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Year 12 2009 Extension 1 Trial Exam

Question 1

Marks

(a) Find $\lim_{x\to 0}\frac{\tan 3x}{2x}$ — 2

(b) Find the acute angle (to nearest minute) between the lines : — 2

$$2x-3y-1=0 \quad \text{and} \quad y=\frac{3x}{5}-7.$$

(c) Divide the interval AB externally in the ratio 3:5 given the points $A(3,-2)$ and $B(-1,7)$. — 2

(d) Expand and simplify $(2x+3y)^4$ — 2

(e) Find the probability of getting 6 heads when a coin is tossed 8 times. — 2

(f) Write $7.\dot{1}\dot{2}$ as the sum of an infinite series. — 2

Hence write $7.\dot{1}\dot{2}$ as a mixed fraction.

Question 2.

The displacement x metres of a particle is given by :

$$x=7+5\sin 3t+6\cos 3t \quad \text{where } t \text{ is the time in seconds.}$$

(a) Show that the particle moves in SHM, stating the centre of motion and period T. — 4

(b) Find the maximum speed of the particle. — 2

(c) Write $5\sin 3t+6\cos 3t$ in the form $R\cos(3t-\alpha)$, where $R>0$ and $0<\alpha<2\pi$. — 2

(d) Graph displacement x versus time t of the particle for $0\le t\le 2\pi$. — 2

(e) Find the first time (to 2 decimal places)when the particle is 14 metres from the origin. — 2

Question 3.

(a) Evaluate $\int_0^4 \frac{2dx}{x^2+16}$ — 2

(b) (i) Factorise $x^3+2x^2-15x-36$ — 3

(ii) Hence solve $x^3+2x^2-15x-36\ge 0$. — 2

(c) The velocity v of a particle is given by : — 3

$$v=5+e^{-x} \quad \text{where } x \text{ is the displacement of the particle.}$$

Find the displacement x as a function of time t if the particle is initially at the origin.

(d) Find the rate of change $\frac{dF}{dt}$ (to 3 significant figures) if $F=G\frac{m_1 m_2}{r^2}$ — 2

where $G=6.67\times10^{-11}$, $m_1=5.97\times10^{24}$, $m_2=1000$, $r=1.5\times10^5$ and $\frac{dr}{dt}=750$.

Question 4.

Marks

(a) Find the area bounded by the lines $x=-1, x=-2$, the x-axis and the curve $y=\frac{1}{x}$. — 2

(b) Find $\int\frac{4x+\sqrt{1-x^2}}{1-x^2}dx$ — 2

(c) Three engineers and nine councillors have a meeting around a circular table. If three councilors are between each engineer find number of possible seating arrangements. — 2

(d) Find the greatest coefficient of $(2x+7)^{13}$. — 3

(e) The velocity v of a body is given by : $v=x\tan^2 x$, where x is the displacement. — 3

Find in simplest terms the acceleration $\ddot{x}$ of the body in terms of the displacement x.

Question 5.

(a) Graph the curve $y=-2\cos^{-1}\left(\frac{x}{3}\right)$. — 3

(b) Solve $\frac{4x-5}{2x+1}\le 3$ — 3

(c) There are 8 red, 9 green and 6 yellow cards in a pack of cards. Five cards are drawn. Find the probability of obtaining 2 red and 3 green cards if it is known that at least one card is green. — 2

Leave the answer in ${}^{n}C_{r}$ form.

(d) The point T lies on the inside of the acute angle XYZ.
From T perpendiculars TV and TW are dropped to the angle arms YX and YZ respectively.
From point Y, the perpendicular YN is dropped to the interval VW.

(i) Draw a diagram showing all the information. — 1

(ii) Prove that $\angle VYN=\angle TYW$. — 3

Question 6.

