Sydney Grammar School
Mathematics Department
Trial Examinations 2008

FORM VI

MATHEMATICS EXTENSION 2

Examination date

Tuesday 5th August 2008

Time allowed

3 hours (plus 5 minutes reading time)

Instructions

All eight questions may be attempted.
All eight questions are of equal value.
All necessary working must be shown.
Marks may not be awarded for careless or badly arranged work.
Approved calculators and templates may be used.
A list of standard integrals is provided at the end of the examination paper.

Collection

Write your candidate number clearly on each booklet.
Hand in the eight questions in a single well-ordered pile.
Hand in a booklet for each question, even if it has not been attempted.
If you use a second booklet for a question, place it inside the first.
Keep the printed examination paper and bring it to your next Mathematics lesson.

Checklist

SGS booklets: 8 per boy. A total of 750 booklets should be sufficient.
Candidature: 74 boys.

Examiner

BDD

QUESTION ONE (15 marks) Use a separate writing booklet. **Marks**

(a) By completing the square, find **2**

$$\int \frac{1}{\sqrt{5-4x-x^2}}\,dx.$$

(b) Use integration by parts to find **2**

$$\int xe^{-x}\,dx.$$

(c) Use log laws to assist in finding **2**

$$\int \frac{\ln x^2}{x}\,dx.$$

(d) Use the substitution $u = \tan x$ to find **3**

$$\int \sec^4 x\,dx.$$

(e) Use the substitution $x = 3\cos\theta$ to find **4**

$$\int \frac{x^2}{\sqrt{9-x^2}}\,dx.$$

(f) Use a careful substitution and a symmetry argument to find **2**

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (\pi - x)^2 \sin x\,dx.$$

Exam continues next page ...

QUESTION TWO (15 marks) Use a separate writing booklet. **Marks**

(a) (i) Express $1 - i\sqrt{3}$ and $1 + i\sqrt{3}$ in modulus–argument form. **1**

(ii) Hence use de Moivre's theorem to evaluate **2**

$$(1 - i\sqrt{3})^{10} + (1 + i\sqrt{3})^{10}.$$

(b) Shade the region of the complex plane described by **3**

$$|z| < 2 \quad \text{and} \quad \text{Re}(z) \leq 1.$$

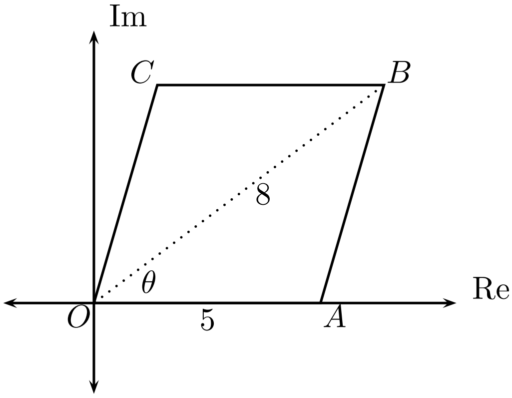
You need not find the coordinates of any points of intersection.

(c) The three complex numbers $-2 + 3i$, $3 - 2i$ and $5 + 4i$ are represented by the points A, B and C respectively in the complex plane.

(i) Show that $\triangle ABC$ is isosceles. **2**

(ii) Find the midpoint M of BC. **1**

(d)



The diagram above shows a rhombus $OABC$ in the first quadrant of the Argand diagram, with the origin O as one vertex and another vertex A lying on the real axis. The longer diagonal OB is 8 units, and each side is 5 units.

Let $\angle AOB = \theta$ and let $z = \cos\theta + i\sin\theta$.

(i) Explain why OC is represented by the complex number $5z^2$. **2**

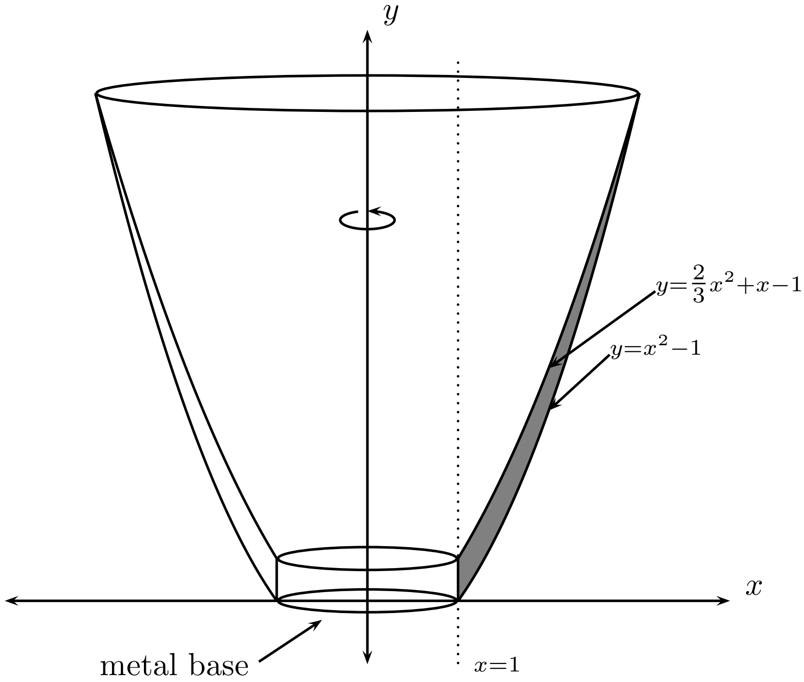
(ii) Show that z satisfies the quadratic equation $5z^2 = 8z - 5$. **1**

(iii) Solve this quadratic equation and then find the complex numbers representing the vertices B and C. **3**

Exam continues overleaf ...

QUESTION THREE (15 marks) Use a separate writing booklet. **Marks**

(a)



A glass is designed by rotating the region bounded by the curves $y = x^2 - 1$ and $y = \frac{2}{3}x^2 + x - 1$ and the line $x = 1$ about the y-axis, as in the diagram above. The resulting volume is to be grafted onto a cylindrical metal base whose volume need not be included in the following calculations.

(i) Find the point where the curves intersect in the first quadrant. **1**

(ii) Use the method of cylindrical shells to determine the volume of glass in the solid. **3**

(b) Draw a neat half-page diagram of the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{2} = 1$ showing foci, directrices and intercepts with the axes. **4**

(c) Let $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ be an ellipse with foci S and S' and eccentricity e. Carefully prove for any point $P(x_1, y_1)$ on the ellipse that $PS + PS' = 2a$. **2**

(d) Let $P(a\cos\theta, b\sin\theta)$ and $Q\big(a\cos(\theta+\alpha), b\sin(\theta+\alpha)\big)$ be two distinct points on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. Let O be the origin.

(i) Show that line OP has equation $(b\sin\theta)x - (a\cos\theta)y = 0$. **2**

(ii) Show that the perpendicular distance from Q to OP is **2**

$$\frac{ab\,\big|\sin\alpha\big|}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}}.$$

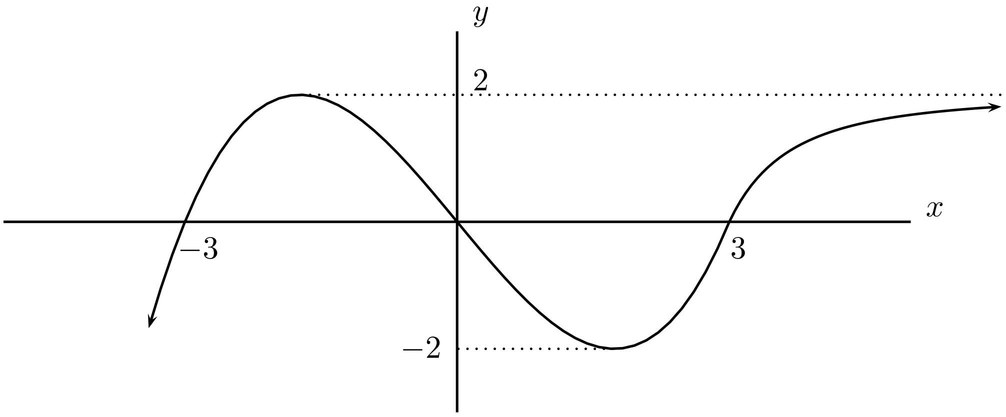
(iii) Hence show that the area of the triangle OPQ is independent of θ. **1**

Exam continues next page ...

QUESTION FOUR (15 marks) Use a separate writing booklet. **Marks**

(a) Factorise $x^5 - 1$ as the product of real linear and quadratic factors. You may leave your answer in terms of trigonometric ratios. **3**

(b)



The graph of a certain function $y = f(x)$ is sketched above. Draw neat half-page sketches of the following graphs.

