Polynomial Theorems

Remainder Theorem

If the polynomial $P(x)$ is divided by $(x - a)$, then the remainder is $P(a)$

Proof:

$$P\left(x\right) = A(x)Q(x) + R(x)$$

let $A(x) = (x - a)$

$$P\left(x\right) = (x - a)Q(x) + R(x)$$

$$P\left(a\right) = (a - a)Q(a) + R(a)$$

$$= R(a)$$

now degree $R(x) < 1$

$\therefore R(x)$ is a constant

$$R(x) = R(a)$$

$$= P(a)$$

e.g. Find the remainder when $P(x) = 5x^3 - 17x^2 - x + 11$ is divided by $(x - 2)$

$$P(x) = 5x^3 - 17x^2 - x + 11$$
$$P(2) = 5(2)^3 - 17(2)^2 - 2 + 11$$
$$= -19$$

$\therefore$ remainder when $P(x)$ is divided by $(x-2)$ is -19

Factor Theorem

If $(x - a)$ is a factor of $P(x)$ then $P(a) = 0$

e.g. (i) Show that $(x - 2)$ is a factor of $P(x) = x^3 - 19x + 30$ and hence factorise $P(x)$.

$$P(2) = (2)^3 - 19(2) + 30$$
$$= 0$$

$\therefore (x-2)$ is a factor

$$\begin{array}{r|llll}
 & x^2 & +2x & -15 & \\
\hline
x-2 & x^3 & +0x^2 & -19x & +30 \\
 & x^3 & -2x^2 & & \\
\hline
 & & 2x^2 & -19x & +30 \\
 & & 2x^2 & -4x & \\
\hline
 & & & -15x & +30 \\
 & & & -15x & +30 \\
\hline
 & & & & 0
\end{array}$$

$$\therefore P(x) = (x-2)\left(x^2 + 2x - 15\right)$$
$$= (x-2)(x+5)(x-3)$$

OR

$$P(x) = x^3 - 19x + 30$$
$$= (x-2)\left(x^2 \quad +2x \quad -15\right)$$

leading term $\times$ leading term constant $\times$ constant

=leading term =constant

If you where to expand out now, how many x would you have? $\underline{-15x}$

How many x do you need? $\underline{-19x}$

How do you get from what you have to what you need? $\underline{-4x}$

$$-4x = -2 \times ?$$

$$\therefore P(x) = (x-2)\left(x^2 + 2x - 15\right)$$
$$\underline{= (x-2)(x+5)(x-3)}$$

(ii) Factorise $P(x) = 4x^3 - 16x^2 - 9x + 36$

Constant factors must be a factor of the constant

Possibilities = 1 , 2 , 3 , 4 , 6 , 9 , 12 , 18 , 36

of course they could be negative!!!

Fractional factors must be of the form $\dfrac{\text{factors of the constant}}{\text{factors of the leading coefficient}}$

$$\text{Possibilities} = \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \cancel{\frac{4}{4}}, \cancel{\frac{6}{4}}, \frac{9}{4}, \cancel{\frac{12}{4}}, \cancel{\frac{18}{4}}, \cancel{\frac{36}{4}}$$

$$= \frac{1}{2}, \cancel{\frac{2}{2}}, \frac{3}{2}, \cancel{\frac{4}{2}}, \cancel{\frac{6}{2}}, \frac{9}{2}, \cancel{\frac{12}{2}}, \cancel{\frac{18}{2}}, \cancel{\frac{36}{2}}$$

they could be negative too

$$P(4) = 4(4)^3 - 16(4)^2 - 9(4) + 36$$
$$= 0$$

$\therefore (x-4)$ is a factor

$$P(x) = 4x^3 - 16x^2 - 9x + 36$$
$$= (x-4)(4x^2 \qquad -9)$$
$$= \underline{(x-4)(2x+3)(2x-3)}$$

2004 Extension 1 HSC Q3b)

Let $P(x) = (x + 1)(x - 3)Q(x) + a(x + 1) + b$, where $Q(x)$ is a polynomial and a and b are real numbers.

When $P(x)$ is divided by $(x + 1)$ the remainder is -11. When $P(x)$ is divided by $(x - 3)$ the remainder is 1.

(i) What is the value of b?

$$P(-1) = -11$$

$$\therefore b = -11$$

(ii) What is the remainder when $P(x)$ is divided by $(x + 1)(x - 3)$?

$$P(3) = 1$$

$$4a + b = 1$$

$$4a = 12$$

$$a = 3$$

$$P(x) = (x+1)(x-3)Q(x) + 3x - 8$$

$$\therefore R(x) = 3x - 8$$

Suppose $x^3 - 2x^2 + a \equiv (x+2)Q(x) + 3$ where $Q(x)$ is a polynomial.

Find the value of a.

$$P(-2) = 3$$

$$(-2)^3 - 2(-2)^2 + a = 3$$

$$-16 + a = 3$$

$$\underline{a = 19}$$

1994 Extension 1 HSC Q4a)

When the polynomial $P(x)$ is divided by $(x + 1)(x – 4)$, the quotient is $Q(x)$ and the remainder is $R(x)$.

(i) Why is the most general form of $R(x)$ given by $R(x) = ax + b$?

The degree of the divisor is 2, therefore the degree of the remainder is at most 1, i.e. a linear function.

(ii) Given that $P(4) = – 5$, show that $R(4) = – 5$

$$P(x) = (x + 1)(x – 4)Q(x) + R(x)$$

$$P(4) = (4 + 1)(4 – 4)Q(4) + R(4)$$

$$\underline{R(4) = – 5}$$

(iii) Further, when $P(x)$ is divided by $(x + 1)$, the remainder is 5. Find $R(x)$

$$R(4)=-5 \qquad P(-1)=5$$

$$4a+b=-5 \qquad -a+b=5$$

$$\therefore 5a=-10$$

$$a=-2 \quad \therefore b=3$$

$$\underline{R(x)=-2x+3}$$

$$2x-1$$

$$\text{use } P\left(\frac{1}{2}\right)$$

Exercise 4D; 1bc, 3ac, 4ac, 7acf, 8bdf, 9b, 12ac, 13, 14, 17, 23*