(a) Using the substitution $x=\frac{1}{y}$ and integration tables find $\int\frac{dx}{x\sqrt{1-x^2}}$. — 4

(b) Prove by Mathematical Induction : — 4

$$1\times3\times5\times\ldots\ldots\ldots\times(2n-1)=\frac{(2n)!}{2^n n!}$$

(c) A man takes out a loan for \$260 000 to be paid in equal monthly payments over 25 years. If the interest on the loan is 8 %p.a. monthly reducible, find the monthly repayment R. — 4

Question 7. **Marks**

(a)

(i) Show that $T = A + Be^{-kt}$ is a solution of $\frac{dT}{dt} = -k(T - A)$. **2**

(ii) A barbeque plate is heated to $85^{\circ}C$ when the ambient temperature is $22^{\circ}C$. **4**
The plate cools to $70^{\circ}C$ in 16 minutes.
Assuming Newton's Law of Cooling find the time for the plate to cool to $30^{\circ}C$.

(b) A projectile is fired with initial speed V *m/s* from the origin O at an angle of α to the horizontal $(0 \le \alpha < 90^{\circ})$. **4**

The trajectory equation is given by : $y = x\tan\alpha - \frac{gx^2}{2V^2}(1 + \tan^2\alpha)$.

The projectile reaches a maximum height, and on the downward motion the projectile hits the target 20 metres above ground level at an angle of 27° to the horizontal.
Find the horizontal distance R that the target is from the Origin O (to nearest cm), if the angle of projection α is 45° and the acceleration due to gravity g is $10m/s^2$.

(c) The sequences $\{1, 3, 5, \ldots\ldots, p\}$ and $\{1, 3, 5, \ldots\ldots\ldots, q\}$ contain the integer values of p and q respectively. **2**

Find the value of $p+q$ if :

$(1+3+5+\ldots\ldots+p)+(1+3+5+\ldots\ldots+q)=(1+3+5+\ldots\ldots+33)$

End of Exam

Ext 1.
2009.
Juse Soln.

(a) $\lim_{x\to 0} \frac{\tan 3x}{2x} = \lim_{x\to 0} \frac{\sin 3x}{3x} \cdot \frac{3}{2\cos 3x}$

$= 1 \times \frac{3}{2\times 1}$

$= \frac{3}{2}$

(b) $m_1 = \frac{2}{3}$ $\quad m_2 = \frac{3}{5}$

$\therefore \tan\theta = \left|\frac{m_1 - m_2}{1+m_1 m_2}\right|$

$= \left|\frac{\frac{2}{3} - \frac{3}{5}}{1+\frac{2}{3}\cdot\frac{3}{5}}\right|$

$= \left|\frac{10-9}{15+6}\right|$

$= \left|\frac{1}{21}\right|$

$\theta = 2^\circ 44'$

(c) $3 : -5$

$A(3,-2)$ $\quad B(-1,7)$

$P = \left(\frac{-3-15}{3-5}, \frac{21+10}{3-5}\right)$

$= \left(9, -15\frac{1}{2}\right)$

(d) $(2x+3y)^4 = 16x^4 + 96x^3y + 216x^2y^2 + 216xy^3 + 81y^4$

(e) $P(6H) = {}^8C_6 \cdot \left(\frac{1}{2}\right)^8$

$= \frac{28}{256}$

$= \frac{7}{64}$

(f) $7.\dot{1}\dot{2} = 7 + 0.12 + 0.0012 + \ldots$

$= 7 + \frac{0.12}{1-\frac{1}{100}}$

$= 7\frac{4}{33}$

2(a) $x = 7 + 5\sin 3t + 6\cos 3t$

$v = 15\cos 3t - 18\sin 3t$

$\ddot{x} = -45\sin 3t - 54\cos 3t$

$= -9\left[5\sin 3t + 6\cos 3t\right]$

$= -9\left[7 + 5\sin 3t + 6\cos 3t - 7\right]$

$\ddot{x} = -9\left[x-7\right]$

which is of the form

$\ddot{x} = -n^2(x-B)$

where $n^2 = 9$

$n = 3$

(a) Centre motion $x = 7$m.