(i) $y = \dfrac{1}{f(x)}$ **2**

(ii) $y = e^{f(x)}$ **3**

(iii) $y^2 = f(x)$ **3**

(c) The line $y = 3x + 4$ is tangent to the cubic $y = 9x^3 - 48x^2 + 55x - 12$ at A, and intersects the cubic again at B.

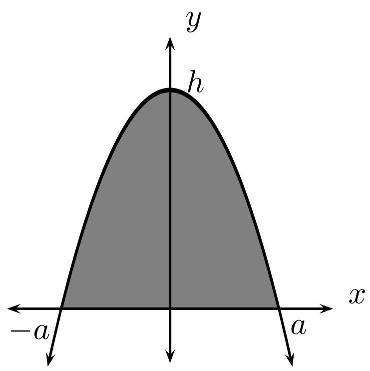
(i) Show that the x-coordinates of the points A and B are the roots of the cubic **1**

$$9x^3 - 48x^2 + 52x - 16 = 0.$$

(ii) Explain briefly why the cubic equation in part (i) must have a double root, and use this fact to find the x-coordinates of the points A and B. **3**

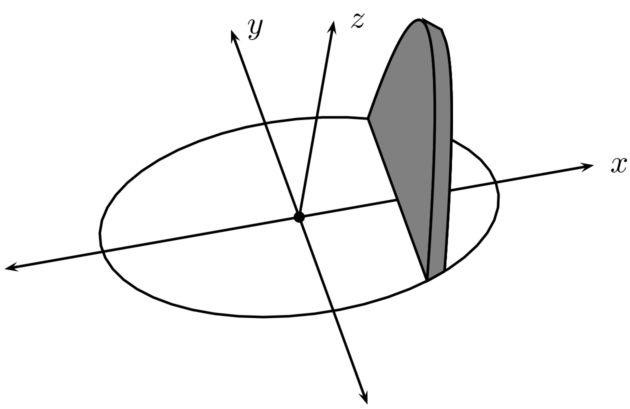
QUESTION FIVE (15 marks) Use a separate writing booklet. **Marks**

(a) **3**



The diagram above shows a parabola with vertex $(0, h)$ and zeroes $x = a$ and $x = -a$. Show that the shaded area is $\frac{4}{3}ha$ square units.

(b)



A certain dome tent is designed with an elliptical base $\dfrac{x^2}{4} + \dfrac{y^2}{1} = 1$, as in the diagram above. Cross-sections perpendicular to the base are segments of a parabola. The height of each parabolic segment is equal to the width of its base.

(i) Use your result in part (a) to show that a typical cross-section has area **1**

$$\frac{8 - 2x^2}{3}.$$

(ii) Show that the volume of the tent is $\dfrac{64}{9}$ cubic units. **2**

Exam continues next page ...

QUESTION FIVE (Continued)

(c) (i) Expand $(\cos\theta + i\sin\theta)^7$ using the binomial theorem. **1**

(ii) Use de Moivre's theorem to establish the identity **2**

$$\tan 7\theta = \frac{7\tan\theta - 35\tan^3\theta + 21\tan^5\theta - \tan^7\theta}{1 - 21\tan^2\theta + 35\tan^4\theta - 7\tan^6\theta}.$$

(iii) Use your result from part (ii) to solve the polynomial equation **2**

$$x^6 - 21x^4 + 35x^2 - 7 = 0.$$

(iv) Use product-of-roots to determine the value of **2**

$$\tan\tfrac{\pi}{7}\,\tan\tfrac{2\pi}{7}\,\tan\tfrac{3\pi}{7}.$$

(v) Use a careful argument involving roots to determine the value of **2**

$$\cot^2\tfrac{\pi}{7} + \cot^2\tfrac{2\pi}{7} + \cot^2\tfrac{3\pi}{7}.$$

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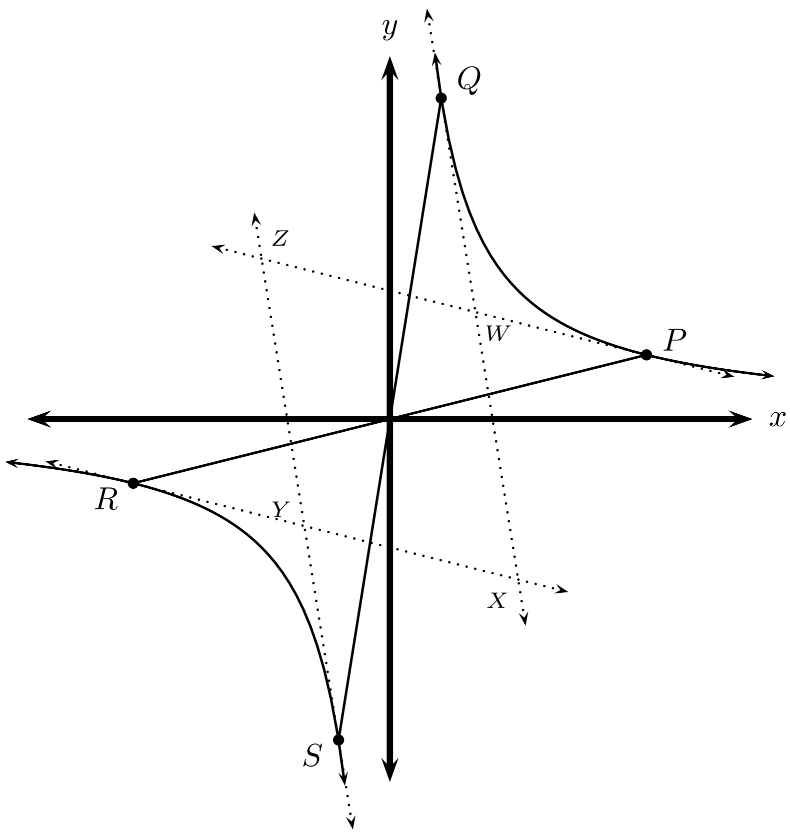
QUESTION SIX (15 marks) Use a separate writing booklet. **Marks**

(a)



Let $P\left(ct_1, \dfrac{c}{t_1}\right)$ and $Q\left(ct_2, \dfrac{c}{t_2}\right)$ be any two points on the rectangular hyperbola $xy = c^2$ with $0 < t_2 < t_1$. Let O be the origin. Let $R\left(-ct_1, -\dfrac{c}{t_1}\right)$ and $S\left(-ct_2, -\dfrac{c}{t_2}\right)$ be the points diametrically opposed to P and Q respectively. (that is, POR and QOS are straight).

(i) Prove that P, Q, R and S are the vertices of a parallelogram. **2**

(ii) Show that the tangent at P has equation $x + {t_1}^2 y = 2ct_1$, and then write down the equation of the tangent at Q. **2**

(iii) Show that the point of intersection W of the tangents at P and at Q has coordinates $\left(\dfrac{2ct_1t_2}{t_1+t_2}, \dfrac{2c}{t_1+t_2}\right)$. **2**

(iv) Let the tangents at Q and R meet at X, the tangents at R and S meet at Y, and the tangents at S and P meet at Z. Write down, without working, the coordinates of X, Y and Z. **1**

(v) Show that $WXYZ$ is also a parallelogram. **1**

Exam continues next page ...

QUESTION SIX (Continued)

(b) Let $I_n = \displaystyle\int_1^2 \frac{(x-1)^{\frac{n}{2}}}{x}\,dx$, where n is any integer.

(i) Show that $I_{-1} = \dfrac{\pi}{2}$. **3**

(ii) Show that $I_n = \frac{2}{n} - I_{n-2}$, for $n \neq 0$. **3**

(HINT: Integration by parts is not required.)

(iii) Hence find I_5. **1**

QUESTION SEVEN (15 marks) Use a separate writing booklet. **Marks**

(a) A particle of mass m kilograms is launched vertically upwards in a highly resistive medium at a velocity 5 m/s. It is subject to the force of gravity and to a resistance due to motion of magnitude $\dfrac{mv^3}{100}$.

Take $g = 10\,\text{m/s}^2$ and upwards as positive.

The equation of motion is $\ddot{x} = -g - \dfrac{v^3}{100}$.

