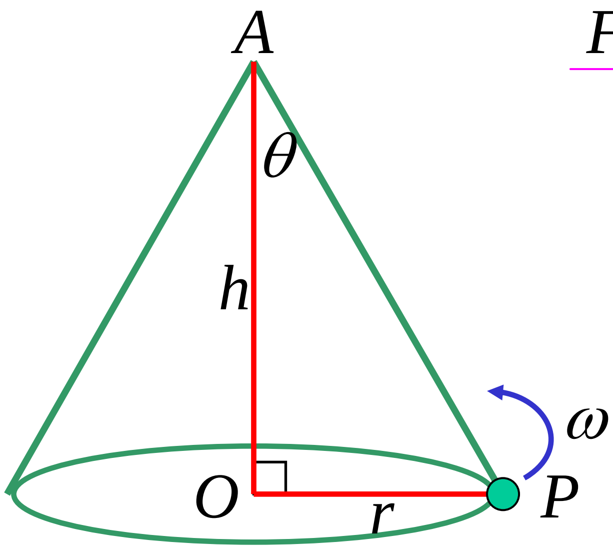
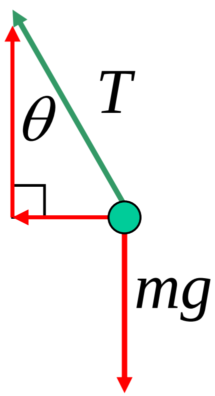
The Conical Pendulum



Force Diagram



T = tension in the string
(always away from object)

$\theta = \angle$ string makes with vertical

$\omega =$ angular velocity of pendulum

Resultant Forces

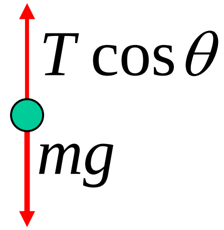
$$m\ddot{x} = \frac{mv^2}{r}$$

$$\text{horizontal forces} = \frac{mv^2}{r}$$

$$T\sin\theta = \frac{mv^2}{r} \qquad \left(= mr\omega^2\right)$$

$$m\ddot{y} = 0$$

$$\text{vertical forces} = 0$$



$$T\cos\theta - mg = 0$$

$$T\cos\theta = mg$$

$$\frac{T\sin\theta}{T\cos\theta} = \frac{mv^2}{r} \times \frac{1}{mg}$$

$$\tan\theta = \frac{v^2}{rg} \qquad \left(= \frac{r\omega^2}{g}\right)$$

But in ΔAOP $\quad \tan\theta = \frac{r}{h}$

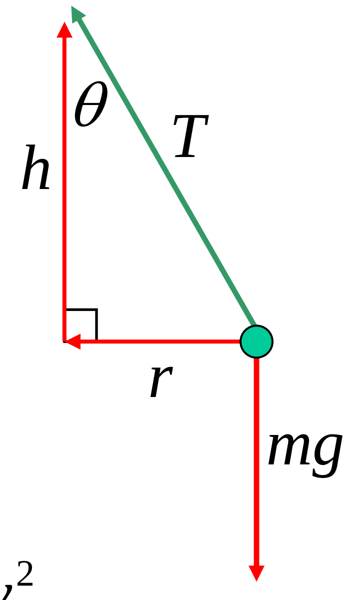
$$\therefore \frac{v^2}{rg} = \frac{r}{h}$$

$$h = \frac{r^2 g}{v^2} \qquad \left(= \frac{g}{\omega^2}\right)$$

Implications

- depth of the pendulum below A is independent of the length of the string.
- as the speed increases, the particle (bob) rises.

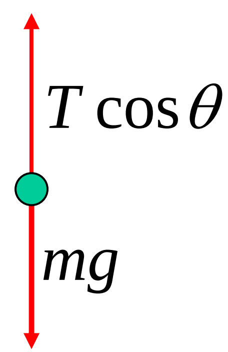
e.g. The number of revolutions per minute of a conical pendulum increases from 60 to 90.
Find the rise in the level of the bob.



