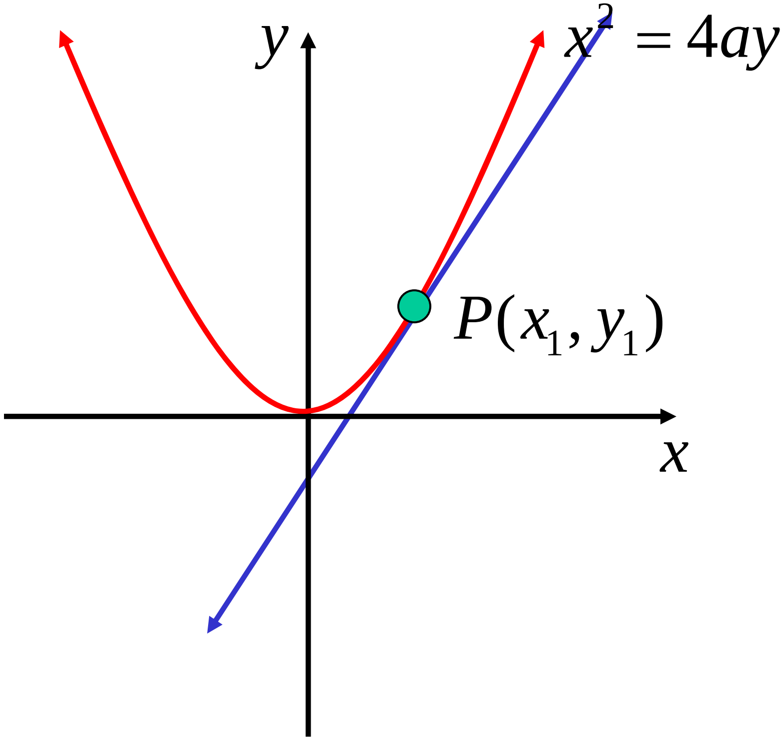
Tangents & Normals

(ii) Using Cartesian

(1) Tangent



$$y = \frac{x^2}{4a}$$

$$\frac{dy}{dx} = \frac{x}{2a}$$

when $x = x_1, \dfrac{dy}{dx} = \dfrac{x_1}{2a}$

$\therefore$ slope of tangent is $\dfrac{x_1}{2a}$

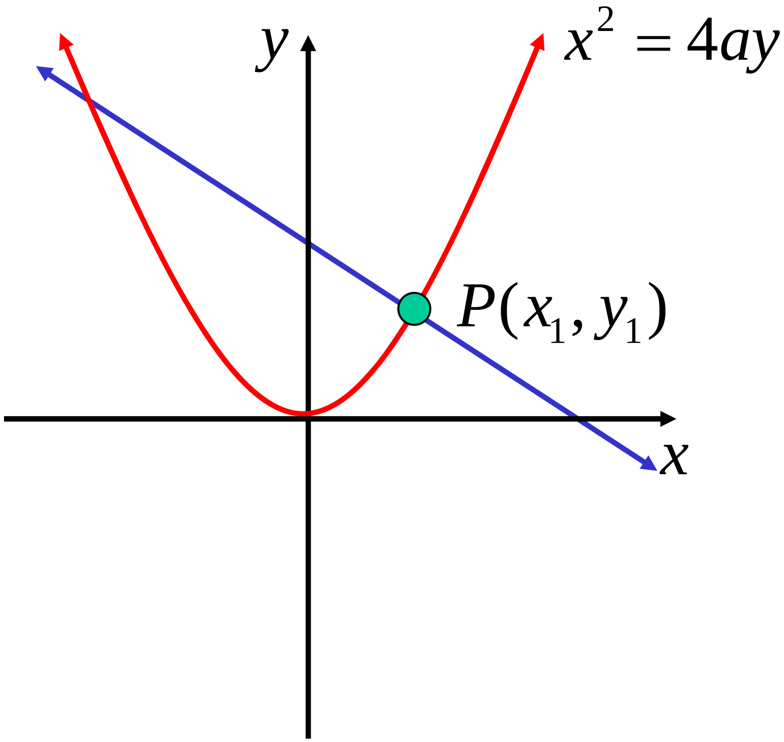
$$y - y_1 = \frac{x_1}{2a}(x - x_1)$$

$$2ay - 2ay_1 = xx_1 - x_1^2$$

$$2ay - 2ay_1 = xx_1 - 4ay_1$$

$$xx_1 = 2a(y + y_1)$$

(2) Normal



1 Show the slope of tangent at P is $\frac{x_1}{2a}$

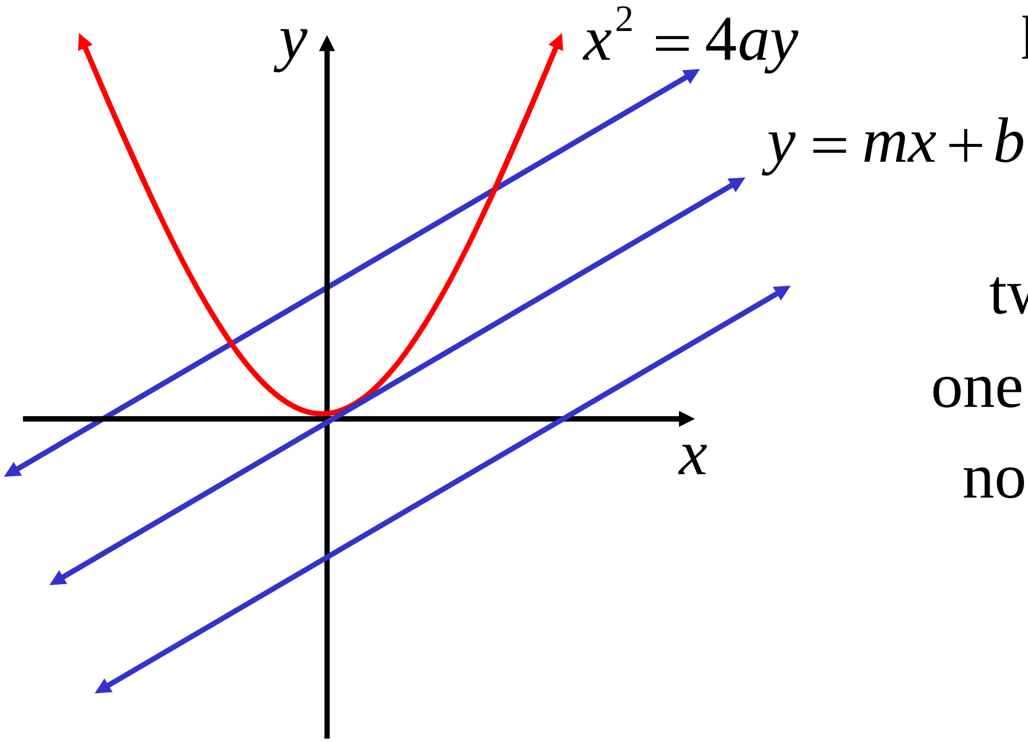
2 $\therefore$ slope of normal is $-\frac{2a}{x_1}$

$$y - y_1 = \frac{-2a}{x_1}(x - x_1)$$

$$x_1 y - x_1 y_1 = -2ax + 2ax_1$$

$$\boxed{2ax + x_1 y = 2ax_1 + x_1 y_1}$$

(3) Line cutting/touching/missing parabola



parabola and tangent meet when;

$$x^2 = 4a(mx + b)$$

$$x^2 - 4amx - 4ab = 0$$

two solutions (cuts) when $\Delta > 0$

one solution (touches) when $\Delta = 0$