(i) Show the height x above the point of launch and the velocity v are related by **1**

$$\frac{dv}{dx} = \frac{v^3 + 1000}{-100v}.$$

(ii) Find constants A, B and C such that **3**

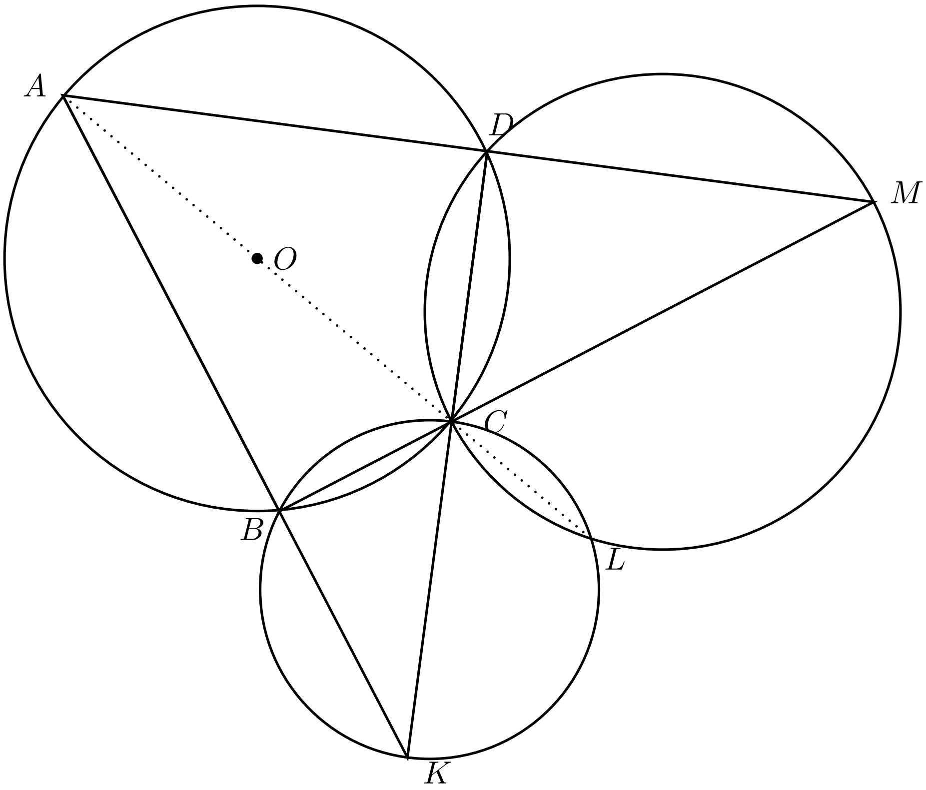
$$\frac{100v}{v^3 + 1000} = \frac{A}{v+10} + \frac{Bv + C}{v^2 - 10v + 100}.$$

(iii) Hence find the maximum height reached by the particle, giving your answer correct to the nearest centimetre. **3**

Exam continues overleaf ...

QUESTION SEVEN (Continued)

(b)



In the diagram above, the points A, B, C and D lie on a circle with centre O. The intervals AB and DC produced meet at K, and the intervals AD and BC produced meet at M. The circles BCK and DCM meet again at L.

Assume additionally that A, O, C and L are collinear.

Copy or trace the diagram into your answer book.

(i) Prove that the centre F of the circle DCM lies on CM. **1**

(ii) Prove that K, L and M are collinear. **2**

(iii) Prove that a circle may be drawn through D, B, K and M. Label the centre of this circle X and describe its location. **2**

(iv) Prove that there is a circle passing through B, X, F and D. **3**

HINT: Let $\angle DMB = \alpha$.

(The circle also passes through O, L and the centre of circle BCK, but you need not prove these facts and should not assume them.)

QUESTION EIGHT (15 marks) Use a separate writing booklet. **Marks**

(a) (i) Show that the polynomial **1**

$$P(z) = 7z^3 + (3i - 48)z^2 + 99z - 64 - i.$$

may be written as a difference of cubes:

$$P(z) = 8(z-2)^3 - (z-i)^3.$$

(ii) Hence solve the polynomial equation $P(z) = 0$. **4**

You may wish to make use of the cube roots of unity 1, ω and ω^2 in your solution, where $\omega = -\dfrac{1}{2} + i\dfrac{\sqrt{3}}{2}$.

Express your solutions in the form $a + ib$, where a and b are real.

(b) Let x be any positive real number.

Define recursively two linked sequences $a_0, a_1, a_2, \ldots$ and $b_0, b_1, b_2, \ldots$ by

$$a_0 = (1+x^2)^{-\frac{1}{2}}, \qquad b_0 = 1$$

$$a_{i+1} = \tfrac{1}{2}(a_i + b_i), \text{ for all } i \geq 0 \qquad b_{i+1} = \sqrt{a_{i+1}b_i}, \text{ for all } i \geq 0.$$

(i) Let $\theta = \tan^{-1} x$, so that $x = \tan\theta$. Prove by induction that for all $n \geq 0$, **4**

$$\frac{a_n}{b_n} = \cos\tfrac{\theta}{2^n} \qquad \text{and} \qquad \frac{\sin\theta}{b_n} = 2^n \sin\tfrac{\theta}{2^n}.$$

(ii) Prove that $\dfrac{\sin\theta}{b_n} \to \theta$ as $n \to \infty$, and write down $\displaystyle\lim_{n\to\infty} \frac{x}{b_n\sqrt{1+x^2}}$. **2**

(iii) Let $x = 1$, and evaluate three iterations to find a_3 and b_3. **1**

You should record each answer correct to four decimal places.

(iv) Hence estimate $\frac{\pi}{4}$. **1**

(v) With $x = 1$, find algebraically how many iterations need to be taken before the ratio $\dfrac{b_{n+1}}{b_n}$ differs from 1 by less than 10^{-10}. **2**

END OF EXAMINATION

The following list of standard integrals may be used:

$$\int x^n \, dx = \frac{1}{n+1} \, x^{n+1}, \;\; n \neq -1; \;\; x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \;\; x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} \, e^{ax}, \;\; a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \;\; a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \;\; a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \;\; a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \;\; a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \;\; a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \;\; a > 0, \;\; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \;\; x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

$$\text{NOTE}: \;\; \ln x = \log_e x, \;\; x > 0$$

QUESTION ONE (15 marks)

Marks

(a) [2]

$$\int \frac{1}{\sqrt{5-4x-x^2}}\,dx = \int \frac{1}{\sqrt{9-(x+2)^2}}\,dx$$
$$= \sin^{-1}\frac{x+2}{3} + C$$

(b) [2]

$$\int xe^{-x}\,dx = x(-e^{-x}) - \int 1(-e^{-x})\,dx$$
$$= -xe^{-x} - e^{-x} + C$$

(c) [2]

$$\int \frac{\ln x^2}{x}\,dx = \int \frac{2\ln x}{x}\,dx$$
$$= \int 2u\,du \quad (\text{where } u = \ln x)$$
$$= u^2 + C$$
$$= (\ln x)^2 + C$$

(d) [3]

$$\int \sec^4 x\,dx = \int \sec^2 x \times \sec^2 x\,du$$
$$= \int (1+u^2)\,du$$
$$= u + \tfrac{1}{3}u^3 + C$$
$$= \tan x + \tfrac{1}{3}\tan^3 x + C$$

Let $u = \tan x$
Then $du = \sec^2 x\,dx$
$\sec^2 x = 1 + \tan^2 x$
$= 1 + u^2$

(e) [4]

$$\int \frac{x^2}{\sqrt{9-x^2}}\,dx = \int \frac{9\cos^2\theta}{3\sin\theta} \times (-3\sin\theta)\,d\theta$$
$$= -\int 9\cos^2\theta\,d\theta$$
$$= \int -\tfrac{9}{2}(1+\cos 2\theta)\,d\theta$$
$$= -\tfrac{9}{2}\theta - \frac{9}{4}\sin 2\theta + C$$
$$= -\tfrac{9}{2}\theta - \frac{9}{2}\cos\theta\sin\theta + C$$
$$= -\tfrac{9}{2}\theta - \frac{1}{2} \times 3\cos\theta \times 3\sin\theta + C$$
$$= -\tfrac{9}{2}\cos^{-1}\frac{x}{3} - \tfrac{1}{2}x\sqrt{9-x^2} + C$$

Let $x = 3\cos\theta$
Then $dx = -3\sin\theta\,d\theta$
$x^2 = 9\cos^2\theta$
$9 - x^2 = 9 - 9\cos^2\theta$
$= 9\sin^2\theta$
$\sqrt{9-x^2} = 3\sin\theta$

(f) Let $u = 2\pi - x$. Then [2]

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (\pi - x)^2 \sin x\,dx = \int_{\frac{3\pi}{2}}^{\frac{\pi}{2}} (-\pi + u)^2 \sin(2\pi - u)\,(-du)$$
$$= -\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (u-\pi)^2 \sin u\,du$$
$$= -\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (\pi - x)^2 \sin x\,dx \quad (\text{relabelling})$$

Hence the value of the integral is 0. (The substitution $u = \pi - x$ and a standard result about odd functions would also work here.)

