SYDNEY GRAMMAR SCHOOL

2010 Trial Examination

3u

Syd Gram 2010 3u (84)

FORM VI
MATHEMATICS EXTENSION 1

Wednesday 11th August 2010

General Instructions

- Reading time — 5 minutes
- Writing time — 2 hours
- Write using black or blue pen.
- Board-approved calculators may be used.
- A list of standard integrals is provided at the end of the examination paper.
- All necessary working should be shown in every question.
- Start each question in a new booklet.

Structure of the paper

- Total marks — 84
- All seven questions may be attempted.
- All seven questions are of equal value.

Collection

- Write your candidate number clearly on each booklet.
- Hand in the seven questions in a single well-ordered pile.
- Hand in a booklet for each question, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Place the question paper inside your answer booklet for Question One.

Checklist

- SGS booklets — 7 per boy
- Candidature — 132 boys

Examiner

SJE

QUESTION ONE (12 marks) Use a separate writing booklet. **Marks**

(a) The polynomial $P(x) = x^4 - x^3 + kx - 4$ has a factor $(x+1)$. Find the value of k. **1**

(b) Differentiate $y = \sin(\log_e x)$. **1**

(c) Find, correct to the nearest degree, the acute angle between the lines **2**

$$x - 3y + 4 = 0 \quad \text{and} \quad 2x + y - 1 = 0.$$

(d) Find the coordinates of the point that divides the interval from $(-3, 4)$ to $(5, -2)$ in the ratio $1:3$. **2**

(e) Find the exact value of $\displaystyle\int_0^2 \frac{4}{4+x^2}\,dx$. **2**

(f) Find $\displaystyle\lim_{x \to 0}\left(\frac{\sin x \cos x}{x}\right)$. **1**

(g) Find the term independent of x in the expansion of $\left(x^2 + \dfrac{2}{x}\right)^{15}$. **3**

QUESTION TWO (12 marks) Use a separate writing booklet. **Marks**

(a) The equation $x^3 + bx^2 + cx + d = 0$ has roots $2+\sqrt{3}$, $2-\sqrt{3}$ and -3. Use the sum and the product of the roots to find b, c and d. **3**

(b) Consider the curve $f(x) = \sin^{-1}(2x)$.

(i) Sketch the curve. **2**

(ii) Find the gradient of the tangent to the curve at the point where $x = \dfrac{1}{4}$. **2**

(c) A particle is undergoing simple harmonic motion subject to the equation $\dfrac{d^2x}{dt^2} = -6x$.

Initially it is at rest at $x = 2$.

(i) In which direction will it start to move? **1**

(ii) Show that $v^2 = 6(4 - x^2)$. **2**

(iii) State the period and the amplitude of the motion. **2**

Exam continues next page ...

QUESTION THREE (12 marks) Use a separate writing booklet. **Marks**

(a) Use the substitution $u = x - 1$ to find $\int x(x-1)^4 \, dx$. **2**

(b) (i) Express $\cos\theta - \sqrt{3}\sin\theta$ in the form $R\cos(\theta + \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. **3**

(ii) Hence, or otherwise, solve $\cos\theta - \sqrt{3}\sin\theta = 1$, for $0 \le \theta \le 2\pi$. **2**

(c) Consider the function $f(x) = \dfrac{x-1}{x-2}$.

(i) Show that $f^{-1}(x) = \dfrac{2x-1}{x-1}$. **1**

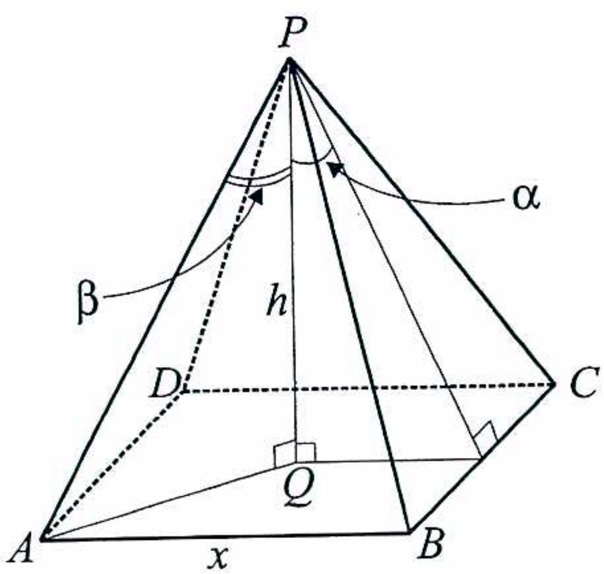
(ii) Find the vertical and horizontal asymptotes of $f^{-1}(x)$. **2**

(iii) Sketch $f^{-1}(x)$ showing the asymptotes and any x or y intercepts. **1**

(iv) Hence, or otherwise, solve $\dfrac{2x-1}{x-1} \ge 1$. **1**

QUESTION FOUR (12 marks) Use a separate writing booklet. **Marks**

(a)



A square pyramid has altitude PQ of length h and base $ABCD$ of side length x, as shown above. Each face makes an angle α with PQ and each edge makes an angle β with PQ. Assume that it is a right pyramid, so that Q lies in the centre of the base.

