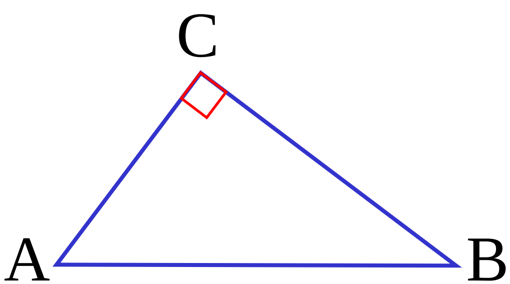
# Harder Extension 1
# Circle Geometry

## Converse Circle Theorems

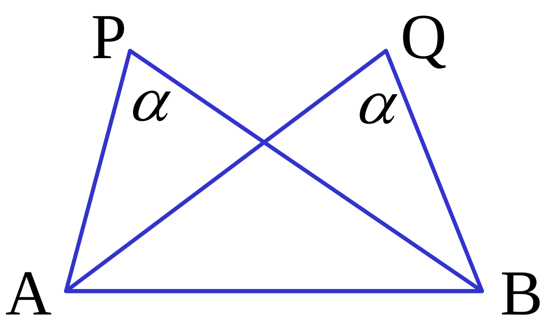
(1) The circle whose diameter is the hypotenuse of a right angled triangle passes through the third vertex.




ABC are concyclic with AB diameter

$\left(\angle \text{ in a semicircle} = 90^\circ\right)$

(2) If an interval AB subtends the same angle at two points P and Q on the same side of AB, then A,B,P,Q are concyclic.



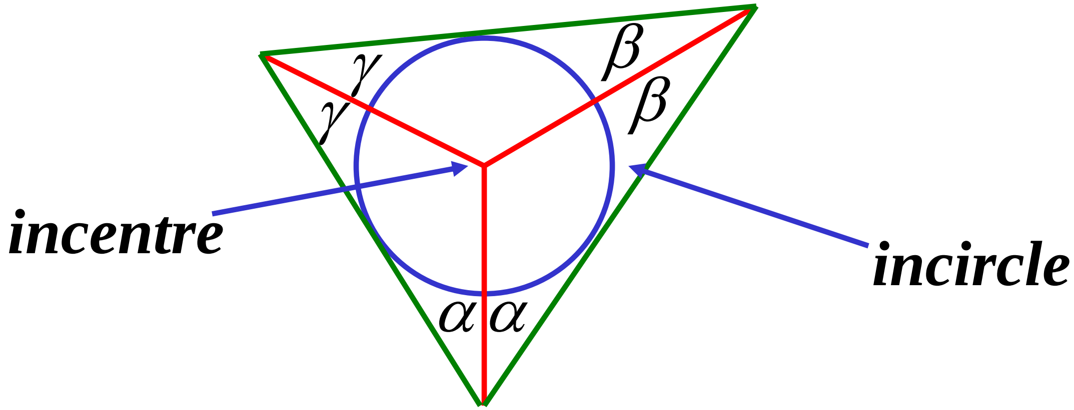

ABQP is a cyclic quadrilateral

$\left(\angle' \text{s in same segment are} = \right)$

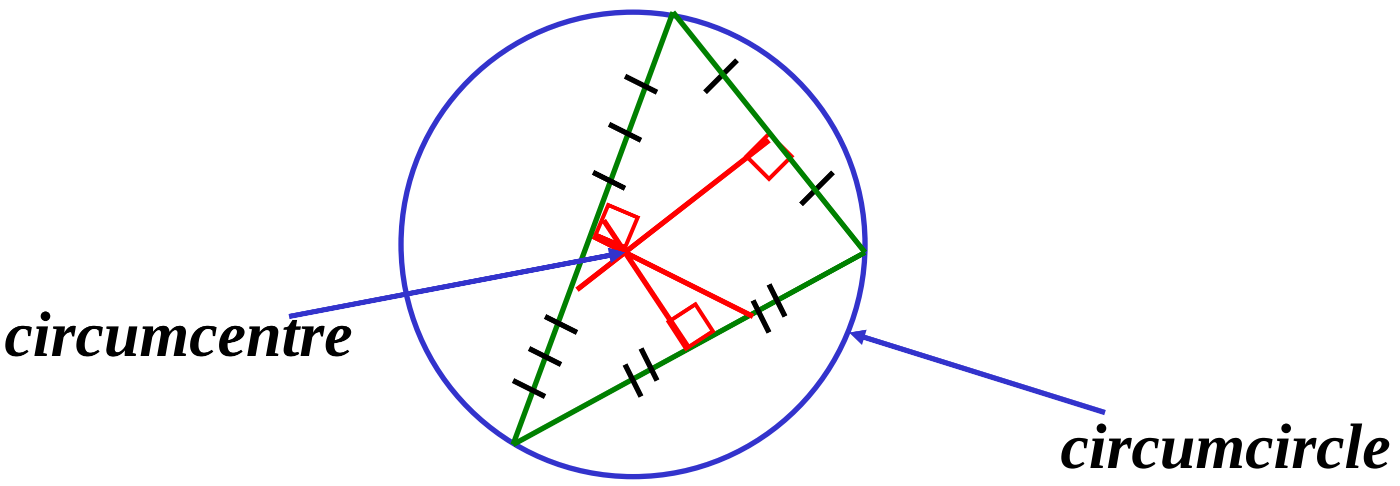
(3) If a pair of opposite angles in a quadrilateral are supplementary (or if an exterior angle equals the opposite interior angle) then the quadrilateral is cyclic.