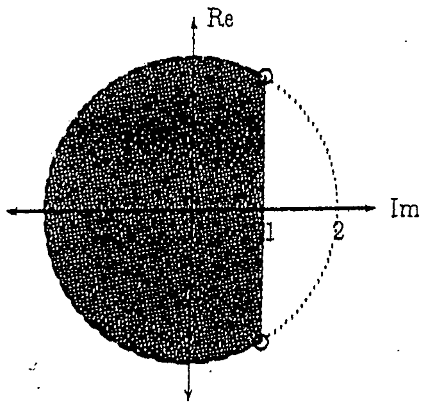
QUESTION TWO (15 marks) Marks

(a) (i) $1 + i\sqrt{3} = 2\operatorname{cis}\frac{\pi}{3}$ and $1 - i\sqrt{3} = 2\operatorname{cis}\frac{-\pi}{3}$ [1]

(ii) [2]

$$\begin{aligned}\text{Hence}(1 - i\sqrt{3})^{10} + (1 + i\sqrt{3})^{10} &= 2^{10}\operatorname{cis}\frac{-10\pi}{3} + 2^{10}\operatorname{cis}\frac{10\pi}{3}\\ &= 2^{10} \times 2\cos\frac{10\pi}{3}\\ &= 2^{11}\cos\frac{4\pi}{3}\\ &= -2^{10}\\ &= -1024\end{aligned}$$

(b) [3]



(c) (i) Let $a = -2 + 3i$, $b = 3 - 2i$ and $c = 5 + 4i$. Then: [2]

$$\begin{aligned}|a - b| &= |-5 + 5i| &&= \sqrt{50}\\ |a - c| &= |-7 - i| &&= \sqrt{50}\\ |b - c| &= |-2 - 6i| &&= \sqrt{40}\end{aligned}$$

The triangle is isosceles, since $AB = AC = \sqrt{50}$.

(ii) The midpoint M of BC is represented by the complex number [1]

$$\begin{aligned}b + \tfrac{1}{2}(c - b) &= \tfrac{1}{2}(b + c)\\ &= 4 + i\end{aligned}$$

(d) (i) Since the diagonals of a rhombus bisect its vertex angles, $\angle BOC = \theta$. But then $\arg(OC) = 2\theta$ and $|OC| = 5$, so $OC = 5\operatorname{cis}2\theta = 5z^2$, by de Moivre's theorem. [2]

(ii) From part (i), $OC = 5z^2$. But [1]

$$\begin{aligned}OC &= OB + BC\\ &= 8\operatorname{cis}\theta - 5\\ &= 8z - 5\end{aligned}$$

Hence $z^2 = 8z - 5$.

(iii) We shall use the quadratic formula to solve $z^2 - 8z + 5 = 0$. [3]

The discriminant is $b^2 - 4ac = -36$ and so

$$z = \frac{8+6i}{10} \quad \text{or} \quad z = \frac{8-6i}{10}$$
$$= \frac{4+3i}{5} \qquad\qquad = \frac{4-3i}{5}$$

But notice that z lies in quadrant 1, so $z = \dfrac{4+3i}{5}$.

Hence $OB = 8z = \dfrac{32+24i}{5}$ and $OC = 8z - 5 = \dfrac{7+24i}{5}$.

QUESTION THREE (15 marks)

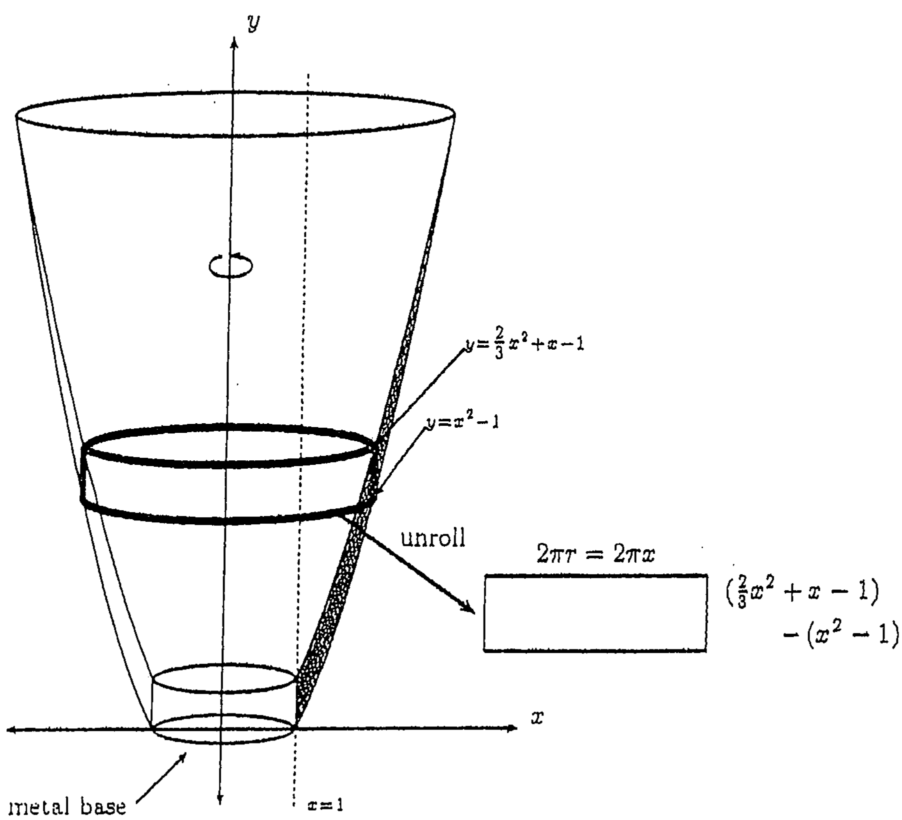
Marks

(a) (i) When they intersect, [1]

$$\begin{aligned} x^2 - 1 &= \tfrac{2}{3}x^2 + x - 1 \\ 3x^2 - 3 &= 2x^2 + 3x - 3 \\ x^2 - 3x &= 0 \\ x(x-3) &= 0 \\ x &= 3 \quad (x > 1) \end{aligned}$$

So they intersect at $(3, 8)$

(ii) [3]

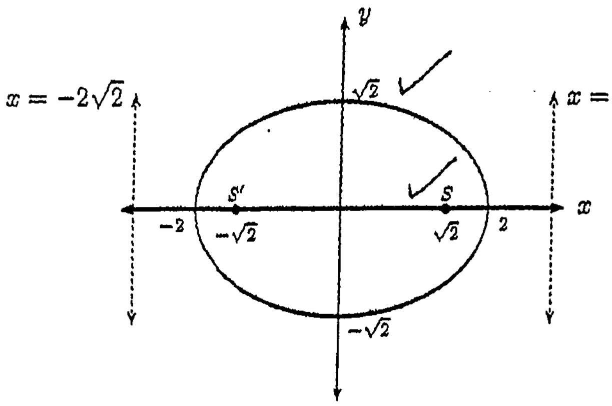


The volume of a cylindrical slice is $\delta V = 2\pi x \times \left(-\frac{1}{3}x^2 + x\right)\delta x$. Hence the total volume required is

$$\begin{aligned} V &= \lim_{\delta x \to 0} \sum_{x=1}^{3} 2\pi x\left(-\tfrac{1}{3}x^2 + x\right)\delta x \\ &= \frac{2\pi}{3}\int_1^3 \left(-x^3 + 3x^2\right)dx \\ &= \frac{2\pi}{3}\left[-\tfrac{1}{4}x^4 + x^3\right]_1^3 \\ &= \frac{2\pi}{3}\left(\frac{27}{4} - \frac{3}{4}\right) \\ &= 4\pi \text{ cubic units} \end{aligned}$$

✓ method.