(i) Show that $AQ = \dfrac{x}{\sqrt{2}}$. **1**

(ii) Hence express x in terms of h and β. **1**

(iii) Show that $\sqrt{2}\tan\alpha = \tan\beta$. **1**

Exam continues overleaf ...

QUESTION FOUR (Continued)

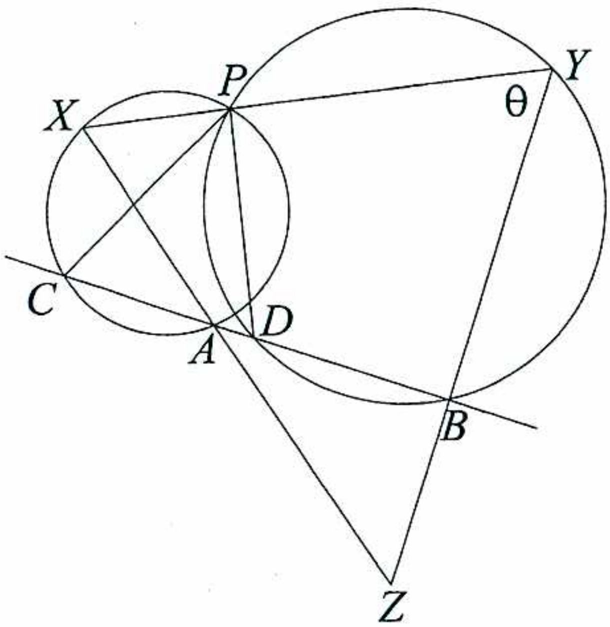
(b) Solve for x and y: **3**

$$\log_3 x + \log_3 y = 6$$
$$\log_2 x - \log_2 y = 4$$

(c) (i) Write $\cos^2 x$ in terms of $\cos 2x$. **1**

(ii) Hence evaluate $\displaystyle\int_0^{\frac{\pi}{3}} \sin 2x \cos^2 x \, dx$. **2**

(d) **3**



In the diagram above, two circles intersect and P is one of the points of intersection. A straight line is drawn through P cutting the two circles at X and Y. An isosceles triangle XYZ is constructed with $XZ = YZ$. Suppose that XZ cuts the smaller circle at A and YZ cuts the larger circle at B. Suppose also that the line AB cuts the circles at C and D. Let $\angle XYZ$ be θ.

Prove that $\triangle CPD$ is isosceles.

NOTE: You do not have to copy the diagram above. It has been reproduced for you on a tear-off sheet at the end of this paper. Insert this sheet into your answer booklet.

Marks

QUESTION FIVE (12 marks) Use a separate writing booklet.

(a) Consider the two points $P(4t, 2t^2)$ and $Q(8t, 8t^2)$ on the parabola $x^2 = 8y$. The tangents at P and Q intersect at R.

(i) Find the equations of the tangents at P and Q. **2**

(ii) Find the coordinates of R. **2**

(iii) Hence find the locus of R. **1**

(b) By substituting a suitable value for x in the expansion of $(1+x)^n$, show that **2**

$$1 + 2\binom{n}{1} + 4\binom{n}{2} + \ldots + 2^{n-1}\binom{n}{n-1} = 3^n - 2^n.$$

(c) A forensic scientist is called upon to determine the time of death of a corpse found in a room which is maintained at a constant temperature of 20°C. The temperature T of the corpse was initially measured at midnight to be 29°C. The scientist measured the temperature of the corpse one hour later and it had fallen to 26°C. Assume that the temperature of the body at the time of death was 36·8°C and that the rate of temperature decrease obeys Newton's law of cooling. Let t be the number of hours after midnight.

(i) Show that $T = 20 + 9e^{-kt}$ satisfies the cooling equation $\dfrac{dT}{dt} = -k(T - 20)$. **1**

(ii) Show that $k = \ln \frac{3}{2}$. **2**

(iii) Hence estimate the time of death, to the nearest minute. **2**

QUESTION SIX (12 marks) Use a separate writing booklet. **Marks**

(a) Prove by mathematical induction that for all positive integer values of n, **3**

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \ldots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}.$$

(b) A film director must decide whether a stuntman is able to perform a dangerous stunt. The stuntman must leap from a building onto the centre of some erected scaffolding. The centre of the scaffolding is 5 m below his initial position and at a horizontal distance of 14 m. The stuntman jumps at an angle of 30° above the horizontal. Let the stuntman's initial velocity be V, and let x and y be his horizontal and vertical displacements respectively from his initial position. You may assume that the velocity and displacement equations are:

$$\dot{x} = V\cos 30^\circ \qquad \dot{y} = -10t + V\sin 30^\circ$$

$$x = Vt\cos 30^\circ \qquad y = -5t^2 + Vt\sin 30^\circ$$

(i) Show that the Cartesian equation of the stuntman's path is **2**

$$y = -\frac{20x^2}{3V^2} + \frac{x}{\sqrt{3}}.$$

(ii) Hence determine the required initial velocity V so that he lands in the centre of the scaffolding. Write your answer to the nearest m/s. **2**