## The Four Centres Of A Triangle

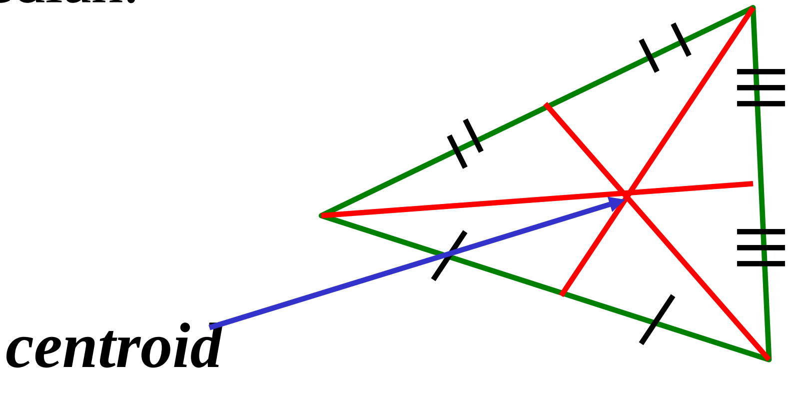
(1) The angle bisectors of the vertices are concurrent at the ***incentre*** which is the centre of the ***incircle***, tangent to all three sides.



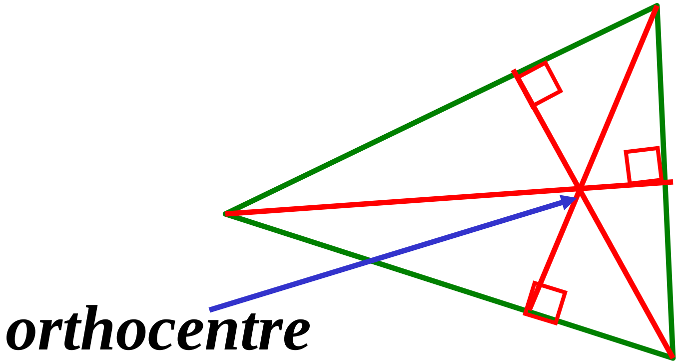
(2) The perpendicular bisectors of the sides are concurrent at the ***circumcentre*** which is the centre of the ***circumcircle***, passing through all three vertices.




(3) The medians are concurrent at the ***centroid***, and the centroid trisects each median.



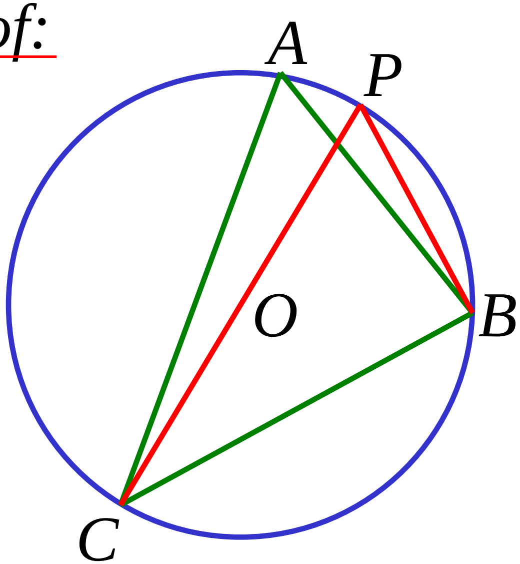
(4) The altitudes are concurrent at the ***orthocentre***.




