Concavity

The second deriviative measures the change in slope with respect to *x*, this is known as **concavity**

$$\text{If } f''(x) > 0, \text{ the curve is concave up}$$
$$\text{If } f''(x) < 0, \text{ the curve is concave down}$$
$$\text{If } f''(x) = 0, \text{ possible point of inflection}$$

e.g. By looking at the second derivative sketch $y = x^3 + 5x^2 + 3x + 2$

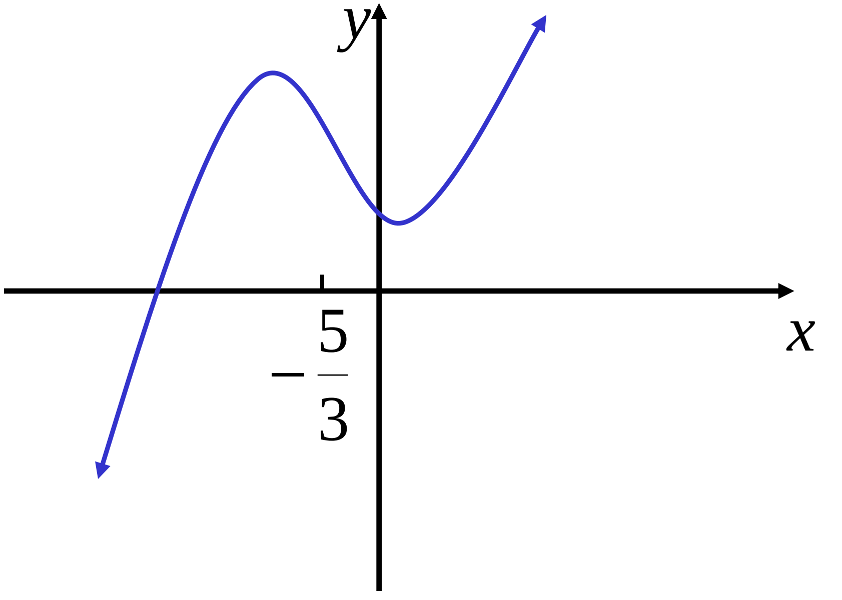
$$\frac{dy}{dx} = 3x^2 + 10x + 3$$

$$\frac{d^2y}{dx^2} = 6x + 10$$

Curve is concave up when $\dfrac{d^2y}{dx^2} > 0$

i.e. $6x + 10 > 0$

$$x > -\frac{5}{3}$$



Turning Points

All turning points are stationary points.

$$\text{If } f''(x) > 0, \text{ minimum turning point}$$
$$\text{If } f''(x) < 0, \text{ maximum turning point}$$

e.g. Find the turning points of $y = x^3 + x^2 - x + 1$

$$\frac{dy}{dx} = 3x^2 + 2x - 1$$

$$\frac{d^2y}{dx^2} = 6x + 2$$

Stationary points occur when $\dfrac{dy}{dx} = 0$

i.e. $3x^2 + 2x - 1 = 0$

$$(3x-1)(x+1) = 0$$

$$x = \frac{1}{3} \quad \text{or} \quad x = -1$$

$$\text{when } x = -1, \frac{d^2y}{dx^2} = 6(-1) + 2$$

$$= -4 < 0$$

$$\therefore (-1,2) \text{ is a maximum turning point}$$

$$\text{when } x = \frac{1}{3}, \frac{d^2y}{dx^2} = 6\left(\frac{1}{3}\right) + 2$$

$$= 4 > 0$$

$$\therefore \left(\frac{1}{3}, \frac{22}{27}\right) \text{ is a minimum turning point}$$

Inflection Points

A point of inflection is where there is a **change in concavity**, to see if there is a change, check either side of the point.

e.g. Find the inflection point(s) of $y = 4x^3 + 6x^2 + 2$

$$\frac{dy}{dx} = 12x^2 + 12x \qquad \frac{d^2y}{dx^2} = 24x + 12$$

Possible points of inflection occur when $\frac{d^2y}{dx^2} = 0$

$i.e.\ \ 24x + 12 = 0$

$$x = -\frac{1}{2}$$

x	$-\frac{1}{2}^-$ (-1)	$-\frac{1}{2}$	$-\frac{1}{2}^+$ (0)
$\frac{d^2y}{dx^2}$	(-12)	0	(12)

$\therefore$ there is a change in concavity

$\therefore \left(-\frac{1}{2}, 3\right)$ is a point of inflection

Horizontal Point of Inflection; $\frac{dy}{dx} = 0 \quad \frac{d^2y}{dx^2} = 0 \quad \frac{d^3y}{dx^3} \neq 0$

Alternative Way of Finding Inflection Points

Possible points of inflection occur when $\frac{d^2y}{dx^2} = 0$

If the first non-zero derivative is of odd order,

$$\text{i.e } \frac{d^3y}{dx^3} \neq 0 \text{ or } \frac{d^5y}{dx^5} \neq 0 \text{ or } \frac{d^7y}{dx^7} \neq 0 \text{ etc}$$

then it is a point of inflection

If the first non-zero derivative is of even order,

$$\text{i.e } \frac{d^4y}{dx^4} \neq 0 \text{ or } \frac{d^6y}{dx^6} \neq 0 \text{ or } \frac{d^8y}{dx^8} \neq 0 \text{ etc}$$

then it is not a point of inflection

$$e.g.\ \frac{d^3y}{dx^3} = 24$$

$$\text{when } x = -\frac{1}{2}, \frac{d^3y}{dx^3} = 24 \neq 0$$

$\therefore$ there is a change in concavity

$\therefore \left(-\frac{1}{2}, 3\right)$ is a point of inflection

Exercise 10E; 1, 2bc, 3, 6ac, 7bd, 8, 10, 12, 14, 16, 18