(iii) Safety requirements are such that if the impact velocity is greater than 15 m/s, then padding must be placed on the scaffolding. Assuming that the stuntman leaps at the required speed, determine whether or not padding is needed. **2**

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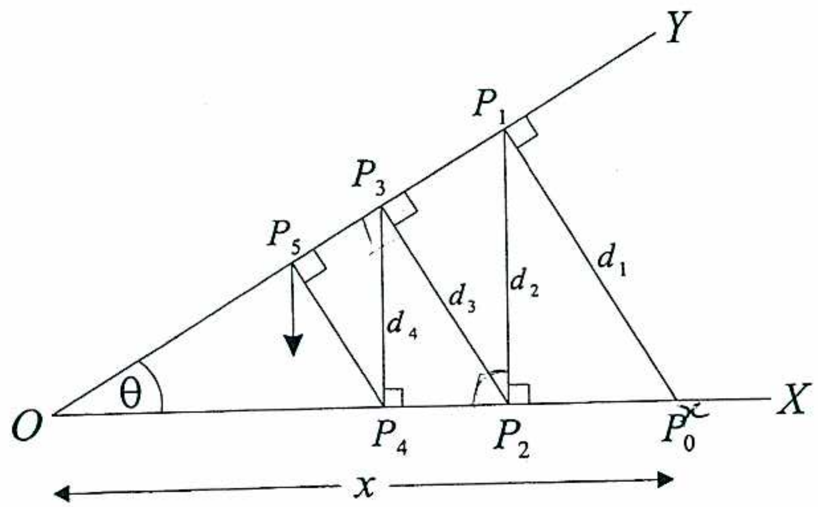
QUESTION SIX (Continued)

(c)



The diagram above shows two straight lines OX and OY. The points $P_0, P_2, P_4, \ldots$ lie on OX, while the points $P_1, P_3, P_5, \ldots$ lie on OY.

P_1 is the foot of the perpendicular from P_0 to OY,
P_2 is the foot of the perpendicular from P_1 to OX,
P_3 is the foot of the perpendicular from P_2 to OY, and so on.

Let $\angle XOY = \theta$, where $0^\circ < \theta < 90^\circ$, let $OP_0 = x$, and let the length of the line joining P_{r-1} to P_r be denoted by d_r, for $r = 1, 2, 3, \ldots$.

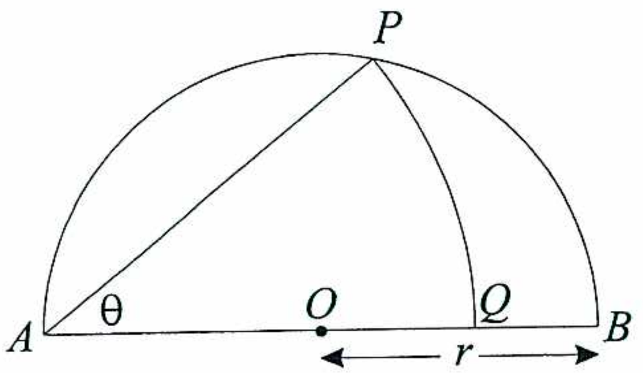
(i) Show that the lengths $d_1, d_2, d_3, \ldots$ form a geometric series. **2**

(ii) Hence prove that $\displaystyle\sum_{r=1}^{\infty} d_r = x \cot \tfrac{\theta}{2}$. **1**

$\frac{d_2}{d_1} = P_1 - P_0 \quad \frac{P_2 - P_1}{P_1 - P_0} =$

QUESTION SEVEN (12 marks) Use a separate writing booklet. **Marks**

(a)



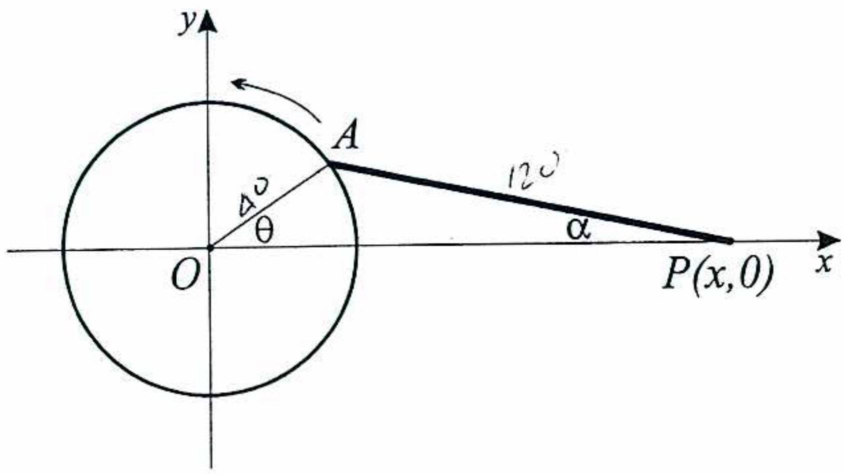
The diagram above shows a semi-circle with centre O, radius r and diameter AB. Let P be a point on arc AB. The arc PQ has centre A and Q lies on AB. Let $\angle PAQ = \theta$.