$$\text{horizontal forces} = \frac{mv^2}{r}$$

$$T\sin\theta = mr\omega^2$$

$$\text{vertical forces} = 0$$



$$T\cos\theta - mg = 0$$

$$T\cos\theta = mg$$

$$\therefore \tan\theta = mr\omega^2 \times \frac{1}{mg}$$

$$= \frac{r\omega^2}{g}$$

$$\text{But } \tan\theta = \frac{r}{h}$$

$$\therefore \frac{r\omega^2}{g} = \frac{r}{h}$$

$$h = \frac{g}{\omega^2}$$

$$\text{when } \omega = 60\text{rev/min}$$

$$= \frac{120\pi}{60}\text{rad/s}$$

$$= 2\pi\text{rad/s}$$

$$h = \frac{g}{(2\pi)^2}$$

$$= \frac{g}{4\pi^2}\text{m}$$

$$\text{when } \omega = 90\text{rev/min}$$

$$= \frac{180\pi}{60}\text{rad/s}$$

$$= 3\pi\text{rad/s}$$

$$h = \frac{g}{(3\pi)^2}$$

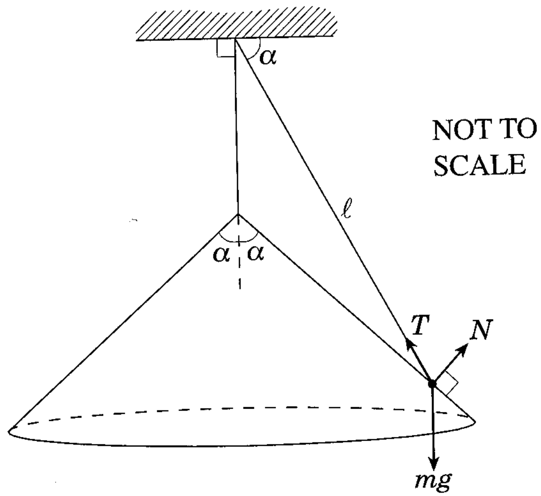
$$= \frac{g}{9\pi^2}\text{m}$$

$$\therefore \text{rise in height} = \left[\frac{g}{4\pi^2} - \frac{g}{9\pi^2}\right]\text{m}$$

$$= 0.14\text{m}$$

(*ii*) (2002)

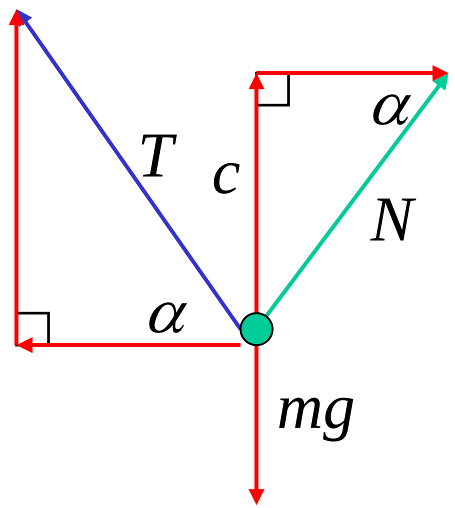
A particle of mass m is suspended by a string of length l from a point directly above the vertex of a smooth cone, which has a vertical axis. The particle remains in contact with the cone and rotates as a conical pendulum with angular velocity ω .



The angle of the cone at its vertex is 2α where $\alpha > \frac{\pi}{4}$, and the string makes an angle of α with the horizontal as shown in the diagram. The forces acting on the particle are the tension in the string T, the normal reaction N and the gravitational force mg.

Note: whenever a particle makes contact with a surface there will be a normal force perpendicular to the surface.

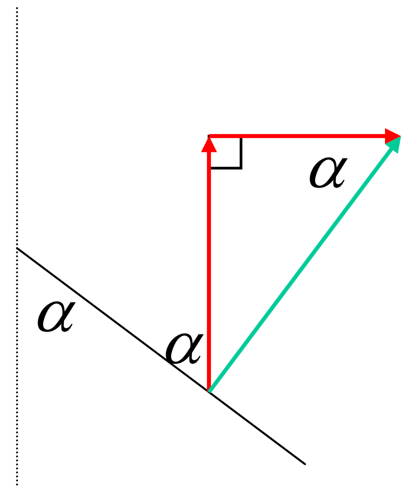
a) Show, with the aid of a diagram, that the vertical component of N is $N\sin\alpha$



$$\frac{c}{N} = \sin\alpha$$

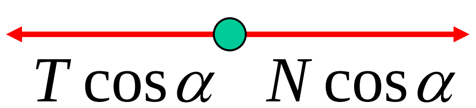
$$c = N\sin\alpha$$

$\therefore$ the vertical component of N is $N\sin\alpha$



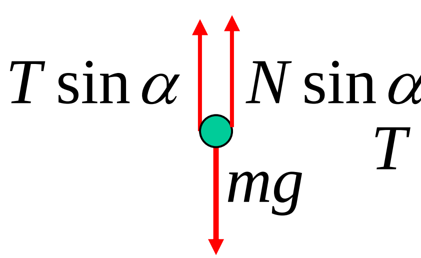
b) Show that $T + N = \dfrac{mg}{\sin\alpha}$, and find an expression for $T - N$ in terms of m, l and ω

horizontal forces $= mr\omega^2$



$$T\cos\alpha - N\cos\alpha = mr\omega^2$$

vertical forces $= 0$



$$T\sin\alpha + N\sin\alpha - mg = 0$$

$$T\sin\alpha + N\sin\alpha = mg$$

$$T\sin\alpha + N\sin\alpha = mg$$

$$(T+N)\sin\alpha = mg$$

$$T + N = \frac{mg}{\sin\alpha}$$

$$T\cos\alpha - N\cos\alpha = mr\omega^2$$

$$(T-N)\cos\alpha = mr\omega^2$$

$$T - N = \frac{mr\omega^2}{\cos\alpha}$$

But $\frac{r}{l} = \cos\alpha$ $\quad \therefore T - N = ml\omega^2$

c) The angular velocity is increased until $N = 0$, that is, when the particle is about to lose contact with the cone.

Find an expression for this value of ω in terms of α, l and g

When $N = 0$;

$$T = \frac{mg}{\sin\alpha} \text{ and } T = ml\omega^2$$

$$\therefore \frac{mg}{\sin\alpha} = ml\omega^2$$

$$\omega^2 = \frac{g}{l\sin\alpha}$$

$$\omega = \sqrt{\frac{g}{l\sin\alpha}}$$

Exercise 9C; all