(b) **4**



(c) Let ℓ and ℓ' be the directrices corresponding to the focus S and S' respectively and let e be the eccentricity. Then using the eccentricity definition of a conic, **2**

$$\begin{aligned} PS + PS' &= eP\ell + eP\ell' \quad \text{(Using the notation } P\ell \text{ for the distance from } P \text{ to } \ell \text{ etc)} \\ &= e(P\ell + P\ell') \\ &= e \times \text{distance between directrices} \\ &= e \times 2 \times \frac{a}{e} \\ &= 2a \end{aligned}$$

(d) (i) **2**

$$\begin{aligned} \text{gradient } OP &= \frac{\text{rise}}{\text{run}} \\ &= \frac{b\sin\theta - 0}{a\cos\theta - 0} \\ &= \frac{b\sin\theta}{a\cos\theta} \end{aligned}$$

Using the point-gradient form $y - y_1 = m(x - x_1)$ gives

$$y - 0 = \frac{b\sin\theta}{a\cos\theta}(x - 0)$$

$$a\cos\theta\, y = b\sin\theta\, x$$

$$b\sin\theta\, x - a\cos\theta\, y = 0$$

(ii) Using the perpendicular distance formula, the perpendicular distance from OP to Q is **2**

$$\frac{\left|b\sin\theta\left(a\cos(\theta+\alpha)\right)-a\cos\theta\left(b\sin(\theta+\alpha)\right)\right|}{\sqrt{(a\cos\theta)^2+(b\sin\theta)^2}}$$

$$=\frac{ab\left|\sin\theta\cos(\theta+\alpha)-\cos\theta\sin(\theta+\alpha)\right|}{\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}}$$

$$=\frac{ab\left|\sin(\theta-(\theta+\alpha))\right|}{\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}}$$

$$=\frac{ab\left|\sin\alpha\right|}{\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}}$$

(iii) The area of $\triangle OPQ$ is **1**

$$\frac{1}{2}\frac{ab|\sin\alpha|}{\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}}\times OP$$

$$=\frac{1}{2}\times\frac{ab|\sin\alpha|}{\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}}\times\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}$$

$$=\tfrac{1}{2}ab|\sin\alpha|$$

which is independent of θ.

QUESTION FOUR (15 marks)

Marks

(a) First we need to solve $x^5=1$. Let $x=\text{cis}\,\theta$ and then by de Moivre's theorem **3**

$$\text{cis}\,5\theta=\text{cis}\,2k\pi \qquad \text{for } k\in\mathbb{Z}$$

Hence $5\theta=2k\pi$

$\theta=\frac{2}{5}k\pi$

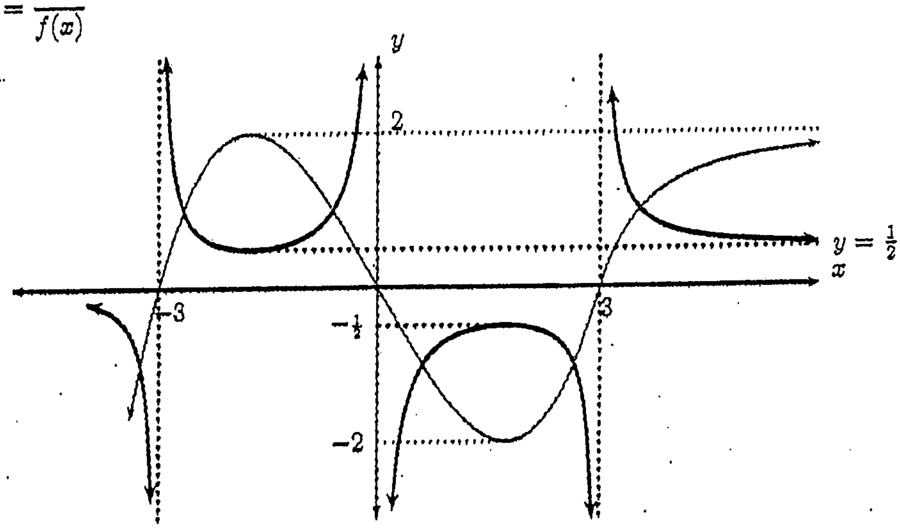
As our five roots we shall take $\text{cis}\,0=1$ and $\text{cis}\,\frac{2\pi}{5},\text{cis}(-\frac{2\pi}{5})$ and $\text{cis}\,\frac{4\pi}{5},\text{cis}(-\frac{4\pi}{5})$ (arranged with conjugates paired). We may factor x^5-1 over the complex numbers;

$$(x-1)\left(x-\text{cis}\,\tfrac{2\pi}{5}\right)\left(x-\text{cis}(-\tfrac{2\pi}{5})\right)\left(x-\text{cis}\,\tfrac{4\pi}{5}\right)\left(x-\text{cis}(-\tfrac{4\pi}{5})\right)$$

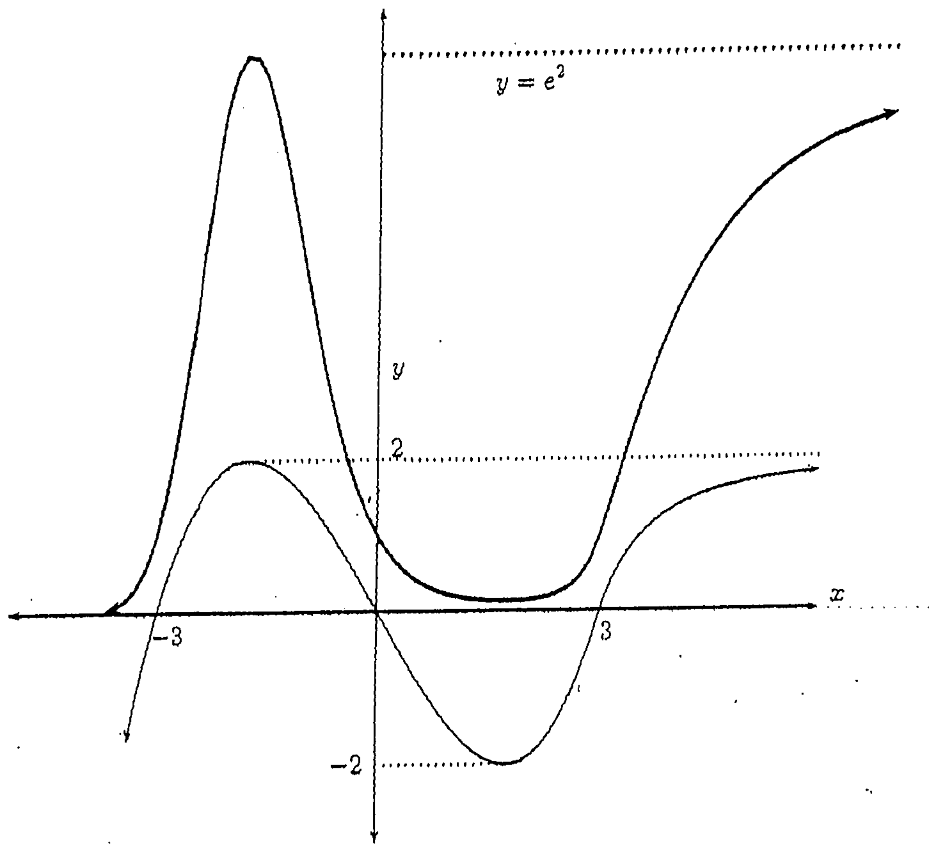
multiplying out the conjugate pairs gives;

$$(x-1)\left(x^2-2x\cos\tfrac{2\pi}{5}+1\right)\left(x^2-2x\cos\tfrac{4\pi}{5}+1\right)$$

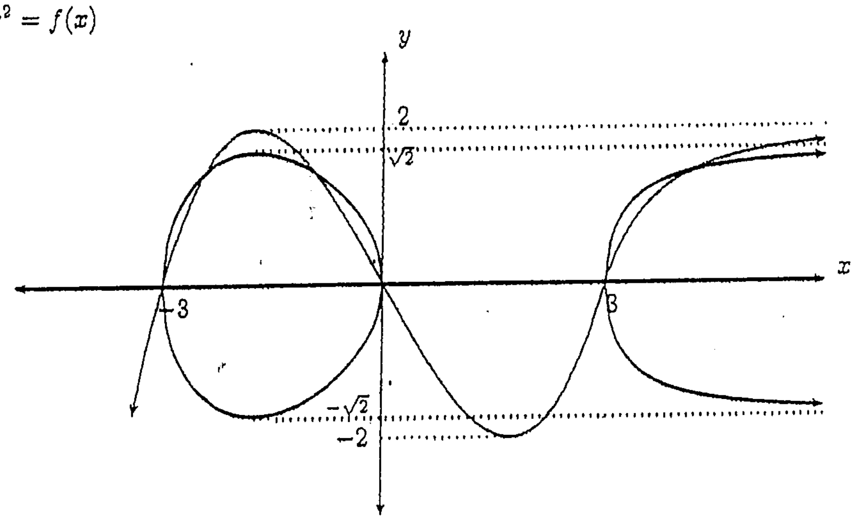
(b) (i) $y=\dfrac{1}{f(x)}$ **2**



(ii) $y = e^{f(x)}$ [3]