(i) Show that $AP = 2r\cos\theta$. **1**

(ii) Prove that as θ varies, the arc PQ will have maximum length when $\theta\sin\theta = \cos\theta$. **3**

(iii) Taking $\theta = 1$ as a first approximation to the value of θ that maximises the arc PQ, use one application of Newton's method to find a better approximation. Round your answer to two decimal places. **2**

(b)



The diagram above shows a rotating wheel with radius 40 cm and a connecting rod AP with length 120 cm. The pin P slides back and forth along the x-axis as the wheel rotates anticlockwise at a rate of 6 revolutions per second. In each part below you need only address the case where A is in the first quadrant.

(i) Show that $\alpha = \sin^{-1}\left(\dfrac{\sin\theta}{3}\right)$. **1**

(ii) Use the chain rule to show that $\dfrac{d\alpha}{dt} = \dfrac{12\pi\cos\theta}{\sqrt{9-\sin^2\theta}}$ radians per second. **1**

(iii) Show that $x = 40\left(\cos\theta + \sqrt{9-\sin^2\theta}\right)$. **2**

(iv) Find an expression for the velocity of the pin P in terms of θ. **2**

END OF EXAMINATION

Extension 1

Question 1

(a) $P(-1) = (-1)^4 - (-1)^3 + kx - 4 = 0$

$1 + 1 - k - 4 = 0$

$-k - 2 = 0$

$k = -2$ ✓ (1)

(b) $y = \sin(\log_e x)$

$\frac{dy}{dx} = \frac{\cos(\log_e x)}{x}$ ✓ (1)

(c) $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$= \left|\frac{\frac{1}{3} - -2}{1 + \frac{1}{3}(-2)}\right|$ ✓

$= \left|\frac{\frac{7}{3}}{\frac{1}{3}}\right|$

$= 7$

$\therefore \theta \doteqdot 82^\circ$ (nearest degree) ✓ (2)

(d) $x = \frac{1(5) + 3(-3)}{1+3}$ $\quad y = \frac{1(-2) + 3(4)}{1+3}$ ✓

$= -1$ $\quad = \frac{5}{2}$

$\therefore$ Coordinates are $\left(-1, \frac{5}{2}\right)$ ✓ (2)

(e) $\int_0 \frac{4}{4+x^2}\,dx = 4\left[\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]_0$ ✓

$= 2\left[\tan^{-1}(1) - \tan^{-1} 0\right]$

$= 2 \cdot \frac{\pi}{4}$

$= \frac{\pi}{2}$ ✓ (2)

(f) $\lim_{x\to 0}\left(\frac{\sin x \cos x}{x}\right) = \lim_{x\to 0}\left(\frac{\sin x}{x}\right) \times \lim_{x\to 0}(\cos x)$

$= 1 \times 1$

$= 1$ ✓

(g) $\left(x^2 + \frac{2}{x}\right)^{15}$ General Term: ${}^{15}C_r (x^2)^r \left(\frac{2}{x}\right)^{15-r}$

$= {}^{15}C_r\, x^{2r}\, x^{-15+r}\, 2^{15-r}$

$= {}^{15}C_r\, x^{3r-15}\, 2^{15-r}$

So Term independent of x has

$3r - 15 = 0$

$r = 5$

$\therefore$ Term is ${}^{15}C_5\, 2^{10} = 3075072$

$\boxed{12}$

Question 2

(a) Sum of roots: $2+\sqrt{3}+2-\sqrt{3}-3 = -\frac{b}{1}$

$1 = -b$

$b = -1$ ✓

Product of roots: $(2+\sqrt{3})(2-\sqrt{3})(-3) = -d$

$(4-3)(-3) = -d$

$d = 3$ ✓

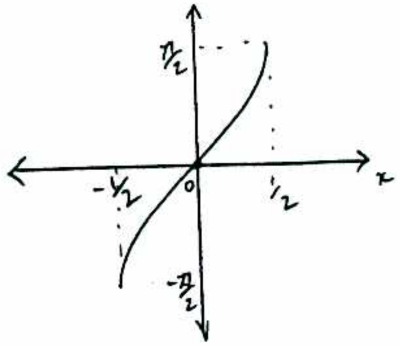
Also, $(2+\sqrt{3})(2-\sqrt{3}) + (2-\sqrt{3})(-3) + (-3)(2+\sqrt{3}) = c$

$4-3 + -6+3\sqrt{3} - 6 - 3\sqrt{3} = c$

$-11 = c$ ✓

3

(b) (i)



