# *Applications of Arithmetic & Geometric Series*

## Formulae for Arithmetic Series

A series is called an ***arithmetic series*** if;

$$d = T_n - T_{n-1}$$

where ***d*** is a constant, called the ***common difference***

The ***general term*** of an arithmetic series with first term ***a*** and common difference ***d*** is;

$$T_n = a + (n-1)d$$

Three numbers $a$, $x$ and $b$ are terms in an arithmetic series if;

$$b - x = x - a \qquad \text{i.e. } x = \frac{a+b}{2}$$

The sum of the first $n$ terms of an arithmetic series is;

| | |
|---|---|
| $S_n = \frac{n}{2}(a + l)$ | (use when the last term $l = T_n$ is known) |
| $S_n = \frac{n}{2}\{2a + (n-1)d\}$ | (use when the difference $d$ is known) |

2007 HSC Question 3b)

Heather decides to swim every day to improve her fitness level.

On the first day she swims 750 metres, and on each day after that she swims 100 metres more than the previous day.

That is she swims 850 metres on the second day, 950 metres on the third day and so on.

(i) Write down a formula for the distance she swims on he $n$th day.

$a = 750$ Find a particular term, $T_n$ $T_n = 750 + (n-1)100$

$d = 100$ $T_n = a + (n-1)d$ $T_n = 650 + 100n$

(ii) How far does she swim on the 10th day? **(1)**

Find a particular term, $T_{10}$

$$T_n = 650 + 100n$$
$$T_{10} = 650 + 100(10)$$
$$T_{10} = 1650$$

$\therefore$ she swims 1650 metres on the 10th day

(iii) What is the total distance she swims in the first 10 days? **(1)**

Find a total amount, $S_{10}$

$$S_n = \frac{n}{2}\{2a + (n-1)d\}$$
$$S_{10} = \frac{10}{2}\{2(750) + 9(100)\}$$
$$S_{10} = 5(1500 + 900)$$
$$= 12000$$

***OR***

$$S_n = \frac{n}{2}(a + l)$$
$$S_{10} = \frac{10}{2}(750 + 1650)$$
$$= 12000$$

$\therefore$ she swims a total of 12000 metres in 10 days

(iv) After how many days does the total distance she has swum equal **(2)**
the width of the English Channel, a distance of 34 kilometres?

Find a total amount, $S_n$

$$S_n = \frac{n}{2}\left\{2a + (n-1)d\right\}$$

$$34000 = \frac{n}{2}\left\{1500 + (n-1)100\right\}$$

$$68000 = n(1400 + 100n)$$

$$68000 = 1400n + 100n^2$$

$$100n^2 + 1400n - 68000 = 0$$

$$n^2 + 14n - 680 = 0$$

$$(n+34)(n-20) = 0$$

$$\therefore n = -34 \quad \text{or} \quad n = 20$$

$\therefore$ it takes 20 days to swim 34 kilometres

## Formulae for Geometric Series

A series is called a ***geometric series*** if;

$$r = \frac{T_n}{T_{n-1}}$$

where ***r*** is a constant, called the ***common ratio***

The ***general term*** of a geometric series with first term ***a*** and common ratio ***r*** is;

$$T_n = ar^{n-1}$$

Three numbers $a$, $x$ and $b$ are terms in a geometric series if;

$$\frac{b}{x} = \frac{x}{a} \qquad \text{i.e. } x = \pm\sqrt{ab}$$

The sum of the first $n$ terms of a geometric series is;

$$S_n = \frac{a\left(r^n - 1\right)}{r - 1} \qquad \text{(easier when } r > 1\text{)}$$

$$S_n = \frac{a\left(1 - r^n\right)}{1 - r} \qquad \text{(easier when } r < 1\text{)}$$

e.g. A company's sales are declining by 6% every year, with 50000 items sold in 2001.

During which year will sales first fall below 20000?

$$a = 50000$$
$$r = 0.94$$
$$T_n < 20000$$

$$T_n = ar^{n-1}$$
$$20000 > 50000(0.94)^{n-1}$$
$$0.4 > (0.94)^{n-1}$$
$$(0.94)^{n-1} < 0.4$$
$$\log(0.94)^{n-1} < \log 0.4$$
$$(n-1)\log(0.94) < \log 0.4$$
$$(n-1) > \frac{\log 0.4}{\log 0.94}$$
$$(n-1) > 14.80864248$$
$$n > 15.80864248$$

$\therefore$ during the 16th year (i.e. 2016) sales will fall below 20000

2005 HSC Question 7a)

Anne and Kay are employed by an accounting firm.

Anne accepts employment with an initial annual salary of \$50000. In each of the following years her annual salary is increased by \$2500.