Interaction Between Geometry & Trigonometry

$$\boxed{\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = \text{diameter if circumcircle}}$$

*Proof:*




$$\angle A = \angle P \qquad (\angle \text{ in same segment} =)$$

$$\therefore \sin A = \sin P$$

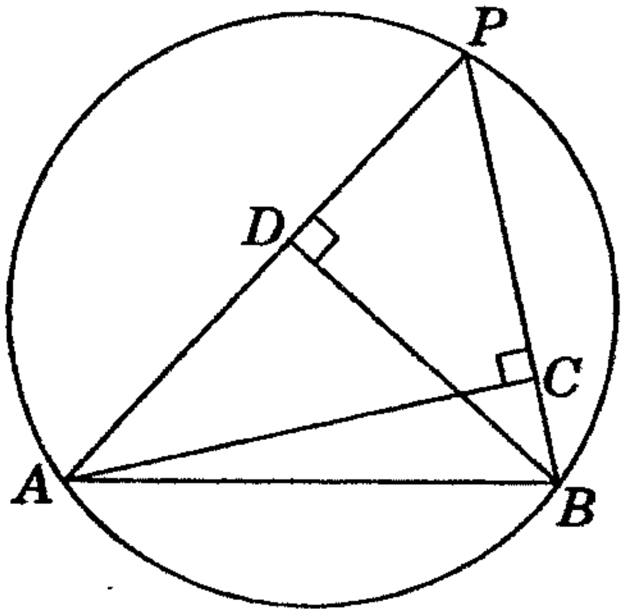
$$\angle PBC = 90^\circ \qquad (\angle \text{ in semicircle} = 90^\circ)$$

$$\frac{BC}{PC} = \sin P$$

$$\frac{BC}{\sin P} = PC$$

$$\therefore PC = \frac{BC}{\sin A}$$

e.g. (1990)



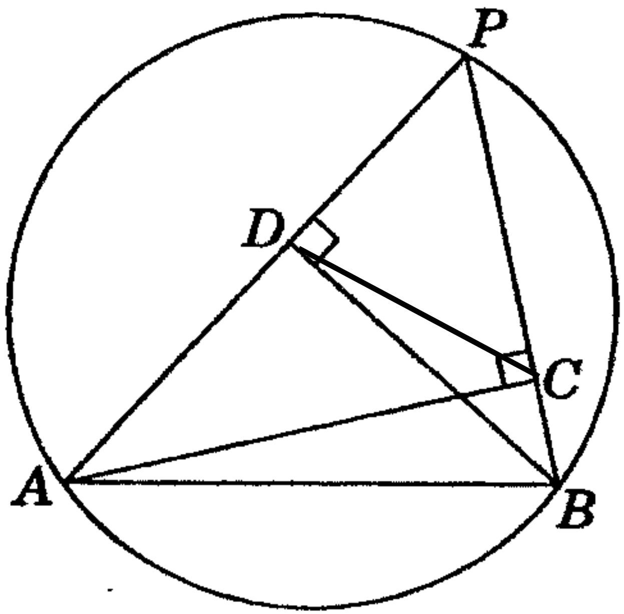

In the diagram, $AB$ is a fixed chord of a circle, $P$ a variable point in the circle and $AC$ and $BD$ are perpendicular to $BP$ and $AP$ respectively.

($i$) Show that $ABCD$ is a cyclic quadrilateral on a circle with $AB$ as diameter.

$\angle BDA = \angle ACB = 90^\circ$ (given)

$\therefore$ $ABCD$ is a cyclic quadrilateral ($\angle$'s in same segment are $=$)