✓✓

(ii) $f'(x) = \frac{2}{\sqrt{1-(2x)^2}}$ ✓

$f'(\frac{1}{4}) = \frac{2}{\sqrt{1-\frac{1}{4}}}$

$= \frac{2}{\frac{\sqrt{3}}{2}}$

$= \frac{4}{\sqrt{3}}$ ✓

4

(c) $\frac{d^2x}{dt^2} = -6x$ $t=0$, $x=2$, $v=0$

(i) Negative direction towards the origin. ✓

(ii) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -6x$ ✓

$\frac{1}{2}v^2 = -3x^2 + c_1$

$v^2 = -6x^2 + c_2$

Now $v=0$, $x=2$

$0 = -24 + c_2$

$\therefore c_2 = 24$ ✓

So $v^2 = -6x^2 + 24$

$= 6(4-x^2)$ as required

(iii) $\frac{d^2x}{dt^2} = -n^2x$

$\therefore n^2 = 6$

$n = \sqrt{6}$

Period $= \frac{2\pi}{n}$

$= \frac{2\pi}{\sqrt{6}}$ ✓

$= \frac{\pi\sqrt{6}}{3}$

Amplitude $= 2$ ✓

5

12

Question 3

(a) $\int x(x-1)^4 \, dx$ $\qquad u = x-1, \quad x = u+1$

$\qquad du = dx$

$= \int (u+1)u^4 \, du$ ✓

$= \int u^5 du + \int u^4 du$

$= \frac{u^6}{6} + \frac{u^5}{5} + c$

$= \frac{(x-1)^6}{6} + \frac{(x-1)^5}{5} + c$ ✓ $\quad$ 2

(b)

(i) $\cos\theta - \sqrt{3}\sin\theta = R\cos(\theta + \alpha)$

$= R[\cos\theta\cos\alpha - \sin\theta\sin\alpha]$ ✓

Equating coefficients:

$R\cos\alpha = 1$

$R\sin\alpha = \sqrt{3}$

Squaring and adding $\qquad R^2 = 4$

$\therefore R = 2$ ✓

$\cos\alpha = \frac{1}{2} \quad (\alpha \text{ acute})$

$\therefore \alpha = \frac{\pi}{3}$ ✓

So $\cos\theta - \sqrt{3}\sin\theta = 2\cos\left(\theta + \frac{\pi}{3}\right)$

(ii) $2\cos\left(\theta + \frac{\pi}{3}\right) = 1 \qquad \frac{\pi}{3} \leq \theta + \frac{\pi}{3} \leq \frac{7\pi}{3}$

$\cos\left(\theta + \frac{\pi}{3}\right) = \frac{1}{2}$ ✓

$\theta + \frac{\pi}{3} = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}$

$\therefore \theta = 0, \frac{4\pi}{3}, 2\pi$ ✓ $\quad$ 5

(c) (i) $f(x) = y = \frac{\quad}{x-2}$

Interchange x and y

$x = \frac{y-1}{y-2}$

$x(y-2) = y-1$

$xy - y = -1 + 2x$ ✓

$y(x-1) = 2x-1$

$y = \frac{2x-1}{x-1}$

$\therefore f^{-1}(x) = \frac{2x-1}{x-1}$ as required. ✓

(ii) Vertical asymptote: $x = 1$ ✓

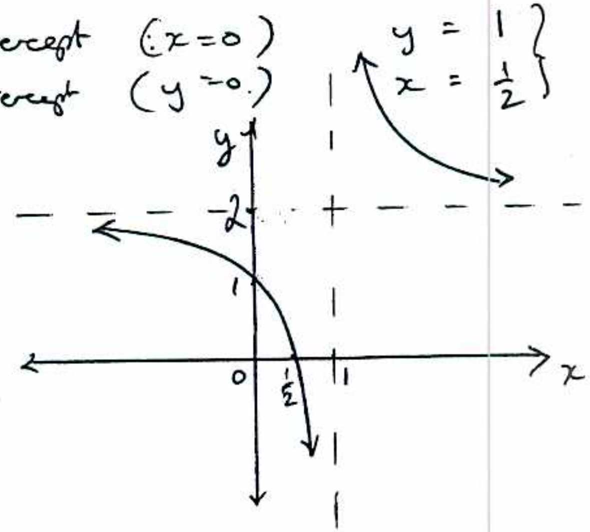
$f^{-1}(x) = \frac{2 - \frac{1}{x}}{1 - \frac{1}{x}}$

as $x \to \pm\infty \qquad f^{-1}(x) \to 2$

$\therefore$ Horizontal asymptote: $y = 2$ ✓

(iii) y-intercept $(x=0) \quad y = 1$

x-intercept $(y=0) \quad x = \frac{1}{2}$



✓

(iv) $\frac{2x-1}{x-1} \geq 1$ when $x \leq 0$ or $x > 1$ ✓

12

Question 4

(a) (i) $AC^2 = x^2 + x^2$

$= 2x^2$

$AC = \sqrt{2}x$

$AQ = \frac{\sqrt{2}x}{2}$

$= \frac{x}{\sqrt{2}}$ as required ✓

(ii) $\tan\beta = \frac{AQ}{h}$

$= \frac{\frac{x}{\sqrt{2}}}{h}$

$\therefore x = \sqrt{2}h\tan\beta$ ✓

(iii) $\tan\alpha = \frac{\frac{x}{2}}{h}$

$\therefore 2\tan\alpha = \frac{x}{h}$ and $\sqrt{2}\tan\beta = \frac{x}{h}$ ✓

$\therefore 2\tan\alpha = \sqrt{2}\tan\beta$

$\sqrt{2}\tan\alpha = \tan\beta$

3

(b) $\log_3 x + \log_3 y = 6$

$\log_2 x - \log_2 y = 4$

$xy = 3^6$ ✓

$\frac{x}{y} = 2^4$

Solving simultaneously $x^2 = 11664$

$x = 108$ $(x > 0)$ ✓

$y = \frac{3^6}{108}$

$= 6^{3/4}$ ✓ 2

(c) (i) $\cos^2 x = \frac{1}{2}\cos 2x + \frac{1}{2}$ ✓

(ii) $\int_0^{\pi/3} \sin 2x \cos^2 x \, dx = \frac{1}{2}\int_0^{\pi/3} (\sin 2x \cos 2x + \sin 2x)\, dx$