Kay accepts employment with an initial annual salary of \$50000. in each of the following years her annual salary is increased by 4%

***arithmetic***

$a = 50000$ $d = 2500$

***geometric***

$a = 50000$ $r = 1.04$

(i) What is Anne's salary in her thirteenth year? **(2)**

Find a particular term, $T_{13}$

$$T_n = a + (n-1)d$$

$$T_{13} = 50000 + 12(2500)$$

$$T_{13} = 80000$$

$\therefore$ Anne earns \$80000 in her 13th year

(ii) What is Kay's annual salary in her thirteenth year? **(2)**

Find a particular term, $T_{13}$

$$T_n = ar^{n-1}$$
$$T_{13} = 50000(1.04)^{12}$$
$$T_{13} = 80051.61093\ldots$$

$\therefore$ Kay earns \$80051.61 in her thirteenth year

(iii) By what amount does the total amount paid to Kay in her first twenty years exceed that paid to Anne in her first twenty years? **(3)**

Find a total amount, $S_{20}$

***Anne*** $S_n = \frac{n}{2}\{2a+(n-1)d\}$

$$S_{20} = \frac{20}{2}\{2(50000)+19(2500)\}$$
$$S_{10} = 10(100000+47500)$$
$$= 1475000$$

***Kay*** $S_n = \frac{a(r^n-1)}{n-1}$

$$S_{20} = \frac{50000(1.04^{20}-1)}{0.04}$$
$$= 1488903.93$$

$\therefore$ Kay is paid \$13903.93 more than Anne

The limiting sum $S_\infty$ exists if and only if $-1 < r < 1$, in which case;

$$S_\infty = \frac{a}{1-r}$$

2007 HSC Question 1d) **(2)**

Find the limiting sum of the geometric series

$$\frac{3}{4} + \frac{3}{16} + \frac{3}{64} + \dots$$

$$a = \frac{3}{4}$$

$$r = \frac{3}{16} \times \frac{4}{3}$$

$$= \frac{1}{4}$$

$$S_\infty = \frac{a}{1-r}$$

$$S_\infty = \frac{\frac{3}{4}}{1 - \frac{1}{4}}$$

$$= 1$$

2003 HSC Question 7a)

(i) Find the limiting sum of the geometric series **(2)**

$$2 + \frac{2}{\sqrt{2}+1} + \frac{2}{\left(\sqrt{2}+1\right)^2} + \dots$$

$$a = 2$$

$$r = \frac{2}{\sqrt{2}+1} \times \frac{1}{2}$$

$$= \frac{1}{\sqrt{2}+1}$$

$$S_\infty = \frac{a}{1-r}$$

$$S_\infty = \frac{2}{1 - \frac{1}{\sqrt{2}+1}}$$

$$S_\infty = \frac{2}{1} \times \frac{\sqrt{2}+1}{\sqrt{2}+1-1}$$

$$= \frac{2}{1} \times \frac{\sqrt{2}+1}{\sqrt{2}}$$

$$= \sqrt{2}\left(\sqrt{2}+1\right)$$

$$= 2+\sqrt{2}$$

(ii) Explain why the geometric series

$$2 + \frac{2}{\sqrt{2}-1} + \frac{2}{\left(\sqrt{2}-1\right)^2} + \ldots$$ does NOT have a limiting sum. **(1)**

Limiting sums only occur when $-1 < r < 1$

$$r = \frac{2}{\sqrt{2}-1} \times \frac{1}{2}$$

$$= \frac{1}{\sqrt{2}-1}$$

$$\therefore r = 2.414\ldots > 1$$

as $r > 1$, no limiting sum exists

2004 HSC Question 9a)

Consider the series $1 - \tan^2\theta + \tan^4\theta - \ldots$

$a = 1$ $\qquad$ $r = -\tan^2\theta$

(i) When the limiting sum exists, find its value in simplest form. **(2)**

$$S_\infty = \frac{a}{1-r}$$

$$S_\infty = \frac{1}{1+\tan^2\theta}$$

$$= \frac{1}{\sec^2\theta}$$

$$= \cos^2\theta$$

(ii) For what values of $\theta$ in the interval $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ does the limiting sum of the series exist? **(2)**

Limiting sums only occur when $-1 < r < 1$

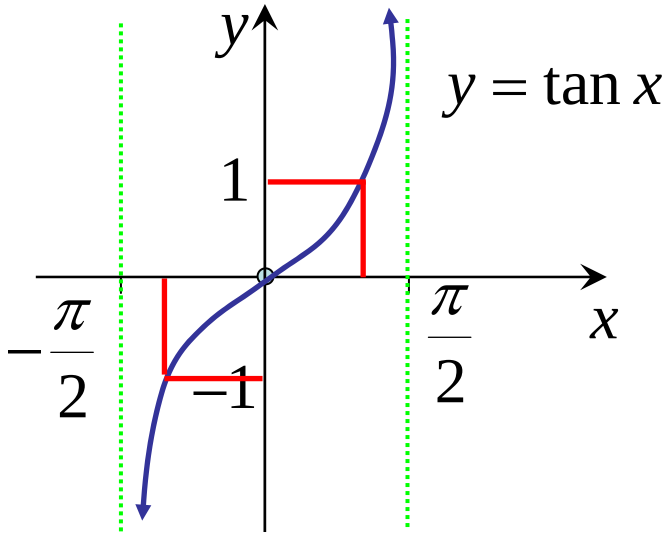
$$-1 < -\tan^2\theta < 1$$

$$1 > \tan^2\theta > -1$$

$$-1 < \tan^2\theta < 1$$

$$-1 < \tan\theta < 1$$

Now $\tan\theta = \pm 1$, when $\theta = \pm\frac{\pi}{4}$




$$\therefore -\frac{\pi}{4} < \theta < \frac{\pi}{4}$$

**Exercise 7A; 3, 4, 5, 10, 11, 16, 17, 20***