$AB$ is diameter as $\angle$ in semicircle $= 90^\circ$

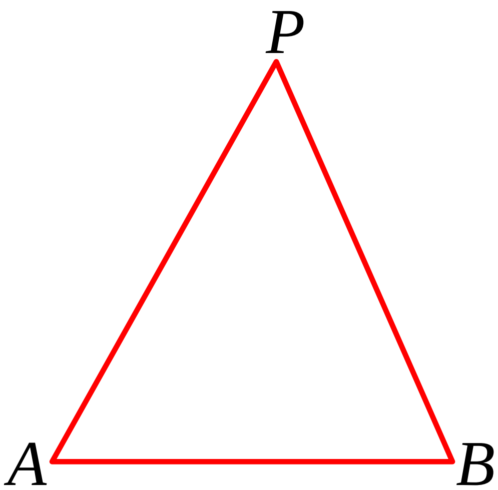


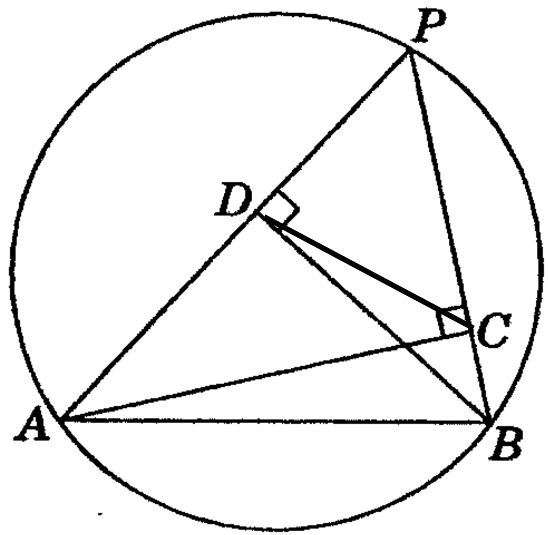

($ii$) Show that triangles $PCD$ and $APB$ are similar

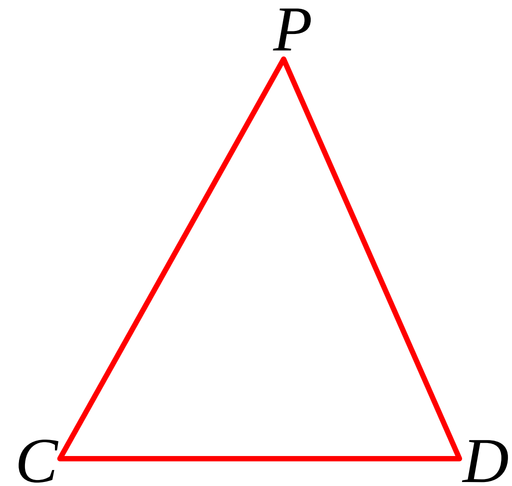
$\angle APB = \angle DPC$ (common $\angle$'s)

$\angle PDC = \angle PBA$ (exterior$\angle$ cyclic quadrilateral)

$\therefore \Delta PDC \,|||\, \Delta PBA$ (equiangular)








($iii$) Show that as $P$ varies, the segment $CD$ has constant length.

$$\frac{CD}{AB} = \frac{PC}{AP} \qquad \text{(ratio of sides in } ||| \ \Delta\text{'s)}$$

$$\text{In } \Delta PCA, \frac{PC}{AP} = \cos P$$

$$\therefore \frac{CD}{AB} = \cos P$$

$$CD = AB \cos P$$

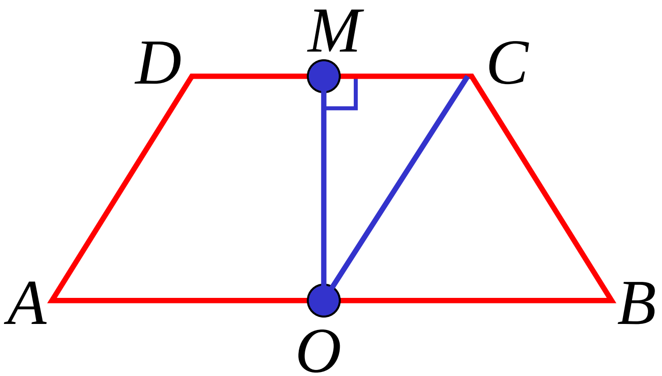
Now, $\angle P$ is constant $\qquad$ ($\angle$'s in same segment are $=$)

and $AB$ is fixed $\qquad$ (given)

$\therefore CD$ is constant

($iv$) Find the locus of the midpoint of $CD$.

$ABCD$ is a cyclic quadrilateral with $AB$ diameter.