no solutions (misses) when $\Delta < 0$

$$\begin{aligned}\Delta &= b^2 - 4ac \\ &= (-4am)^2 - 4(1)(-4ab) \\ &= 16a^2m^2 + 16ab \\ &= 16a(am^2 + b)\end{aligned}$$

$\therefore$ two solutions (cuts) when $am^2 + b > 0$

one solution (touches) when $am^2 + b = 0$ ← *(common idea)*

no solutions (misses) when $am^2 + b < 0$

e.g. Find the equation of the two tangents to the parabola $x^2 = 4y$ passing through the point (3,2).

tangent will be of the form $y = mx + b$

$$\therefore 2 = 3m + b$$

$$b = 2 - 3m$$

tangents are $y = mx + 2 - 3m$

$$x^2 = 4y$$

$$x^2 = 4(mx + 2 - 3m)$$

$$x^2 - 4mx + (12m - 8) = 0$$

line is a tangent if $\Delta = 0$

$$(-4m)^2 - 4(1)(12m - 8) = 0$$

$$16m^2 - 48m + 32 = 0$$

$$m^2 - 3m + 2 = 0$$

$$(m - 1)(m - 2) = 0$$

$$m = 1 \quad \text{or} \quad m = 2$$

$\therefore$ tangents are $y = x - 1$ and $y = 2x - 4$

Exercise 9G; 1ac, 2ac, 3a, 4, 7, 9, 11, 12, 13, 15, 17, 18