

Mathematical Induction

e.g. (i) Prove $1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$

Test: $n = 1$ $L.H.S = \frac{1}{1^2}$ $R.H.S = 2 - \frac{1}{1}$
 $= 1$ $= 1$
 $\therefore L.H.S \leq R.H.S$

$(A) (n = k) 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{k^2} \leq 2 - \frac{1}{k}$

$(P) (n = k + 1) 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{(k + 1)^2} \leq 2 - \frac{1}{k + 1}$

Proof:

$$\begin{aligned}1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{(k+1)^2} &= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{k^2} + \frac{1}{(k+1)^2} \\ &\leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2} \\ &= 2 - \frac{(k+1)^2 - k}{k(k+1)^2} \\ &= 2 - \frac{k^2 + k + 1}{k(k+1)^2} \\ &= 2 - \frac{k^2 + k}{k(k+1)^2} - \frac{1}{k(k+1)^2} \\ &< 2 - \frac{k(k+1)}{k(k+1)^2} \\ &= 2 - \frac{1}{k+1} \\ \therefore 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{(k+1)^2} &\leq 2 - \frac{1}{k+1}\end{aligned}$$

(ii) A sequence is defined by;

$$a_1 = \sqrt{2} \qquad a_{n+1} = \sqrt{2 + a_n} \text{ for } n \geq 1$$

Show that $a_n < 2$ for $n \geq 1$

Test: $n = 1$ $a_1 = \sqrt{2} < 2$

\textcircled{A} ($n = k$) $a_k < 2$

\textcircled{P} ($n = k + 1$) $a_{k+1} < 2$

Proof:

$$a_{k+1} = \sqrt{2 + a_k}$$

$$< \sqrt{2 + 2}$$

$$= \sqrt{4}$$

$$= 2$$

$\therefore a_{k+1} < 2$

(iii) The sequences x_n and y_n are defined by;

$$x_1 = 5, y_1 = 2 \qquad x_{n+1} = \frac{x_n + y_n}{2}, y_{n+1} = \frac{2x_n y_n}{x_n + y_n}$$

Prove $x_n y_n = 10$ for $n \geq 1$

Test: $n = 1$ $x_1 y_1 = (5)(2)$
 $= 10$

A ($n = k$) $x_k y_k = 10$

P ($n = k + 1$) $x_{k+1} y_{k+1} = 10$

Proof:

$$x_{k+1} y_{k+1} = \left(\frac{x_k + y_k}{2} \right) \left(\frac{2x_k y_k}{x_k + y_k} \right)$$

$$= x_k y_k$$

$$= 10$$

$\therefore x_{k+1} y_{k+1} = 10$

(iv) The Fibonacci sequence is defined by;

$$a_1 = a_2 = 1 \quad a_{n+1} = a_n + a_{n-1} \quad \text{for } n > 1$$

Prove that $a_n < \left(\frac{1+\sqrt{5}}{2}\right)^n$ for $n \geq 1$

Test: $n = 1$ and $n = 2$

$$\begin{aligned} L.H.S &= a_1 \\ &= 1 \end{aligned} \qquad \begin{aligned} R.H.S &= \left(\frac{1+\sqrt{5}}{2}\right)^1 \\ &= 1.62\dots \end{aligned}$$

$$\therefore L.H.S \leq R.H.S$$

$$\begin{aligned} L.H.S &= a_2 \\ &= 1 \end{aligned} \qquad \begin{aligned} R.H.S &= \left(\frac{1+\sqrt{5}}{2}\right)^2 \\ &= 2.62\dots \end{aligned}$$

$$\therefore L.H.S \leq R.H.S$$

$$\textcircled{A} \quad (n = k - 1 \& n = k) \quad a_{k-1} < \left(\frac{1+\sqrt{5}}{2}\right)^{k-1} \quad \& \quad a_k < \left(\frac{1+\sqrt{5}}{2}\right)^k$$

$$\textcircled{P} \quad (n = k + 1) \quad a_{k+1} < \left(\frac{1+\sqrt{5}}{2}\right)^{k+1}$$

Proof:

$$a_{k+1} = a_k + a_{k-1}$$

$$< \left(\frac{1 + \sqrt{5}}{2} \right)^k + \left(\frac{1 + \sqrt{5}}{2} \right)^{k-1}$$

$$= \left(\frac{1 + \sqrt{5}}{2} \right)^{k+1} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^{-1} + \left(\frac{1 + \sqrt{5}}{2} \right)^{-2} \right]$$

$$= \left(\frac{1 + \sqrt{5}}{2} \right)^{k+1} \left[\frac{2}{1 + \sqrt{5}} + \frac{4}{(1 + \sqrt{5})^2} \right]$$

$$= \left(\frac{1 + \sqrt{5}}{2} \right)^{k+1} \left[\frac{2 + 2\sqrt{5} + 4}{(1 + \sqrt{5})^2} \right]$$

$$= \left(\frac{1 + \sqrt{5}}{2} \right)^{k+1} \left[\frac{6 + 2\sqrt{5}}{(1 + \sqrt{5})^2} \right]$$

$$= \left(\frac{1 + \sqrt{5}}{2} \right)^{k+1}$$

$$\therefore a_{k+1} < \left(\frac{1 + \sqrt{5}}{2} \right)^{k+1}$$

Sheets

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Exercise 10E*