Let $M$ be the midpoint of $CD$

$O$ is the midpoint of $AB$

$OM$ is constant

(= chords are equidistant from the centre)

$\therefore M$ is a fixed distance from $O$

$$OM^2 = OC^2 - MC^2$$

$$= \left(\frac{1}{2}AB\right)^2 - \left(\frac{1}{2}AB\cos P\right)^2$$

$$= \frac{1}{4}AB^2 - \frac{1}{4}AB^2\cos^2 P$$

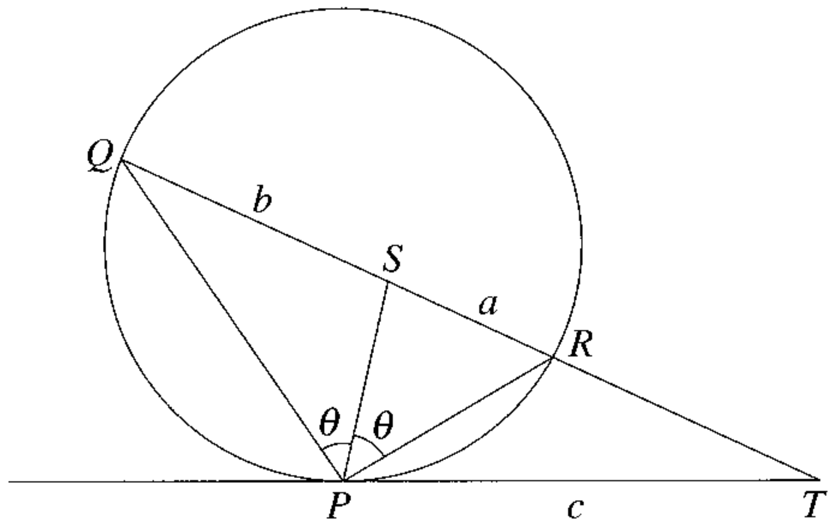
$$= \frac{1}{4}AB^2\sin^2 P$$

$$OM = \frac{1}{2}AB\sin P$$

$\therefore$ locus is circle, centre $O$

and radius $= \dfrac{1}{2}AB\sin P$

# 2008 Extension 2 Question 7b)




In the diagram, the points $P$, $Q$ and $R$ lie on a circle. The tangent at $P$ and the secant $QR$ intersect at $T$. The bisector of $\angle PQR$ meets $QR$ at $S$ so that $\angle QPS = \angle RPS = \theta$. The intervals $RS$, $SQ$ and $PT$ have lengths $a$, $b$ and $c$ respectively.

$(i)$ Show that $\angle TSP = \angle TPS$

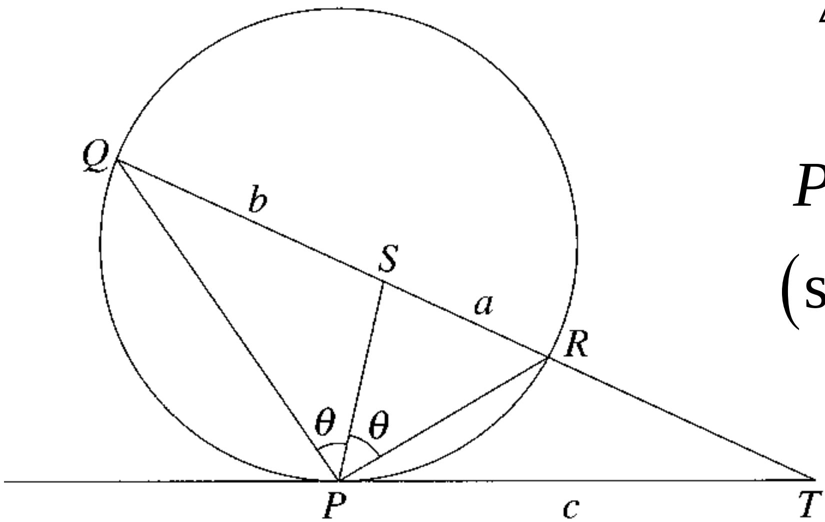
$\angle RQP = \angle RPT$ (alternate segment theorem)

$\angle TSP = \angle RQP + \angle SPQ$ (exterior $\angle, \Delta SPQ$)

$\angle TSP = \angle RPT + \theta$

$\therefore \angle SPT = \angle TSP$

$(ii)$ Hence show that $\dfrac{1}{a} = \dfrac{1}{b} + \dfrac{1}{c}$




$\Delta TPS$ is isosceles $\quad (2 = \angle\text{'s})$

$\therefore ST = c \quad (= \text{sides in isosceles } \Delta)$

$PT^2 = QT \times RT \quad (\text{common } \angle)$

(square of tangents=products of intercepts)

$$c^2 = (c+b)(c-a)$$

$$c^2 = c^2 - ac + bc - ab$$

$$bc = ac + ab$$

$$\frac{1}{a} = \frac{1}{b} + \frac{1}{c}$$

**Past HSC Papers**

**Exercise 10C***