Period $T = \frac{2\pi}{3}$ s

(b) $V_{max} = \sqrt{15^2+18^2}$

$= 3\sqrt{61}$ m/s

(c) $5\sin 3t + 6\cos 3t = R\cos(3t-\alpha)$

$= R\cos 3t\cos\alpha + R\sin 3t\sin\alpha$

$\therefore R\sin\alpha = 5$ $\quad R>0$ $\quad \sin\alpha > 0$ $\quad 0 < \alpha < \frac{\pi}{2}$

$R\cos\alpha = 6$ $\quad \cos\alpha > 0$

$R = \sqrt{5^2+6^2}$

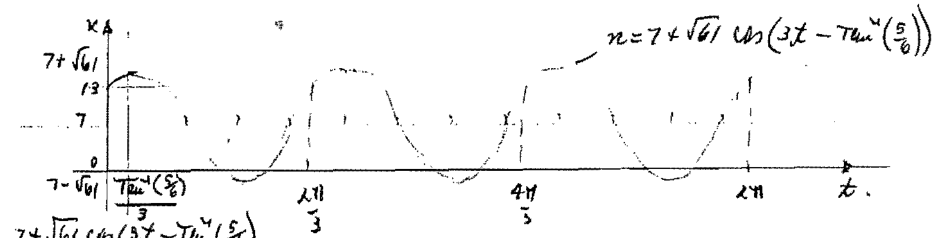
$= \sqrt{61}$

$\tan\alpha = \frac{5}{6}$

$\alpha = \tan^{-1}\left(\frac{5}{6}\right)$

$\therefore 5\sin 3t + 6\cos 3t = \sqrt{61}\cos\left(3t - \tan^{-1}\left(\frac{5}{6}\right)\right)$

(d)



$14 = 7+\sqrt{61}\cos\left(3t - \tan^{-1}\left(\frac{5}{6}\right)\right)$

$t = \frac{\tan^{-1}\frac{5}{6} - \cos^{-1}\left(\frac{7}{\sqrt{61}}\right)}{3} = 0.08$ s

3 (a) $\int_0^4 \frac{2\,dx}{x^2+16} = 2.\frac{1}{4}\left[\tan^{-1}\frac{x}{4}\right]_0^4$

$= \frac{1}{2}\left(\tan^{-1}1 - 0\right)$

$= \frac{\pi}{8}$

b) Let $P(x) = x^3 + 2x^2 - 15x - 36$

$P(1) = -48$

$P(-1) = -20$

$P(2) = -50$

$P(-2) = -6$

$P(3) = -36$

$P(-3) = -27 + 18 + 45 - 36$ ~1

$= 0$

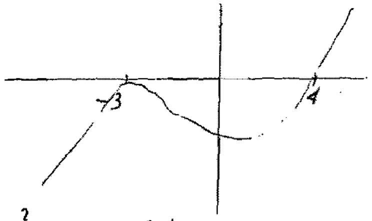
$\therefore$ $x+3$ is a factor of $P(x)$

$\therefore$ $P(x) = (x+3)(x^2 - x - 12)$ ~1

$= (x+3)(x+3)(x-4)$

ie $x^3+2x^2-15x-36 = (x+3)^2(x-4)$ ~1

(ii)



$x^3+2x^2-15x-36 \geq 0$

$x = -3$ or $x \geq 4$. ~2

3 (c) $v = 5 + e^{-x}$

$\frac{dx}{dt} = 5 + e^{-x}$

$\frac{dt}{dx} = \frac{1}{5+e^{-x}}$

$= \frac{e^x}{5e^x+1}$

$\therefore$ $t = \frac{1}{5}\ln(5e^x+1) + c$

But $t=0$ $x=0$ $\Rightarrow$ $c = -\frac{1}{5}\ln 6$

$\therefore$ $t = \frac{1}{5}\ln\left(\frac{5e^x+1}{6}\right)$

$x = \ln\left(\frac{6e^{5t}-1}{5}\right)$

(d) $F = G\frac{m_1 m_2}{r^2}$

$\frac{dF}{dt} = -2G.\frac{m_1 m_2}{r^3}\frac{dr}{dt}$

$= \frac{-2\times 6.67\times10^{-11}.\ 5.97\times10^{24}\times10^{3}\times750}{(1.5\times10^5)^3}$