(iii) $y^2 = f(x)$ [3]



(c)

(i) Solving these equations simultaneously to find the point of intersection gives; [1]

$$9x^3 - 48x^2 + 55x - 12 = 3x + 4.$$

And rearranging gives the required equation;

$$P(x) = 9x^3 - 48x^2 + 52x - 16 = 0.$$

(ii) At A the curves are tangential, which means that the equation will have a double root. This must be a single root of $P'(x) = 27x^2 - 96x + 52$. This has roots **3**

$$x = \frac{96 \pm \sqrt{3600}}{54}$$
$$= \frac{26}{9} \quad \text{or} \quad \frac{2}{3}$$

Checking these roots in $P(x)$ we find only $x = \frac{2}{3}$ is a root of both the original polynomial *and* its derivative, so is the required x-coordinate of A. Since the roots of $P(x)$ multiply to $\frac{16}{9}$, the other root is

$$\frac{16}{9} \div \left(\frac{2}{3}\right)^2 = 4$$

QUESTION FIVE (15 marks) **Marks**

(a) The equation of this parabola is $y = -\frac{h}{a^2}(x^2 - a^2)$. **3**

$$\begin{aligned}\text{Area} &= \frac{2h}{a^2}\int_0^a (a^2 - x^2)\,dx \\ &= \frac{2h}{a^2}\left[a^2x - \tfrac{1}{3}x^3\right]_0^a \\ &= \frac{2h}{a^2}\left(a^3 - \tfrac{1}{3}a^3\right). \\ &= \frac{2h}{a^2} \times \frac{a^3}{3} \\ &= \frac{4}{3}ha\end{aligned}$$

(b) (i) A slice taken at x has base running from $y = -\sqrt{1 - \frac{x^2}{4}}$ to $y = +\sqrt{1 - \frac{x^2}{4}}$. We can use part (a) with $a = \sqrt{1 - \frac{x^2}{4}}$ and $h = 2a$. Thus the area of a slice is **1**

$$\begin{aligned}\frac{4}{3}ha &= \frac{4}{3} \times 2\sqrt{1 - \tfrac{x^2}{4}} \times \sqrt{1 - \tfrac{x^2}{4}} \\ &= \frac{8}{3}\left(1 - \tfrac{x^2}{4}\right) \\ &= \frac{8 - 2x^2}{3}\end{aligned}$$

(ii) The volume required is **2**

$$\begin{aligned}V &= \lim_{\delta x \to 0} 2 \times \sum_{x=0}^{2} \frac{8 - 2x^2}{3}\,\delta x \\ &= 2\int_0^2 \frac{8 - 2x^2}{3}\,dx \\ &= \frac{2}{3}\left[8x - \frac{2}{3}x^3\right]_0^2 \\ &= \frac{2}{3}\left(16 - \frac{16}{3}\right) \\ &= \frac{64}{9}\end{aligned}$$

(c) (i) Using the binomial expansion on $(\cos\theta + i\sin\theta)^7$ gives; 1

$$\cos^7\theta + 7\cos^6\theta\, i\sin\theta + 21\cos^5\theta\, i^2\sin^2\theta + 35\cos^4\theta\, i^3\sin^3\theta$$
$$+ 35\cos^3\theta\, i^4\sin^4\theta + 21\cos^2\theta\, i^5\sin^5\theta + 7\cos^1\theta\, i^6\sin^6\theta + i^7\sin^7\theta$$

(ii) Simplifying our result from part (i); 2

$$\cos^7\theta - 21\cos^5\theta\sin^2\theta + 35\cos^3\theta\, i^4\sin^4\theta - 7\cos^1\theta\sin^6\theta$$
$$+ i(7\cos^6\theta\sin\theta - 35\cos^4\theta\sin^3\theta + 21\cos^2\theta\sin^5\theta - \sin^7\theta)$$

By de Moivre's theorem this is $\cos 7\theta + i\sin 7\theta$, hence

$$\tan 7\theta = \frac{\sin 7\theta}{\cos 7\theta}$$
$$= \frac{7\cos^6\theta\sin\theta - 35\cos^4\theta\sin^3\theta + 21\cos^2\theta\sin^5\theta - \sin^7\theta}{\cos^7\theta - 21\cos^5\theta\sin^2\theta + 35\cos^3\theta\, i^4\sin^4\theta - 7\cos^1\theta\sin^6\theta}$$

If we now divide top and bottom by $\cos^7\theta$, we get the required expression;

$$\tan 7\theta = \frac{7\tan\theta - 35\tan^3\theta + 21\tan^5\theta - \tan^7\theta}{1 - 21\tan^2\theta + 35\tan^4\theta - 7\tan^6\theta}$$

(iii) If we let $x = \tan\theta$ then 2

$$\tan 7\theta = \frac{7x - 35x^3\theta + 21x^5 - x^7}{1 - 21x^2 + 35x^4 - 7x^6}$$

Thus the solutions of $\tan 7\theta = 0$ are the solutions of $7x - 35x^3\theta + 21x^5 - x^7 = 0$. This seventh degree polynomial factorises as $-x(x^6 - 21x^4 + 35x^2 - 7) = 0$. One solution is the trivial solution $x = 0$ and other other solutions will be the non-trivial solutions of $\tan 7\theta = 0$.

The solutions of $\tan 7\theta = 0$ are

$$7\theta = k\pi, \qquad k \in \mathbb{Z}$$

seven distinct solutions are $\theta = 0,\ \frac{\pi}{7}, -\frac{\pi}{7}, \frac{2\pi}{7}, -\frac{2\pi}{7}, \frac{3\pi}{7}, -\frac{3\pi}{7}$. Thus the required roots of the degree six polynomials are

$$\pm\tan\frac{\pi}{7}, \qquad \pm\tan\frac{2\pi}{7}, \qquad \pm\tan\frac{3\pi}{7}$$

(iv) The product of roots for the degree six polynomial in part (iii) is 2

$$-\tan^2\frac{\pi}{7} \times -\tan^2\frac{2\pi}{7} \times -\tan^2\frac{3\pi}{7} = -\frac{7}{1}$$

Hence $\tan\frac{\pi}{7}\tan\frac{2\pi}{7}\tan\frac{3\pi}{7} = \sqrt{7}$ (we take the positive root since all the angles are first quadrant).

(v) One way to tackle this problem is to notice that the cubic in x^2 2

$$(x^2)^3 - 21(x^2)^2 + 35(x^2) - 7 = 0$$

has roots $\tan^2\frac{\pi}{7}$, $\tan^2\frac{2\pi}{7}$ and $\tan^2\frac{3\pi}{7}$. If we label these roots α, β and γ then the required expression is

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$$
$$= \frac{35}{+7}$$
$$= 5$$

Alternative method: The polynomial with reciprocal roots is

$$\left(\tfrac{1}{x}\right)^6 - 21\left(\tfrac{1}{x}\right)^4 + 35\left(\tfrac{1}{x}\right)^2 - 7 = 0$$

i.e. $1 - 21x^2 + 35x^4 - 7x^6 = 0$. Regarding this as a cubic in x^2, the required expression is the sum of the roots i.e. $-\frac{35}{-7} = 5$, as before.

QUESTION SIX (15 marks) Marks

(a) (i) It is enough to show that the diagonals bisect, i.e that $PO = RO$ and $QO = SO$. **2**

$$RO^2 = (-ct_1)^2 + \left(-\frac{c}{t_1}\right)^2 \qquad QO^2 = (-ct_2)^2 + \left(-\frac{c}{t_2}\right)^2$$

$$= (ct_1)^2 + \left(\frac{c}{t_1}\right)^2 \qquad = (ct_2)^2 + \left(\frac{c}{t_2}\right)^2$$

$$= PO^2 \qquad = SO^2$$

Hence it is a parallelogram.

Alternative Method: Show that the midpoint of both PR and QS is O.

Alternative Method: Consider gradients.

