Mod-Arg Relations

$$(1)\ |z_1 z_2| = |z_1||z_2|$$

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2$$

Proof: let $z_1 = r_1 cis\theta_1$ and $z_2 = r_2 cis\theta_2$

$$z_1 z_2 = r_1(\cos\theta_1 + i\sin\theta_1)\times r_2(\cos\theta_2 + i\sin\theta_2)$$

$$= r_1 r_2(\cos\theta_1\cos\theta_2 + i\sin\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 - \sin\theta_1\sin\theta_2)$$

$$= r_1 r_2\{(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\}$$

$$= r_1 r_2\{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\}$$

$$\therefore |z_1 z_2| = r_1 r_2 \qquad\qquad \arg(z_1 z_2) = \theta_1 + \theta_2$$

$$= |z_1||z_2| \qquad\qquad = \arg z_1 + \arg z_2$$

NOTE: it follows that;

$$|z_1 z_2 z_3 \ldots z_n| = |z_1||z_2||z_3|\ldots|z_n|$$

$$\arg(z_1 z_2 z_3 \ldots z_n) = \arg z_1 + \arg z_2 + \arg z_3 + \ldots + \arg z_n$$

$$(2)\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

$$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$$

Proof:

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \times \frac{\cos\theta_2 - i\sin\theta_2}{\cos\theta_2 - i\sin\theta_2}$$

$$= \frac{r_1}{r_2}\left(\frac{\cos\theta_1\cos\theta_2 + i\sin\theta_1\cos\theta_2 - i\cos\theta_1\sin\theta_2 + \sin\theta_1\sin\theta_2}{\cos^2\theta_2 + \sin^2\theta_2}\right)$$

$$= \frac{r_1}{r_2}\{(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)\}$$

$$= \frac{r_1}{r_2}\{\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\}$$

$$\therefore \left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|}$$

$$\arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \arg z_1 - \arg z_2$$

NOTE: it follows that;

$$\left|\frac{z_1 z_2}{z_3 z_4}\right| = \frac{|z_1||z_2|}{|z_3||z_4|}$$

$$\arg\left(\frac{z_1 z_2}{z_3 z_4}\right) = \arg z_1 + \arg z_2 - \arg z_3 - \arg z_4$$

$$(3)\ |z^n| = |z|^n$$

$$\arg(z^n) = n \arg z$$

e.g. Find the modulus and argument of $z = \dfrac{(5+i)(-2-i)}{3+2i}$

$$|z| = \frac{\sqrt{5^2+1^2}\sqrt{(-2)^2+(-1)^2}}{\sqrt{3^2+2^2}}$$

$$= \frac{\sqrt{26}\sqrt{5}}{\sqrt{13}}$$

$$= \sqrt{10}$$

$$\arg z = \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{-1}{-2}\right) - \tan^{-1}\left(\frac{2}{3}\right)$$

$$= 11^\circ 19' + (-153^\circ 26') - 33^\circ 41'$$

$$= -175^\circ 48'$$

Exercise 4B; evens

Exercise 4C; 1 to 10 evens, 12 to 15 all