$= -1.77\times10^5$

4 (a) $$\text{Area} = \int_{-1}^{-2} \frac{1}{x}\,dx$$
$$= \Big[\ln(-x)\Big]_{-1}^{-2}$$
$$= \underline{\ln 2}\ \text{unit}^2$$

b) $$\int \frac{4x + \sqrt{1-x^2}}{1-x^2}\,dx = \int \frac{4x}{1-x^2} + \frac{1}{\sqrt{1-x^2}}\,dx$$
$$= -2\ln(1-x^2) + \sin^{-1}x + C.$$

c) E x x x x E x x x E x x x x

Ways E = 2!
Ways C = 9!
Total ways 2! × 9!

d) $$T_{r+1} = {}^{13}C_r\,(2x)^{13-r}\,7^r$$
$$T_r = \binom{13}{r-1}(2x)^{14-r}\,7^{r-1}$$
$$\frac{T_{r+1}}{T_r} = \frac{13!}{r!\,(13-r)!}\cdot\frac{(r-1)!\,(14-r)!}{13!}\cdot\frac{2^{13-r}}{2^{14-r}}\cdot\frac{7^r}{7^{r-1}}\cdot\frac{x^{13-r}}{x^{14-r}}$$
$$= \frac{14-r}{r}\cdot\frac{7}{2}\cdot\frac{1}{x}.$$

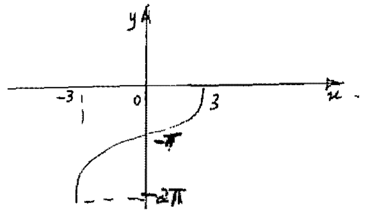
For greatest coefficient $\frac{T_{r+1}}{T_r} \geq 1$
$$\frac{7(14-r)}{2r} \geq 1$$
$$98 - 7r \geq 2r$$
$$9r \leq 98$$
$$r \leq 10\tfrac{8}{9}$$

$\Rightarrow r \leq 10$ $T_{11} > T_{10}$, $T_{10} > T_9$, $T_9 > T_8$; $T_{12} < T_{11}$, $T_{13} < T_{12}$, $T_{14} < T_{13}$

Greatest coefficient $r = 10$
$$T_{11} = \binom{13}{10} 2^{13-10}\cdot 7^{10}$$
$$= 2288 \times 7^{10}.$$

(e) $$\ddot{x} = v\frac{dv}{dx}$$
$$= x\tan^2 x\,\frac{d}{dx}\left[x\tan^2 x\right]$$
$$= x\tan^2 x\left[\tan^2 x + 2x\tan x\sec^2 x\right]$$
$$= x\tan^3 x\left[\tan x + 2x\sec^2 x\right].$$

5 (a)

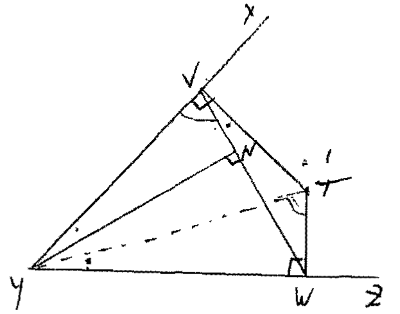


(b) $$\frac{4x-5}{2x+1} \leq 3 \qquad x \neq -\tfrac{1}{2}$$
$$(2x+1)(4x-5) \leq 3(2x+1)^2$$
$$(2x+1)(4x-5-6x-3) \leq 0.$$
$$(2x+1)(-2x-8) \leq 0$$
$$(2x+1)(x+4) \geq 0. \qquad -4 \quad -\tfrac{1}{2}$$
$$x \leq -4 \text{ or } x > -\tfrac{1}{2}.$$