$= \frac{1}{2}\left[\frac{\sin^2 2x}{4}\right]_0^{\pi/3} + \frac{1}{2}\left[-\frac{\cos 2x}{2}\right]_0^{\pi/3}$ ✓

$= \frac{1}{8}\left[\left(\frac{\sqrt{3}}{2}\right)^2 - 0\right] + \frac{1}{4}\left[- -\frac{1}{2} - -1\right]$

$= \frac{3}{32} + \frac{3}{8}$

$= \frac{15}{32}$ ✓ 3

(c) Alternate Solution

$\int_0^{\pi/3} \sin 2x \cos^2 x \, dx = \frac{1}{2}\int_0^{\pi/3} \sin 2x (1 + \cos 2x)\, dx$

$= \frac{1}{2}\int_0^{\pi/3} \sin 2x \, dx + \frac{1}{2}\int_0^{\pi/3} \sin 2x \cos 2x \, dx$

$= -\frac{1}{4}\left[\cos 2x\right]_0^{\pi/3} + \frac{1}{4}\int_0^{\pi/3} \sin 4x \, dx$

$= \left[\frac{-\cos 2x}{4}\right]_0^{\pi/3} + \left[\frac{-\cos 4x}{16}\right]_0^{\pi/3}$

$= \frac{1}{8} + \frac{1}{4} + \frac{1}{32} + \frac{1}{16}$

$= \frac{15}{32}$

(c) Alternate Solution.

$$\int_0^{\pi/3} \sin 2x \cos^2 x \, dx = 2\int_0^{\pi/3} \cos^3 x \sin x \, dx$$

$$= -2\left[\frac{\cos^4 x}{4}\right]_0^{\pi/3}$$

$$= -2\left(\frac{1}{64} - \frac{1}{4}\right) = \frac{15}{32}$$

(d)

$\angle YXZ = \theta$ (base angles of isosceles triangle XYZ)

$\angle PXA = \angle PCA$ (angles subtended at the circumference by arc PA)

$= \theta$

$\angle PDC = \theta$ (exterior angle of a cyclic quadrilateral PYBD)

$\therefore \angle PCD = \angle PDC = \theta$

$\therefore \triangle CPD$ is isosceles

3

12

Question

(a) $x^2 = 8y$, $\therefore a = 2$

(i) gradient of tangent at $P(4t, 2t^2)$ is t

equation of tangent at P: $y - 2t^2 = t(x - 4t)$

$y = tx - 2t^2$

gradient of tangent at $Q(8t, 8t^2)$ is $2t$

equation of tangent at Q: $y - 8t^2 = 2t(x - 8t)$

$y = 2tx - 8t^2$

(ii) Solving the equations of the tangents simultaneous

$tx - 2t^2 = 2tx - 8t^2$

$6t^2 = tx$

$x = 6t$

$y = t(6t) - 2t^2$

$= 4t^2$

$\therefore$ Coordinates of R are $(6t, 4t^2)$

(iii) Locus of R: $t = \frac{x}{6}$

$y = 4\left(\frac{x}{6}\right)^2$

$= \frac{4x^2}{36}$

$\therefore x^2 = 9y$

5

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{n}x^n$$

Substitute $x = 2$

$$3^n = 1 + \binom{n}{1}2 + \binom{n}{2}4 + \dots + \binom{n}{n-1}2^{n-1} + \binom{n}{n}2^n$$

Now $\binom{n}{n} = 1$

$$\therefore \quad 3^n = 1 + 2\binom{n}{1} + 4\binom{n}{2} + \dots + 2^{n-1}\binom{n}{n-1} + 2^n$$

$$3^n - 2^n = 1 + 2\binom{n}{1} + 4\binom{n}{2} + \dots + 2^{n-1}\binom{n}{n-1}$$

as required 2

(e) At midnight: $t=0$, $T = 29°C$

$t = 1$, $T = 26°C$

(i) $T = 20 + 9e^{-kt} \Rightarrow T - 20 = 9e^{-kt}$

$$\frac{dT}{dt} = -k.9e^{-kt}$$

$$= -k(T-20) \quad \text{as required.}$$

(ii) $T = 20 + 9e^{-kt}$

$t = 1, T = 26$

$$26 = 20 + 9e^{-k}$$
$$6 = 9e^{-k}$$
$$\frac{2}{3} = e^{-k}$$
$$\ln\left(\frac{2}{3}\right) = -k$$
$$\therefore k = \ln\left(\frac{3}{2}\right)$$

Question 5. (cont.)