(ii) **2**

$$\text{The gradient at } P = \frac{\dot{y}}{\dot{x}} = \frac{-\frac{c}{t_1^2}}{c} = -\frac{1}{t_1^2}$$

Hence

$$y - y_1 = m(x - x_1)$$

$$y - \frac{c}{t_1} = -\frac{1}{t_1^2}(x - ct_1)$$

$$t_1^2 y - ct_1 = -x + ct_1$$

$$x + t_1^2 y = 2ct_1$$

Hence we have the tangents:

at P: $x + t_1^2 y = 2ct_1$ .....(1)

at Q: $x + t_2^2 y = 2ct_2$ .....(2)

(iii) We shall solve (1) and (2) simultaneously. **2**

(1) − (2) gives

$$(t_1^2 - t_2^2)y = 2c(t_1 - t_2)$$

$$(t_1 + t_2)y = 2c$$

$$y = \frac{2c}{t_1 + t_2}$$

$t_2^2(1) - t_1^2(2)$ gives

$$(t_2^2 - t_1^2)x = 2c(t_1 t_2^2 - t_1^2 t_2)$$

$$(t_2^2 - t_1^2)x = 2ct_1 t_2(t_2 - t_1)$$

$$(t_1 + t_2)x = 2ct_1 t_2$$

$$x = \frac{2ct_1 t_2}{t_1 + t_2}$$

(iv) The coordinates of W, X, Y and Z are: **1**

$$W\left(\frac{2ct_1t_2}{t_1+t_2}, \frac{2c}{t_1+t_2}\right) \qquad X\left(\frac{-2ct_1t_2}{-t_1+t_2}, \frac{2c}{-t_1+t_2}\right)$$

$$Y\left(\frac{2ct_1t_2}{-t_1-t_2}, \frac{2c}{-t_1-t_2}\right) \qquad Z\left(\frac{-2ct_1t_2}{t_1-t_2}, \frac{2c}{t_1-t_2}\right)$$

These expressions may be simplified to:

$$W\left(\frac{2ct_1t_2}{t_1+t_2}, \frac{2c}{t_1+t_2}\right) \qquad X\left(\frac{2ct_1t_2}{t_1-t_2}, \frac{-2c}{t_1-t_2}\right)$$

$$Y\left(\frac{-2ct_1t_2}{t_1+t_2}, \frac{-2c}{t_1+t_2}\right) \qquad Z\left(\frac{-2ct_1t_2}{t_1-t_2}, \frac{2c}{t_1-t_2}\right)$$

(v) It is evident from the expressions above that the midpoint of both WY and XZ is the origin O. Hence the diagonals bisect each other and the quadrilateral is a parallelogram. [1]

(b) (i) [3]

$$
\begin{aligned}
I_{-1} &= \int_1^2 \frac{(x-1)^{-\frac{1}{2}}}{x}\,dx \\
&= \int_1^2 \frac{dx}{x\sqrt{x-1}} \\
&= \int_0^1 \frac{2u\,du}{(u^2+1)u} \\
&= \Big[\tan^{-1} u\Big]_0^1 \\
&= \frac{\pi}{2}
\end{aligned}
$$

Let $u^2 = x - 1$
Then $2u\,du = dx$
and $x = u^2 + 1$
and $\sqrt{x-1} = u$

(ii) [3]

$$
\begin{aligned}
I_n &= \int_1^2 \frac{(x-1)(x-1)^{\frac{n}{2}-1}}{x}\,dx \\
&= \int_1^2 \frac{x(x-1)^{\frac{n}{2}-1}}{x}\,dx - \int_1^2 \frac{(x-1)^{\frac{n}{2}-1}}{x}\,dx \\
&= \left[\frac{2}{n}(x-1)^{\frac{n}{2}}\right]_1^2 - I_{n-2} \\
&= \frac{2}{n} - I_{n-2}
\end{aligned}
$$

(iii) [1]

$$
\begin{aligned}
I_5 &= \frac{2}{5} - I_3 \\
&= \frac{2}{5} - \left(\frac{2}{3} - I_1\right) \\
&= \frac{2}{5} - \frac{2}{3} + I_1 \\
&= \frac{2}{5} - \frac{2}{3} + \frac{2}{1} - I_{-1} \\
&= \frac{26}{15} - \frac{\pi}{2}
\end{aligned}
$$

Marks

QUESTION SEVEN (15 marks)

1

(a) (i)

$$m\ddot{x} = -mg - \frac{mv^3}{100}$$
$$v\frac{dv}{dx} = -g - \frac{v^3}{100}$$
$$v\frac{dv}{dx} = \frac{-100g - v^3}{100}$$
$$\frac{dv}{dx} = \frac{100g + v^3}{-100v}$$

3

(ii)

$$\frac{100v}{v^3 + 1000} = \frac{A}{v+10} + \frac{Bv + C}{v^2 - 10v + 100}$$
$$100v = A(v^2 - 10v + 100) + (Bv + C)(v + 10)$$

Let $v = -10$: it follows that $-1000 = 300A + 0$ so $A = -\frac{10}{3}$.

Let $v = 0$: it follows that
$$0 = 100A + 10C$$
$$= -\frac{1000}{3} + 10C$$

Hence $C = \frac{100}{3}$.

Finally, equating coefficients of v^2 gives $0 = A + B$, hence $B = -A = \frac{10}{3}$.

3

(iii)

$$x = \frac{10}{3}\int \frac{1}{v+10}\,dv - \frac{10}{3}\int \frac{v+10}{v^2 - 10v + 100}\,dv$$
$$= \frac{10}{3}\ln|v+10| - \frac{10}{6}\int \frac{2v - 10}{v^2 - 10v + 100}\,dv - 50\int \frac{dv}{v^2 - 10v + 100}$$
$$= \frac{10}{3}\ln|v+10| - \frac{10}{6}\ln|v^2 - 10v + 100| - \frac{50}{5\sqrt{3}}\tan^{-1}\frac{v-5}{5\sqrt{3}} + C$$

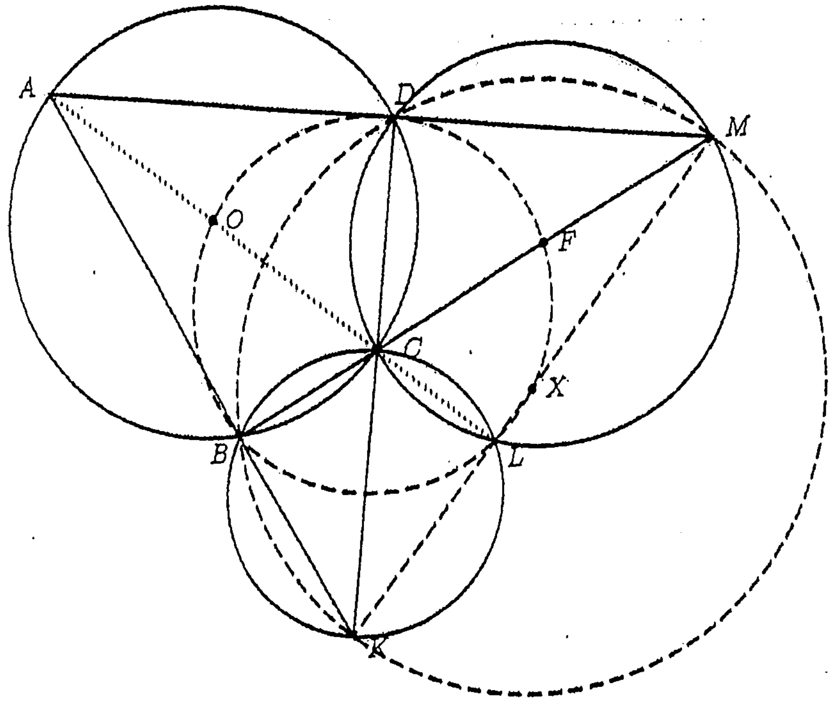
Using the initial condition $v = 5$ when $x = 0$,

$$0 = \frac{10}{3}\ln 15 - \frac{10}{6}\ln 75 - 0 + C$$

Hence $C = -\frac{5}{3}\ln\frac{15^2}{75} = -\frac{5}{3}\ln 3$. The maximum height is obtained when $v = 0$, hence

$$\text{Maximum height} = \frac{10}{3}\ln 10 - \frac{5}{3}\ln 100 - \frac{10}{\sqrt{3}}\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) - \frac{5}{3}\ln 3$$
$$= -\frac{5}{3}\ln 3 + \frac{10}{6\sqrt{3}}\pi$$
$$\doteqdot 1{\cdot}19 \text{ metres}$$

(b)



(i) $\angle CDA = 90°$ (angle in a semicircle) and so $\angle CDM = 90°$ (straight angle). Thus CM is the diameter of circle CDM and the centre F is the midpoint of CM. **1**

(ii) **2**

$\angle KLC = \angle CBA$ (exterior opposite angle of cyclic quadrilateral KLCB)

$= 90°$ (angle in semicircle)

$\angle MLC = \angle CDA$ (exterior opposite angle of cyclic quadrilateral MLCD)

$= 90°$ (angle in semicircle)

Hence $\angle KLC + \angle MLC = 180°$ and so KLM is straight. **2**

(iii) We have already shown that $\angle CBA = 90°$. Hence $\angle KBM = 90°$ (straight angle). Thus KM is the diameter of a circle through KMB (converse to angle in a semicircle theorem). Similarly, $\angle KDM = 90°$ (straight angle) and KM is the diameter of a circle through KMD. This must be the same circle, since they have a common diameter. The centre X of this circle is the midpoint of KM. **3**

(iv)

$\angle DFC = 2\alpha$ (angle at centre and circumference on arc DC)

$\angle DXB = 2\alpha$ (angle at centre and circumference on arc DB of circle $DBKM$)

Since DB subtends equal angles at F and X there must be a circle through $DBXF$.