(c) $$P(2R\,3G) = \frac{{}^8C_2\cdot{}^9C_3}{{}^{23}C_5 - {}^{14}C_5}$$

5(d)

(i) ✓



(ii) $\angle YVT + \angle YWT = 90° + 90°$

$= 180°$

$\therefore YVTW$ is a cyclic quadrilateral (opposite angles are supplementary) -1

In ΔYTW and ΔYVN

$\angle YTW = \angle YVN$ (Angles at the circumference in the same segment are equal) -1

$\angle YWT = \angle YNV$ (Both right angles) -1

$\therefore \Delta YTW \parallel\!\!\!\mid \Delta YVN$ (Equiangular)

$\therefore \angle VYN = \angle TYW$ (corresponding angles of similar triangles are equal)

7

5 (e) $x = \frac{1}{y}$

$dx = \frac{-1}{y^2}\,dy$

$$\therefore \int \frac{dx}{x\sqrt{1-x^2}} = \int \frac{-1}{y^2} \cdot \frac{dy}{\frac{1}{y}\sqrt{1-\frac{1}{y^2}}}$$

$$= \int \frac{-1}{y^2} \cdot \frac{y\,dy}{\frac{\sqrt{y^2-1}}{y}}$$

$$= -\int \frac{dy}{\sqrt{y^2-1}}$$

$$= -\ln\left(y + \sqrt{y^2-1}\right)$$

$$= -\ln\left(\frac{1}{x} + \sqrt{\frac{1}{x^2}-1}\right)$$

$$= \ln\left(\frac{x}{1+\sqrt{1-x^2}}\right) + C.$$

5(b) Step 1.

$n=1 \quad LHS=1 \quad RHS = \frac{(2n)!}{2^n n!}$

$= \frac{2!}{2.1!}$

$= \frac{2}{2}$

$=1$

$\therefore LHS = RHS$ True $n=1$

Step 2 Assume statement is true $n=k$

$1 \times 3 \times 5 \times \ldots \times (2k-1) = \frac{(2k)!}{2^k k!}$

To prove statement is true $n=k+1$

$1 \times 3 \times 5 \ldots \times (2k-1)(2k+1) = \frac{[2(k+1)]!}{2^{k+1}(k+1)!}$

Now $1 \times 3 \times \ldots \times (2k-1)(2k+1) = \frac{(2k)!}{2^k k!} \cdot (2k+1)$ (By assumption)

$= \frac{(2k)!\,(2k+1)(2k+2)}{2^k k!\; 2k+2}$

$= \frac{(2k+2)!}{2^k . k!\; 2.(k+1)}$

$= \frac{[2(k+1)]!}{2^{k+1} . (k+1)!}$

$\therefore$ If statement true $n=k$ it is also true $n=k+1$.
Since statement is true $n=1$ it also true $n=1+1=2$,
$n=2+1=3$ and so on for all positive integer n.

6(c) monthly interest $= \frac{8}{1200} = \frac{1}{150}$ $\quad n = 25 \times 12 = 300$.

$R \equiv$ Repayment

Amount owing end 1st Month $= 260000 \times (1+\frac{1}{150}) - R$

Amount owing end 2nd Month $= \left[260000\left(1+\frac{1}{150}\right) - R\right]\left(1+\frac{1}{150}\right) - R$

$= 260000 \cdot \left(\frac{151}{150}\right)^2 - R\left[1 + \frac{151}{150}\right]$.

