

# *Inequality Techniques*

To prove  $x \geq y$ , it can be easier to prove  $x - y \geq 0$

e.g. (i) (1995) Prove  $pq \leq \frac{p^2 + q^2}{2}$

$$\frac{p^2 + q^2}{2} - pq = \frac{p^2 - 2pq + q^2}{2}$$
$$= \frac{(p - q)^2}{2}$$

$$\geq 0$$

$$\therefore \underline{\underline{\frac{p^2 + q^2}{2} \geq pq}}$$

(ii) (1994) a) Prove  $a^2 + b^2 + c^2 > ab + bc + ac$

$$(a - b)^2 > 0$$

$$a^2 - 2ab + b^2 > 0$$

$$\therefore a^2 + b^2 > 2ab$$

$$a^2 + c^2 > 2ac$$

$$b^2 + c^2 > 2bc$$

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$$2a^2 + 2b^2 + 2c^2 > 2ab + 2ac + 2bc$$

$$\underline{a^2 + b^2 + c^2 > ab + ac + bc}$$

b) If  $a + b + c = 1$ , prove  $ab + ac + bc < \frac{1}{3}$

$$a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + ac + bc)$$

$$\therefore (a + b + c)^2 - 2(ab + ac + bc) > ab + ac + bc$$

$$3(ab + ac + bc) < (a + b + c)^2$$

$$3(ab + ac + bc) < 1$$

$$\underline{ab + ac + bc < \frac{1}{3}}$$

c) Prove  $\frac{1}{3}(a+b+c) \geq \sqrt[3]{abc}$

$$a^2 + b^2 + c^2 \geq ab + ac + bc$$

$$a^2 + b^2 + c^2 - ab - ac - bc \geq 0$$

$$(a+b+c)(a^2 + b^2 + c^2 - ab - ac - bc) \geq 0$$

$$a^3 + ab^2 + ac^2 - a^2b - a^2c - abc + a^2b + b^3 + bc^2 - ab^2 - abc - b^2c \\ + a^2c + b^2c + c^3 - abc - ac^2 - bc^2 \geq 0$$

$$a^3 + b^3 + c^3 - 3abc \geq 0$$

$$\frac{1}{3}(a^3 + b^3 + c^3) \geq abc$$

let  $a = a^{\frac{1}{3}}, b = b^{\frac{1}{3}}, c = c^{\frac{1}{3}}$

$$\frac{1}{3}(a+b+c) \geq a^{\frac{1}{3}}b^{\frac{1}{3}}c^{\frac{1}{3}}$$

$$\frac{1}{3}(a+b+c) \geq \sqrt[3]{abc}$$

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## Arithmetic Mean $\geq$ Geometric Mean

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}$$

d) Suppose  $(1+x)(1+y)(1+z) = 8$ , prove  $xyz \leq 1$

$$(1+x)(1+y)(1+z) = 8$$

$$1 + x + y + xy + z + xz + yz + xyz = 8$$

$$\frac{1}{3}(x + y + z) \geq \sqrt[3]{xyz}$$

$$\underline{x + y + z \geq 3\sqrt[3]{xyz}}$$

$$xy + yz + xz \geq 3\sqrt[3]{(xy)(yz)(xz)}$$

$$xy + yz + xz \geq 3\sqrt[3]{x^2 y^2 z^2}$$

$$\underline{xy + yz + xz \geq 3\left(\sqrt[3]{xyz}\right)^2}$$

$$1 + x + y + z + xy + xz + yz + xyz = 8$$

$$1 + 3\sqrt[3]{xyz} + 3(\sqrt[3]{xyz})^2 + xyz \leq 8$$

$$1 + 3\sqrt[3]{xyz} + 3(\sqrt[3]{xyz})^2 + (\sqrt[3]{xyz})^3 \leq 8$$

$$(1 + \sqrt[3]{xyz})^3 \leq 8$$

$$1 + \sqrt[3]{xyz} \leq 2$$

$$\sqrt[3]{xyz} \leq 1$$

$$\underline{xyz \leq 1}$$

## Inequalities Sheet

### Exercise 10D

*Note: Cambridge 8H (Book 1); 28*