QUESTION EIGHT (15 marks) **Marks**

(a) (i) Expanding, [1]

$$8(z-2)^3-(z-i)^3 = 8(z^3-6z^2+12z-8)-(z^3-3iz^2+3i^2z-i^3)$$
$$= (8-1)z^3+z^2(-48+3i)+z(96+3)-64-i$$

(ii) [4]

$$8(z-2)^3-(z-i)^3=0$$
$$8(z-2)^3=(z-i)^3$$
$$\left(\frac{2z-4}{z-i}\right)^3=1$$

Hence $\dfrac{2z-4}{z-i}=1,\ \omega$ or ω^2.

The first solution: $\dfrac{2z-4}{z-i}=1$

$$2z-4=z-i$$
$$z=4-i$$

The second solution: $\dfrac{2z-4}{z-i}=\omega$

$$2z-4=\omega z-\omega i$$
$$z=\frac{4-i\omega}{2-\omega}$$
$$z=\frac{(4-i\omega)(2-\omega^2)}{(2-\omega)(2-\omega^2)}$$
$$z=\frac{(8+i-2i\omega-4\omega^2)}{4+1+2}$$

where we have used the identities $\omega+\omega^2=-1$ and $\omega\times\omega^2=\omega^3=1$. Since $\omega=\frac{1}{2}+i\frac{\sqrt{3}}{2}$ and $\omega^2=\frac{1}{2}-i\frac{\sqrt{3}}{2}$ the second solution simplifies to

$$\frac{8+2+2i\sqrt{3}+i+\sqrt{3}+i}{7}=\frac{10+\sqrt{3}+2i(\sqrt{3}+1)}{7}$$

The calculation for the third solution is identical except ω and ω^2 are swapped, leading to

$$z=\frac{(8+i-2i\omega^2-4\omega)}{7}$$
$$=\frac{8+2-2i\sqrt{3}+i-\sqrt{3}+i}{7}$$
$$=\frac{10-\sqrt{3}+2i(1-\sqrt{3})}{7}$$

(The second and third solutions are like conjugates in the sense of swapping $\sqrt{3}$ and $-\sqrt{3}$.)

(b) (i) [4]

When $n=0$,

$$\frac{a_n}{b_n}=\frac{1}{\sqrt{1+\tan^2\theta}}=\frac{1}{\sqrt{\sec^2\theta}}=\cos\theta \quad \text{(as required)}$$

and

$$\frac{\sin\theta}{b_n}=\frac{\sin\theta}{1} \quad \text{(as required)}$$

Assume the result holds for $n=k$. That is, assume that

$$\frac{a_k}{b_k}=\cos\frac{\theta}{2^k} \qquad \text{and} \qquad \frac{\sin\theta}{b_k}=2^k\sin\frac{\theta}{2^k}.$$

We need to show that the result holds for $n = k+1$, that is to prove that

$$\frac{a_{k+1}}{b_{k+1}} = \cos\tfrac{\theta}{2^{k+1}} \qquad \text{and} \qquad \frac{\sin\theta}{b_{k+1}} = 2^{k+1}\sin\tfrac{\theta}{2^{k+1}}.$$

Consider the LHS of the first expression;

$$\begin{aligned}
\frac{a_{k+1}}{b_{k+1}} &= \frac{a_{k+1}}{\sqrt{a_{k+1}b_k}} \\
&= \sqrt{\frac{a_{k+1}}{b_k}} \\
&= \sqrt{\frac{\frac{1}{2}(a_k+b_k)}{b_k}} \\
&= \sqrt{\tfrac{1}{2}(\frac{a_k}{b_k}+1)} \\
&= \sqrt{\tfrac{1}{2}(1+\cos\tfrac{\theta}{2^k})} \quad \text{(By the inductive assumption)} \\
&= \sqrt{\cos^2\tfrac{1}{2}\tfrac{\theta}{2^k}} \quad \text{(By a double angle formula for cos)} \\
&= \cos\tfrac{\theta}{2^{k+1}} \quad (\star\star)
\end{aligned}$$

as required. For the second result, notice the recurrence relation $b_{k+1} = \sqrt{a_{k+1}b_k}$ may be rewritten $\frac{b_{k+1}}{b_k} = \frac{a_{k+1}}{b_{k+1}}$. Now consider the LHS of the second expression;

$$\begin{aligned}
\frac{\sin\theta}{b_{k+1}} &= \frac{\sin\theta}{b_k} \times \frac{b_k}{b_{k+1}} \\
&= 2^k\sin\tfrac{\theta}{2^k} \times \frac{b_k}{b_{k+1}} \quad \text{(By the inductive assumption)} \\
&= 2^k\sin\tfrac{\theta}{2^k} \times \frac{b_{k+1}}{a_{k+1}} \\
&= 2^k\sin\tfrac{\theta}{2^k} \times \sec\tfrac{\theta}{2^{k+1}} \quad \text{(By } \star\star\text{)} \\
&= 2^k \times 2\sin\left(\tfrac{1}{2}\tfrac{\theta}{2^k}\right)\cos\left(\tfrac{1}{2}\tfrac{\theta}{2^k}\right) \times \sec\tfrac{\theta}{2^{k+1}} \quad \text{(by the double angle formula for sin)} \\
&= 2^{k+1}\sin\tfrac{\theta}{2^{k+1}}
\end{aligned}$$

as required.

Hence the two results hold for all integers $n \geq 0$ by mathematical induction.

2

(ii)

$$\begin{aligned}
\frac{\sin\theta}{b_n} &= 2^n\sin\frac{\theta}{2^n} \\
&= \theta \times \frac{\sin\frac{\theta}{2^n}}{\frac{\theta}{2^n}} \\
&\to \theta \times 1 \quad (\text{as } \frac{\theta}{2^n} \to 0, \text{ i.e. as } n \to \infty)
\end{aligned}$$

Hence $\lim_{n\to\infty} \frac{x}{b_n\sqrt{1+x^2}} = \tan^{-1}x$, since $\frac{x}{\sqrt{1+x^2}} = \sin\theta$.

1

(iii)

n	0	1	2	3
a_n	0.7071	0.8536	0.8887	0.8974
b_n	1	0.9239	0.9061	0.9018

(iv) Hence $\frac{\pi}{4} \doteqdot \frac{1}{b_3\sqrt{2}} \doteqdot 0{\cdot}7851$. (Compare this with $\frac{\pi}{4} \doteqdot 0{\cdot}785398$ using the calculators value of π.) [1]

[2]

(v) Notice again that

$$\frac{b_{n+1}}{b_n} = \frac{a_{n+1}}{b_{n+1}} = \cos\frac{\theta}{2^{n+1}}$$

Hence
$$\begin{aligned} 0 &< 1 - \frac{b_{n+1}}{b_n} \\ &= 1 - \cos\frac{\theta}{2^{n+1}} \\ &= 2\sin^2\frac{\theta}{2^{n+2}} \end{aligned}$$

Hence we require
$$\begin{aligned} 2\sin^2\frac{\theta}{2^{n+2}} &< 10^{-10} \\ \sin\frac{\theta}{2^{n+2}} &< \frac{10^{-5}}{\sqrt{2}} \\ \frac{\theta}{2^{n+2}} &< \sin^{-1}\frac{10^{-5}}{\sqrt{2}} \\ 2^{n+2} &> \frac{\pi}{4\sin^{-1}\frac{10^{-5}}{\sqrt{2}}} \\ n &> \log_2\frac{\pi}{4\sin^{-1}\frac{10^{-5}}{\sqrt{2}}} - 2 \\ n &> 14{\cdot}8 \quad \text{(approx)} \end{aligned}$$

Thus 15 iterations will give the desired accuracy.

BDD