Amount owing end 3rd Month $= \left[260000\left(\frac{151}{150}\right)^2 - R\left[1+\frac{151}{150}\right]\right]\frac{151}{150} - R$

$= 260000\left(\frac{151}{150}\right)^3 - R\left[1 + \frac{151}{150} + \left(\frac{151}{150}\right)^2\right]$. ①

end 300 months $0 = 260000\left(\frac{151}{150}\right)^{300} - R\left[1 + \left(\frac{151}{150}\right) + \left(\frac{151}{150}\right)^2 + \left(\frac{151}{150}\right)^{299}\right]$ ②

$R\left[\frac{\left(\frac{151}{150}\right)^{300} - 1}{\frac{151}{150} - 1}\right] = 260000\left(\frac{151}{150}\right)^{300}$

$R = \frac{260000 \cdot \frac{1}{150} \cdot \left(\frac{151}{150}\right)^{300}}{\left(\frac{151}{150}\right)^{300} - 1}$

Repayment $= \$2006.72$ per month. ③

Ans:

a) (i) $T = A + Be^{-kt}$

$\frac{dT}{dt} = -kBe^{-kt}$ but $Be^{-kt} = T - A$

$= -k[T - A]$. — (2)

ii) $t = 0$ $T = 85$ $A = 22$

$t = 16$ $T = 70$

$\therefore T = 22 + Be^{-kt}$ — (1)

$85 = 22 + B$

$B = 63$ — (1)

$T = 22 + 63e^{-kt}$

$70 = 22 + 63e^{-16k}$

$k = \frac{1}{16}\ln\left(\frac{48}{63}\right)$ — (1)

$\therefore T = 22 + 63e^{-\frac{t}{16}\ln\frac{63}{48}}$

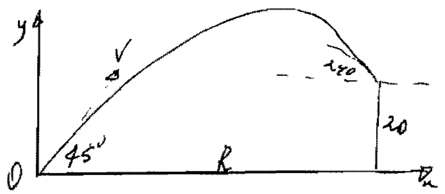
$30 = 22 + 63e^{-\frac{t}{16}\ln\left(\frac{63}{48}\right)}$

$e^{-\frac{t}{16}\ln\frac{63}{48}} = \frac{8}{63}$

$t = \frac{-16\ln\frac{8}{63}}{\ln\frac{63}{48}}$ — (1)

$= 121$ mins.

b)



$y = x\tan\alpha - \frac{gx^2}{2V^2}(1 + \tan^2\alpha)$

$= x - \frac{5x^2}{V^2}(1 + 1)$

$y = x - \frac{10x^2}{V^2}$

$20 = R - \frac{10R^2}{V^2}$ — ① — (1)

Also $\frac{dy}{dx} = \tan\alpha - \frac{gx}{V^2}(1 + \tan^2\alpha)$

$-\tan 27^\circ = 1 - \frac{10R}{V^2} \cdot 2$

$\frac{R}{V^2} = \frac{1 + \tan 27^\circ}{20}$ — ② — (1)

$\therefore$ from ① $20 = R - 10R \cdot \left[\frac{1 + \tan 27^\circ}{20}\right]$ — (1)

$R = \frac{20}{1 - \frac{1}{2} - \frac{1}{2}\tan 27^\circ}$

$= \frac{40}{1 - \tan 27^\circ}$

Range $R = 81.55$ m. — (1)

(c) $(1 + 3 + 5 + p) \times (1 + 3 + 5 + q) = 1 + 3 + 5 + \ldots + 33$

Now $2n - 1 = p$

Number terms $n = \frac{p+1}{2}$

$\therefore \frac{1}{2}\left(\frac{p+1}{2}\right)(1 + p) \times \frac{1}{2}\left(\frac{q+1}{2}\right)(1 + q) = \frac{1}{2} \cdot \frac{33+1}{2} \cdot (33 + 1)$

$\left(\frac{p+1}{2}\right)^2 \times \left(\frac{q+1}{2}\right)^2 = 17^2$ — (1)

$\therefore \frac{p+1}{2} = 15$ $\frac{q+1}{2} = 8$

$p = 29$ $q = 15$

$\therefore p + q = 44$ — (1)