(c) (iii) For time of death solve for t when $T = 36.8$.

$$36.8 = 20 + 9e^{-kt} \qquad \left(k = \ln\tfrac{3}{2}\right)$$

$$\frac{16.8}{9} = e^{-kt}$$

$$\ln\left(\frac{16.8}{9}\right) = -kt$$

$$t = -\frac{\ln\left(\frac{16.8}{9}\right)}{\ln\left(\frac{3}{2}\right)}$$

$$\doteqdot -1.54 \text{ hours}$$

$$\doteqdot -1 \text{ hour } 32 \text{ mins.}$$

$\therefore$ Time of death is approximately 10:28 p.m (1 hour 32 minutes before midnight) 5

12

Question 6

a) Step 1 : $n = 1$

$$\text{LHS} = \frac{1}{2!} = \frac{1}{2}$$

$$\text{RHS} = 1 - \frac{1}{2} = \frac{1}{2} = \text{LHS}$$

Hence, the result is true for $n = 1$.

Step 2 : Suppose the result is true for $n = k$.

i.e. $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \dots + \frac{k}{(k+1)!} = 1 - \frac{1}{(k+1)!}$ (*)

We need to show that the result is true for $n = k+1$

i.e. $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \dots + \frac{k}{(k+1)!} + \frac{k+1}{(k+2)!} = 1 - \frac{1}{(k+2)!}$

$$\text{LHS} = 1 - \frac{1}{(k+1)!} + \frac{k+1}{(k+2)!} \quad \text{by } (*)$$

$$= 1 - \frac{(k+2) - (k+1)}{(k+2)!}$$

$$= 1 - \frac{k+2-k-1}{(k+2)!}$$

$$= 1 - \frac{1}{(k+2)!}$$

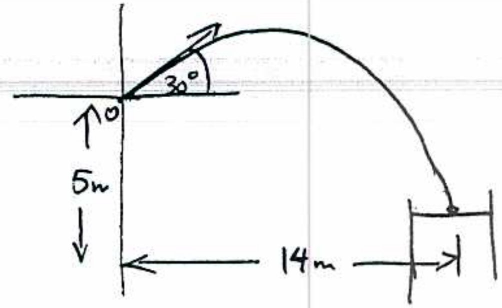
$$= \text{RHS}$$

Step 3 It follows from Step 1 and Step 2 by mathematical induction that it is true for all positive integers.

3

Question 6 (cont.)

(b) (i)



$$x = Vt\cos 30^\circ = \frac{V\sqrt{3}}{2}t$$

$$\therefore t = \frac{2x}{\sqrt{3}V}$$

So $y = -5\left(\frac{2x}{\sqrt{3}V}\right)^2 + \cancel{V}\frac{\cancel{2}x}{\sqrt{3}\cancel{V}} \cdot \frac{1}{\cancel{2}}$

$$= -\frac{20x^2}{3V^2} + \frac{x}{\sqrt{3}} \quad \text{as required.}$$

(ii) Sub. $x = 14$, $y = -5$ and solve for V

$$-5 = -\frac{20(14)^2}{3V^2} + \frac{14}{\sqrt{3}}$$

$$-15V^2 = -20 \times 196 + 14\sqrt{3}V^2$$

$$V^2(14\sqrt{3} + 15) = 20 \times 196$$

$$V^2 = \frac{3920}{14\sqrt{3}+15} \doteqdot 99.87589\dots$$

$$\therefore V \doteqdot 10 \text{ m/s}$$

i) Impact Velocity $= \sqrt{\dot{x}^2 + \dot{y}^2}$

First, calculate time of flight, $x = 14$, $v = 10$

$$t = \frac{2(14)}{\sqrt{3}(10)} = \frac{14}{5\sqrt{3}} \text{ s}$$

$$\dot{x} = \frac{10\sqrt{3}}{2} = 5\sqrt{3}$$

$$\dot{y} = -10\frac{14}{5\sqrt{3}} + 10\tfrac{1}{2} = \frac{-28}{\sqrt{3}} + 5 = \frac{-28\sqrt{3} + 15}{3}$$

$$v = \sqrt{(5\sqrt{3})^2 + \left(\frac{15 - 28\sqrt{3}}{3}\right)^2} = \sqrt{199.99\ldots} \doteqdot 14.14 \text{ m/s}$$

$\therefore$ Padding is not required

6

Question 6 (cont.)

(c) (i) $\angle OP_0P_1 = 90 - \theta$ (angle sum of $\triangle OP_0P_1$)

$\angle P_2P_1P_0 = \theta$ (angle sum of $\triangle P_2P_1P_0$)

Now $\cos\theta = \dfrac{d_2}{d_1}$

Similarly $\angle P_3P_2P_1 = \theta$

and $\cos\theta = \dfrac{d_3}{d_2}$

Geometric series $\therefore$ as $\dfrac{d_3}{d_2} = \dfrac{d_2}{d_1} = \cos\theta$

$a = d_1 = x\sin\theta$ (from $\triangle P_1OP_0$)

$r = \cos\theta$

(ii) $$\sum_{r=1}^{\infty} d_r = \frac{x\sin\theta}{1 - \cos\theta} \quad \text{since } |\cos\theta| < 1$$

$$= x\frac{2\sin\theta/2\cos\theta/2}{2\sin^2\theta/2} = x\frac{\cos\theta/2}{\sin\theta/2} = x\cot\theta/2 \quad \text{as required}$$

3

Question 7

a) (i) Join PB $\therefore$ $\angle APB = 90^\circ$ (angle in a semi-circle)

So $\dfrac{AP}{2r} = \cos\theta$

$AP = 2r\cos\theta$

(ii) Let l be arc length PQ

$$l = AP \times \theta$$
$$= 2r\cos\theta \times \theta \quad (\theta \text{ in radians})$$

$$\frac{dl}{d\theta} = 2r\cos\theta + 2r\theta(-\sin\theta)$$
$$= 2r(\cos\theta - \theta\sin\theta)$$

Stationary point when $\cos\theta - \theta\sin\theta = 0$

$\theta\sin\theta = \cos\theta$

Now $\dfrac{d^2l}{d\theta^2} < 0$ for a maximum

$$\frac{d^2l}{d\theta^2} = 2r(-\sin\theta) - 2r[\sin\theta + \theta\cos\theta]$$
$$= 2r(-2\sin\theta - \cos\theta)$$
$$= -2r(2\sin\theta + \theta\cos\theta) < 0$$

since $0 < \theta < \frac{\pi}{2}$ (θ in a semi-circle)

(iii) let $f(\theta) = \theta\sin\theta - \cos\theta$

$f'(\theta) = \theta\cos\theta + 2\sin\theta$

$\theta_0 = 1$

(a) (iii) cont. $\theta_1 = 1 - \dfrac{f(\theta_0)}{f'(\theta_0)}$

$$= 1 - \frac{1 \times \sin 1 - \cos 1}{1 \times \cos 1 + 2\sin 1}$$
$$= \frac{\cos 1 + 2\sin 1 - \sin 1 + \cos 1}{\cos 1 + 2\sin 1}$$
$$= \frac{2\cos 1 + \sin 1}{\cos 1 + 2\sin 1}$$
$$\doteqdot 0.86 \quad (2 \text{ dec. pl.}).$$

(b) (i) Using sine rule:

$$\frac{\sin\alpha}{40} = \frac{\sin\theta}{120}$$
$$\sin\alpha = \frac{\sin\theta}{3}$$
$$\alpha = \sin^{-1}\left(\frac{\sin\theta}{3}\right) \text{ as required}$$
$$0 \leq \alpha < \tfrac{\pi}{2}.$$

(ii) $\dfrac{d\alpha}{dt} = \dfrac{d\alpha}{d\theta} \cdot \dfrac{d\theta}{dt}$

Now $\dfrac{d\theta}{dt} = 6$ revs/s

$= 12\pi$ radians/sec

$$\frac{d\alpha}{d\theta} = \frac{1}{3}\cos\theta \cdot \frac{1}{\sqrt{1 - \frac{\sin^2\theta}{9}}}$$

$$\frac{d\alpha}{d\theta} = \frac{1}{3}\cos\theta \quad \frac{3}{\sqrt{9-\sin^2\theta}}$$

$$= \frac{\cos\theta}{\sqrt{9-\sin^2\theta}}$$

So $$\frac{d\alpha}{dt} = \frac{\cos\theta}{\sqrt{9-\sin^2\theta}} \cdot 12\pi$$

$$= \frac{12\pi\cos\theta}{\sqrt{9-\sin^2\theta}}$$ as required

(iii) Drop a perpendicular AX from A intersecting OP at X

$x = OX + XP$

Now $OX = 40\cos\theta$

$XP = 120\cos\alpha$

$$\therefore \quad x = 40\cos\theta + 120\sqrt{1-\sin^2\alpha}$$

$$= 40\cos\theta + 120\sqrt{1-\frac{\sin^2\theta}{9}}$$ from (i)

$$= 40\left(\cos\theta + 3\,\frac{\sqrt{9-\sin^2\theta}}{3}\right)$$

$$= 40\left(\cos\theta + \sqrt{9-\sin^2\theta}\right)$$

Alternate solution to (iii) using cosine rule

$$\cos\theta = \frac{40^2 + x^2 - 120^2}{2(40)x}$$

$$x^2 - 80x\cos\theta + (40^2-120^2) = 0$$

Using the quadratic formula

$$x = \frac{80\cos\theta \pm \sqrt{80^2\cos^2\theta - 4(40^2-120^2)}}{2}$$

$$= 40\cos\theta \pm 40\sqrt{\cos^2\theta + 8}$$

$$= 40\left(\cos\theta + \sqrt{\cos^2\theta+8}\right), \quad x > 0$$

$$= 40\left(\cos\theta + \sqrt{9-\sin^2\theta}\right)$$ as required.

(iv) $$\frac{dx}{dt} = \frac{dx}{d\theta}\cdot\frac{d\theta}{dt}$$

$$= 40\left(-\sin\theta + \frac{1}{2}\frac{1}{\sqrt{9-\sin^2\theta}} \cdot -2\sin\theta\cos\theta\right) \times 12\pi$$

$$= 480\pi\left[-\sin\theta - \frac{\cos\theta\sin\theta}{\sqrt{9-\sin^2\theta}}\right]$$

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